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Tempered Fractional Stable Motion

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Abstract Tempered fractional stable motion adds an exponential tempering to the power-law kernel in a linear fractional stable motion, or a shift to the power-law filter in a harmonizable fractional stable motion. Increments from a stationary time series that can exhibit semi-long-range dependence. This paper develops the basic theory of tempered fractional stable processes, including dependence structure, sample path behavior, local times, and local nondeterminism.

Keywords Stable process · Fractional calculus · Long-range dependence · Local times · Sample paths · Local nondeterminism

Mathematics Subject Classification 60G52 · 60G17 · 60E07

1 Introduction

Linear fractional stable motion and real harmonizable fractional stable motion are distinct stochastic processes with stationary increments, see for example Samorodnitsky and Taqqu [16]. They can be constructed using the fractional integral of a symmetric α-stable (SαS) noise [7, Remark 7.30]. These models are useful in practice, because their increments can exhibit the heavy-tailed analog of long-range dependence, see for example Watkins et al. [18]. This paper develops a model extension, based on tempered fractional calculus [14]. The resulting stationary increment processes, termed

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linear tempered fractional stable motion (LTFSM) and real harmonizable tempered fractional stable motion (HTFSM), are obtained by replacing the fractional integral with a tempered fractional integral. The (Riemann–Liouville) tempered fractional integral

$$\mathbb{I}^{\alpha,\lambda} f(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} f(u)(t-u)_+^{\alpha-1} e^{-\lambda(t-u)_+} \mathrm{d}u,$$

with $\alpha > 0$, $\lambda > 0$, and $(x)_+ = xI(x > 0)$, is a convolution with an exponentially tempered power law [9]. It reduces to the traditional Riemann–Liouville fractional integral when $\lambda = 0$ [15, Definition 2.1].

The remainder of this paper is organized as follows. Section 2 develops the LTFSM model, proves a scaling property, and computes the dependence structure of the increment process, which can exhibit the heavy-tailed analog of semi-long-range dependence. Section 3 computes the dependence structure of the TFSM increments and uses this to prove that LTFSM and HTFSM are different processes. Section 4 establishes sample path properties of LTFSM and HTFSM. Section 5 proves the existence of local times and establishes the useful property of local nondeterminism.

2 Moving Average Process

We say that the real-valued random variable X has a symmetric α-stable (SαS) distribution, denoted by $S_\alpha(\sigma, 0, 0)$, if its characteristic function has the form

$$\mathbb{E}\left[\exp\{i(\theta X)\}\right] = \exp\left\{-\sigma^\alpha |\theta|^\alpha\right\},$$

for some constants $\sigma > 0$ and $0 < \alpha \leq 2$. The parameters α and σ are called the index of stability and the scale parameter, respectively [16, Chapter 1]. The formula

$$\|X\|_\alpha = \left(-\log \mathbb{E}[e^{iX}]\right)^{1/\alpha} \tag{2.1}$$

defines a norm (quasinorm if $0 < \alpha < 1$) on the space of SαS random variables, see Nolan [11,12] and Xiao [19] for more details.

Let $L^0(\Omega)$ be the set of all real-valued random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(E, \mathcal{E}, m)$ be a measure space and define $\mathcal{E}_0 = \{A \in \mathcal{E} : m(A) < \infty\}$. An independently scattered set function $M : \mathcal{E}_0 \to L^0(\Omega)$ such that

$$M(A) \sim S_\alpha\left((m(A)^{\frac{1}{\alpha}}), 0, 0\right)$$

for each $A \in \mathcal{E}_0$ is called an SαS random measure on $(E, \mathcal{E})$ with control measure m. Independently scattered means that if $A_1, A_2, \ldots, A_k$ belongs to $\mathcal{E}_0$ and are disjoint, then the random variables $M(A_1), M(A_2), \ldots, M(A_k)$ are independent.

Given an independently scattered SαS random measure $Z_\alpha(\mathrm{d}x)$ on the real line with Lebesgue control measure $\mathrm{d}x$, the stochastic integral

$$I(f) := \int_{-\infty}^{+\infty} f(x)\, Z_\alpha(\mathrm{d}x) \tag{2.2}$$

is defined for all Borel measurable functions $f \in L^\alpha(\mathbb{R})$. Then, [16, Proposition 3.4.1] shows that $I(f)$ is an SαS random variable with characteristic function

$$\mathbb{E}\left[e^{i\theta I(f)}\right] = \exp\left\{-|\theta|^\alpha \int_{-\infty}^{+\infty} |f(x)|^\alpha \, dx\right\},$$

and hence, we have

$$\left\| I(f) \right\|_\alpha^\alpha = \int_{-\infty}^{+\infty} \left| f(x) \right|^\alpha dx \tag{2.3}$$

for any $0 < \alpha < 2$.

Definition 2.1 Given an independently scattered SαS random measure $Z_\alpha(dx)$ on $\mathbb{R}$ with control measure dx, the stochastic integral

$$X_{H,\alpha,\lambda}(t) := \int_{-\infty}^{+\infty} \left[e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{1}{\alpha}} \right] Z_\alpha(dx) \tag{2.4}$$

with $0 < \alpha < 2$, $0 < H < 1$, $\lambda > 0$, $(x)_+ = \max\{x, 0\}$, and $0^0 = 0$ will be called a *linear tempered fractional stable motion* (LTFSM).

Remark 2.2 When $\alpha = 2$, the LTFSM reduces to a tempered fractional Brownian motion, see [8,9]. When $\lambda = 0$, it becomes a linear fractional stable motion [16, Section 7.4]. The stable *Yaglom noise*

$$G_{H,\alpha,\lambda}(t) := \int_{-\infty}^{+\infty} e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha}} \, Z_\alpha(dx)$$

is also well defined, due to the exponential tempering, and clearly, $X_{H,\alpha,\lambda}(t) = G_{H,\alpha,\lambda}(t) - G_{H,\alpha,\lambda}(0)$. Stable Yaglom noise is the tempered fractional integral of the stable noise $Z_\alpha(dx)$, up to a multiplicative constant.

It is easy to check that the function

$$g_{\alpha,\lambda,t}(x) := e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{1}{\alpha}} \tag{2.5}$$

belongs to $L^\alpha(\mathbb{R})$, so that LTFSM is well defined; furthermore,

$$\left\| X_{H,\alpha,\lambda}(t) \right\|_\alpha^\alpha = \int_{\mathbb{R}} \left| g_{\alpha,\lambda,t}(x) \right|^\alpha dx \tag{2.6}$$

for any $0 < \alpha < 2$. The next result shows that LTFSM has a nice scaling property, involving both the time scale and the tempering.

Proposition 2.3 *The* LTFSM (2.4) *is an SαS process with stationary increments, such that*

$$\left\{X_{H,\alpha,\lambda}(ct)\right\}_{t\in\mathbb{R}} \triangleq \left\{c^H X_{H,\alpha,c\lambda}(t)\right\}_{t\in\mathbb{R}} \tag{2.7}$$

for any scale factor $c > 0$*, where* $\triangleq$ *indicates equality in the sense of finite dimensional distributions.*

Proof Since $Z_\alpha(\mathrm{d}x)$ has control measure $\mathrm{d}x$, the random measure $Z_\alpha(c\,\mathrm{d}x)$ has control measure $c^{\frac{1}{\alpha}}\mathrm{d}x$. Note that

$$g_{\alpha,\lambda,ct}(cx) = c^{H-\frac{1}{\alpha}} g_{\alpha,c\lambda,t}(x), \tag{2.8}$$

for all $t, x \in \mathbb{R}$ and all $c > 0$. Given $t_1 < t_2 < \cdots < t_n$, a change of variable $x = cx'$ then yields

$$\begin{aligned}
\big(X_{H,\alpha,\lambda}(ct_i) : i = 1, \ldots, n\big) &= \left(\int g_{\alpha,\lambda,ct_i}(x) Z_\alpha(\mathrm{d}x) : i = 1, \ldots, n\right) \\
&= \left(\int g_{\alpha,\lambda,ct_i}(cx') Z_\alpha(c\,\mathrm{d}x') : i = 1, \ldots, n\right) \\
&\simeq \left(\int c^{H-\frac{1}{\alpha}} g_{\alpha,c\lambda,t_i}(x') c^{\frac{1}{\alpha}} Z_\alpha(\mathrm{d}x') : i = 1, \ldots, n\right) \\
&= \left(c^H X_{H,\alpha,c\lambda}(t_i) : i = 1, \ldots, n\right)
\end{aligned}$$

where $\simeq$ denotes equality in distribution, so that (2.7) holds. For any $s, t \in \mathbb{R}$, the integrand (2.5) satisfies $g_{\alpha,\lambda,s+t}(s+x) - g_{\alpha,\lambda,s}(s+x) = g_{\alpha,\lambda,t}(x)$; hence, a change of variable $x = s + x'$ yields

$$\begin{aligned}
&\big(X_{H,\alpha,\lambda}(s+t_i) - X_{H,\alpha,\lambda}(s) : i = 1, \ldots, n\big) \\
&\quad = \left(\int \left[g_{\alpha,\lambda,s+t_i}(x) - g_{\alpha,\lambda,s}(x)\right] Z_\alpha(\mathrm{d}x) : i = 1, \ldots, n\right) \\
&\quad \simeq \left(\int \left[g_{\alpha,\lambda,s+t_i}(s+x') - g_{\alpha,\lambda,s}(s+x')\right] Z_\alpha(\mathrm{d}x') : i = 1, \ldots, n\right) \\
&\quad = \left(\int g_{\alpha,\lambda,t_i}(x') Z_\alpha(\mathrm{d}x') : i = 1, \ldots, n\right) \\
&\quad = \big(X_{H,\alpha,\lambda}(t_i) : i = 1, \ldots, n\big)
\end{aligned}$$

which shows that LTFSM has stationary increments. □

Next, we consider the increments of LTFSM, which form a stationary stochastic process in view of Proposition 2.3.

Definition 2.4 Given an LTFSM (2.4), we define the tempered fractional stable noise (TFSN)

$$Y_{H,\alpha,\lambda}(t) := X_{H,\alpha,\lambda}(t+1) - X_{H,\alpha,\lambda}(t) \quad \text{for integers } -\infty < t < \infty. \tag{2.9}$$

Astrauskas et al. [1] studied the dependence structure of linear fractional stable noise using the following nonparametric measure of dependence. Given a stationary SαS process $\{Y(t)\}$, we define

$$r(t) = r(\theta_1, \theta_2, t) := \mathbb{E}\left[e^{i(\theta_1 Y(t) + \theta_2 Y(0))}\right] - \mathbb{E}\left[e^{i\theta_1 Y(t)}\right]\mathbb{E}\left[e^{i\theta_2 Y(0)}\right] \tag{2.10}$$

for $t > 0$ and $\theta_1, \theta_2 \in \mathbb{R}$. If we also define

$$I(t) = I(\theta_1, \theta_2, t) := \|\theta_1 Y(t) + \theta_2 Y(0)\|_\alpha^\alpha - \|\theta_1 Y(t)\|_\alpha^\alpha - \|\theta_2 Y(0)\|_\alpha^\alpha \tag{2.11}$$

then we have

$$r(\theta_1, \theta_2, t) = K(\theta_1, \theta_2)\left(e^{-I(\theta_1, \theta_2, t)} - 1\right), \tag{2.12}$$

where

$$K(\theta_1, \theta_2) := \mathbb{E}\left[e^{i\theta_1 Y(t)}\right]\mathbb{E}\left[e^{i\theta_2 Y(0)}\right] = \mathbb{E}\left[e^{i\theta_1 Y(0)}\right]\mathbb{E}\left[e^{i\theta_2 Y(0)}\right] \tag{2.13}$$

since $Y(t)$ is stationary. If $I(t) \to 0$ as $t \to \infty$, then $r(t) \sim -K(\theta_1, \theta_2)I(t)$ as $t \to \infty$. If $\{Y(t)\}_{t\in\mathbb{R}}$ is a stationary Gaussian process, then $-I(1, -1, t) = \mathrm{Cov}[Y(t), Y(0)]$, so that $r(t) \sim K(\theta_1, \theta_2)\mathrm{Cov}[\theta_1 Y(t), \theta_2 Y(0)]$ in this (typical) case; hence, $r(t)$ is a natural extension of the usual autocovariance function.

Next, we compute the dependence structure of TFSN. Given two real-valued functions $f(t)$, $g(t)$ on $\mathbb{R}$, we will write $f(t) \asymp g(t)$ if $C_1 \le |f(t)/g(t)| \le C_2$ for all $t > 0$ sufficiently large, for some $0 < C_1 < C_2 < \infty$.

Theorem 2.5 *Let $Y_{H,\alpha,\lambda}(t)$ be a tempered fractional stable noise (2.9) for some $0 < \alpha \le 1$ and $0 < H < 1$. Then,*

$$r(\theta_1, \theta_2, t) \asymp e^{-\lambda\alpha t} t^{H\alpha - 1} \tag{2.14}$$

as $t \to \infty$ for all $\lambda > 0$.

Proof It follows easily from (2.4) that TFSN has the moving average representation

$$Y_{H,\alpha,\lambda}(t) = \int_{-\infty}^{+\infty} \left[e^{-\lambda(t+1-x)_+}(t+1-x)_+^{H-\frac{1}{\alpha}} - e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha}}\right] Z_\alpha(\mathrm{d}x). \tag{2.15}$$

Define $g_t(x) = (t-x)_+^{H-\frac{1}{\alpha}} e^{-\lambda(t-x)_+}$ for $t \in \mathbb{R}$ and write

$$\begin{aligned} I(\theta_1, \theta_2, t) &= \int_{-\infty}^{+\infty} \left|\theta_1\left[g_{t+1}(x) - g_t(x)\right] + \theta_2\left[g_1(x) - g_0(x)\right]\right|^\alpha \mathrm{d}x \\ &\quad - \int_{-\infty}^{+\infty} \left|\theta_1\left[g_{t+1}(x) - g_t(x)\right]\right|^\alpha \mathrm{d}x - \int_{-\infty}^{+\infty} \left|\theta_2\left[g_1(x) - g_0(x)\right]\right|^\alpha \mathrm{d}x \\ &:= I_1(\theta_1, \theta_2, t) + I_2(\theta_1, \theta_2, t), \end{aligned} \tag{2.16}$$

where

$$I_1(\theta_1,\theta_2,t) = \int_{-\infty}^{0} \Big|\theta_1\left[g_{t+1}(x)-g_t(x)\right]+\theta_2\left[g_1(x)-g_0(x)\right]\Big|^{\alpha}\,\mathrm{d}x$$
$$- \int_{-\infty}^{0} \Big|\theta_1\big[g_{t+1}(x)-g_t(x)\big]\Big|^{\alpha}\,\mathrm{d}x - \int_{-\infty}^{0} \Big|\theta_2\big[g_1(x)-g_0(x)\big]\Big|^{\alpha}\,\mathrm{d}x$$

and

$$I_2(\theta_1,\theta_2,t) = \int_0^1 \Big|\theta_1\left[g_{t+1}(x)-g_t(x)\right]+\theta_2 g_1(x)\Big|^{\alpha}\,\mathrm{d}x$$
$$- \int_0^1 \Big|\theta_1\big[g_{t+1}(x)-g_t(x)\big]\Big|^{\alpha}\,\mathrm{d}x - \int_0^1 \Big|\theta_2 g_1(x)\Big|^{\alpha}\,\mathrm{d}x.$$

Also,

$$\begin{aligned} K(\theta_1,\theta_2) &= \mathbb{E}\Big[e^{i\theta_1 Y(t)}\Big]\mathbb{E}\Big[e^{i\theta_2 Y(0)}\Big] \\ &= \mathbb{E}\Big[e^{i\theta_1 Y(0)}\Big]\mathbb{E}\Big[e^{i\theta_2 Y(0)}\Big] \\ &= \exp\Big\{-(|\theta_1|^{\alpha}+|\theta_2|^{\alpha})\int_{-\infty}^{+\infty}\Big|g_1(x)-g_0(x)\Big|^{\alpha}\,\mathrm{d}x\Big\} \end{aligned} \tag{2.17}$$

by stationarity. Therefore, $I(\theta_1,\theta_2,t) = K(\theta_1,\theta_2)(I_1(t)+I_2(t))$, where we write $I_j(\theta_1,\theta_2,t) = I_j(t)$ for $j = 1, 2$ for brevity. A change of variable in $I_1(t)$ for $t > 1$ gives

$$\begin{aligned} I_1(t) = \int_0^{\infty} &\Big|\theta_1\big[e^{-\lambda(t+1+x)}(t+1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t+x)}(t+x)^{H-\frac{1}{\alpha}}\big] \\ &+\theta_2\big[e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda x}x^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha}\mathrm{d}x \\ &-\int_0^{\infty}\Big|\theta_1\big[e^{-\lambda(t+1+x)}(t+1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t+x)}(t+x)^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha}\mathrm{d}x \\ &-\int_0^{\infty}\Big|\theta_2\big[e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda x}x^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha}\,\mathrm{d}x \end{aligned}$$

Let

$$f_{t+1,t}(x) := \Big|\theta_1\big[e^{-\lambda(t+1+x)}(t+1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t+x)}(t+x)^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha}. \tag{2.18}$$

For every $t > 1$ and $x > 0$, we get

$$\begin{aligned} e^{\alpha\lambda t}t^{-\alpha(H-\frac{1}{\alpha})}f_{t+1,t}(x) &= \Big|\theta_1\Big|^{\alpha}\Big|e^{-\lambda(1+x)}\Big(\frac{t+1+x}{t}\Big)^{H-\frac{1}{\alpha}} - e^{-\lambda x}\Big(\frac{t+x}{t}\Big)^{H-\frac{1}{\alpha}}\Big|^{\alpha} \\ &\to \Big|\theta_1\Big|^{\alpha}\,e^{-\lambda\alpha x}\,\Big|e^{-\lambda}-1\Big|^{\alpha} \text{ as } t\to\infty \end{aligned}$$

and

$$\sup_{t>1}\left|e^{\alpha\lambda t}t^{-\alpha\left(H-\frac{1}{\alpha}\right)}f_{t+1,t}(x)\right| \leq \left|\theta_1(e^{-\lambda}-1)\right|^{\alpha}e^{-\lambda\alpha x}$$

which belongs to $L^1(0,\infty)$. Now we can use the dominated convergence theorem to see that

$$\begin{aligned}\int_0^{\infty} f_{t+1,t}(x)\,\mathrm{d}x &\to \left|\theta_1(e^{-\lambda}-1)\right|^{\alpha}e^{-\lambda\alpha t}t^{\alpha(H-\frac{1}{\alpha})}\int_0^{\infty}e^{-\lambda\alpha x}\,\mathrm{d}x\\ &= \frac{\left|\theta_1(e^{-\lambda}-1)\right|^{\alpha}e^{-\lambda\alpha t}t^{\alpha(H-\frac{1}{\alpha})}}{\lambda\alpha}\end{aligned} \tag{2.19}$$

as $t\to\infty$. Now consider,

$$\begin{aligned}g_{t,t+1,0,1}(x){:=}&\Big|\theta_1\big[e^{-\lambda(t+1+x)}(t+1+x)^{H-\frac{1}{\alpha}}-e^{-\lambda(t+x)}(t+x)^{H-\frac{1}{\alpha}}\big]\\ &+\theta_2\big[e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}}-e^{-\lambda x}x^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha}\\ &-\left|\theta_2\right|^{\alpha}\Big|\big[e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}}-e^{-\lambda x}x^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha}.\end{aligned} \tag{2.20}$$

Then,

$$\begin{aligned}e^{\lambda\alpha t}t^{-\alpha(H-\frac{1}{\alpha})}g_{t,t+1,0,1}(x)=&\Bigg|\theta_1\Bigg[e^{-\lambda(1+x)}\left(\frac{t+1+x}{t}\right)^{H-\frac{1}{\alpha}}-e^{-\lambda x}\left(\frac{t+x}{t}\right)^{H-\frac{1}{\alpha}}\Bigg]\\ &+\theta_2\Bigg[e^{-\lambda(1+x)}e^{\lambda t}\left(\frac{1+x}{t}\right)^{H-\frac{1}{\alpha}}-e^{-\lambda x}e^{\lambda t}\left(\frac{x}{t}\right)^{H-\frac{1}{\alpha}}\Bigg]\Bigg|^{\alpha}\\ &-\Bigg|\theta_2\Bigg[e^{-\lambda(1+x)}e^{\lambda t}\left(\frac{1+x}{t}\right)^{H-\frac{1}{\alpha}}-e^{-\lambda x}e^{\lambda t}\left(\frac{x}{t}\right)^{H-\frac{1}{\alpha}}\Bigg]\Bigg|^{\alpha}\\ &=:\left|a_t+b_t\right|^{\alpha}-\left|b_t\right|^{\alpha}\end{aligned}$$

where

$$a_t=\theta_1\left[e^{-\lambda(1+x)}\left(\frac{t+1+x}{t}\right)^{H-\frac{1}{\alpha}}-e^{-\lambda x}\left(\frac{t+x}{t}\right)^{H-\frac{1}{\alpha}}\right]$$

and

$$b_t=\theta_2\left[e^{-\lambda(1+x)}e^{\lambda t}\left(\frac{1+x}{t}\right)^{H-\frac{1}{\alpha}}-e^{-\lambda x}e^{\lambda t}\left(\frac{x}{t}\right)^{H-\frac{1}{\alpha}}\right].$$

It is obvious that $a_t\to C_x{:=}\theta_1e^{-\lambda x}(e^{-\lambda}-1)$ and $b_t\to-\infty$ as $t\to\infty$. Then, $|a_t+b_t|^{\alpha}-|b_t|^{\alpha}\to 0$ as $t\to\infty$ since $0<\alpha\leq 1$. Therefore,

$$e^{\lambda\alpha t}t^{-\alpha(H-\frac{1}{\alpha})}g_{t,t+1,0,1}\to 0,$$

as $t \to \infty$. Moreover, for any $0 < \alpha \le 1$, using the inequality $\left||a|^\alpha - |b|^\alpha\right| \le \left|a-b\right|^\alpha$ (see [16], Page 211), we get

$$\left|g_{t,t+1,0,1}\right| \le f_{t+1,t},$$

where $g_{t,t+1,0,1}$ and $g_{t,t+1,0,1}$ are defined in (2.18) and (2.20), respectively, if we let $a = \theta_1(g_{t+1} - g_t) + \theta_2(g_1 - g_0)$ and $b = \theta_2(g_1 - g_0)$. Consequently,

$$\begin{aligned}\sup_{t>1}\left|e^{\lambda\alpha t} t^{-\alpha(H-\frac{1}{\alpha})} g_{t,t+1,0,1}\right| &\le \sup_{t>1}\left|e^{\alpha\lambda t} t^{-\alpha(H-\frac{1}{\alpha})} f_{t+1,t}(x)\right| \\ &\le \left|\theta_1(e^{-\lambda}-1)\right|^\alpha e^{-\lambda\alpha x}\end{aligned}$$

which also belongs to $L^1(0,\infty)$. Applying the dominated convergence theorem yields

$$\int_{-\infty}^{+\infty} g_{t,t+1,0,1}(x)\mathrm{d}x \to 0 \quad \text{as } t \to \infty. \tag{2.21}$$

Therefore, from (2.19) and (2.21)

$$I_1(t) \sim -C_1 e^{-\lambda\alpha t} t^{H\alpha-1} \tag{2.22}$$

as $t \to \infty$, where $C_1 := |\theta_1(e^{-\lambda} - 1)|^\alpha/(\lambda\alpha)$.

Next, write

$$\begin{aligned}I_2(t) = &\int_0^1 \left|\theta_1[g_{t+1}(x) - g_t(x)] + \theta_2 g_1(x)\right|^\alpha \mathrm{d}x \\ &- \int_0^1 \left|\theta_1[g_{t+1}(x) - g_t(x)]\right|^\alpha \mathrm{d}x - \int_0^1 \left|\theta_2\, g_1(x)\right|^\alpha \mathrm{d}x,\end{aligned}$$

Define

$$u_t(x) := \theta_1\left[e^{-\lambda(t+1-x)}(t+1-x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t-x)}(t-x)^{H-\frac{1}{\alpha}}\right], \tag{2.23}$$

and

$$v(x) := \theta_2 e^{-\lambda(1-x)}(1-x)^{H-\frac{1}{\alpha}}. \tag{2.24}$$

Rewrite

$$I_2(t) = \int_0^1 \xi(u_t(x) + v(x)) - \xi(u_t(x)) - \xi(v(x))\,\mathrm{d}x$$

where

$$\xi(x) := \left|x\right|^\alpha. \tag{2.25}$$

Using [1, Eq. (3.9)], we have

$$|I_2(t)| \le \int_0^1 |\xi(u_t(x)+v(x)) - \xi(u_t(x)) - \xi(v(x))|\,\mathrm{d}x \le 2\int_0^1 \left|u_t(x)\right|^\alpha \mathrm{d}x. \tag{2.26}$$

On the other hand, $u_t(x) = \theta_1(f_x(t+1) - f_x(t))$ where $f_x(u) = e^{-\lambda(u-x)}(u-x)^{H-\frac{1}{\alpha}}$. Recall that $H - \frac{1}{\alpha} < 0$, and apply the mean value theorem to see that for any $0 < x < 1$ and $t > 2$, we have for some $u \in (t, t+1)$ that

$$
\begin{aligned}
\left|u_t(x)\right| &\leq \left|\theta_1\right|\left| -\lambda e^{-\lambda(u-x)}(u-x)^{H-\frac{1}{\alpha}} + \left(H - \frac{1}{\alpha}\right) e^{-\lambda(u-x)}(u-x)^{H-\frac{1}{\alpha}-1}\right| \\
&\leq \left|\theta_1\right| e^{-\lambda(t-1)}\left[\left(\frac{1}{\alpha} - H\right)|t-1|^{H-\frac{1}{\alpha}-1} + \lambda\,|t-1|^{H-\frac{1}{\alpha}}\right] \\
&\leq \left|\theta_1\right| e^{-\lambda(t-1)}\left[\frac{1}{\alpha} - H + \lambda\right]\left|t-1\right|^{H-\frac{1}{\alpha}}.
\end{aligned}
\tag{2.27}
$$

From (2.26) and (2.27), we get

$$
I_2(t) \leq 2\int_0^1 \left|u_t(x)\right|^{\alpha} \mathrm{d}x \leq 2\left|\theta_1\right|^{\alpha} e^{-\lambda\alpha(t-1)}\left[\frac{1}{\alpha} - H + \lambda\right]^{\alpha}\left|t-1\right|^{H\alpha-1}. \tag{2.28}
$$

Hence, $|I_2(t)| \leq C_2 e^{-\lambda\alpha t} t^{H\alpha-1}$ for $t > 0$ large, where $C_2 := 2|\theta_1|^{\alpha} e^{\lambda\alpha}[\alpha^{-1} - H + \lambda]^{\alpha}$. Then, it follows from (2.22) and (2.28) that

$$
I(t) \asymp e^{-\lambda\alpha t} t^{H\alpha-1}
$$

as $t \to \infty$. Since $I(t) \to 0$ as $t \to \infty$, it follows from (2.12) that $r(t) \sim -K(\theta_1, \theta_2)I(t)$; hence, (2.14) holds. □

Theorem 2.6 *Let $Y_{H,\alpha,\lambda}(t)$ be a tempered fractional stable noise (2.9) for some $1 < \alpha < 2$, $\frac{1}{\alpha} < H < 1$, and $\lambda > 0$. Then,*

$$
r(t) \asymp e^{-\lambda t} t^{H-\frac{1}{\alpha}}
$$

as $t \to \infty$.

Proof Recall that $f_{t+1,t}(x)$ is given by (2.18). Then,

$$
\begin{aligned}
e^{\lambda t}\, t^{-(H-\frac{1}{\alpha})} f_{t+1,t}(x) = &\left|\theta_1\right|^{\alpha} e^{\lambda t} t^{-\left(H-\frac{1}{\alpha}\right)} \\
&\times\left|e^{-\lambda(t+1+x)}(t+1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t+x)}(t+x)^{H-\frac{1}{\alpha}}\right|^{\alpha} = a_t \cdot b_t,
\end{aligned}
$$

where

$$
a_t := \left|\theta_1\right|^{\alpha} e^{-\lambda t(\alpha-1)} t^{\left(H-\frac{1}{\alpha}\right)(\alpha-1)}
$$

and

$$
b_t := \left|e^{-\lambda(1+x)}\left(1 + \frac{1}{t} + \frac{x}{t}\right)^{H-\frac{1}{\alpha}} - e^{-\lambda x}\left(1 + \frac{x}{t}\right)^{H-\frac{1}{\alpha}}\right|^{\alpha}.
$$

Note that $a_t \to 0$ (since $1 < \alpha < 2$) and $b_t \to \left|e^{-\lambda(1+x)} - e^{-\lambda x}\right|^\alpha$ as $t \to \infty$. Now, let $h(t) = e^{-\lambda t(\alpha-1)} t^{(\alpha-1)(H-\frac{1}{\alpha})}$. Observe that $h(t)$ attains its maximum at $t = \frac{1}{\lambda}(H - \frac{1}{\alpha})$. Moreover, since $H - \frac{1}{\alpha} > 0$ we have for any fixed $x > 0$ and all $t \geq 1$ that

$$\begin{aligned} d(t) &:= \left|e^{-\lambda(1+x)}\big(1+\frac{1}{t}+\frac{x}{t}\big)^{H-\frac{1}{\alpha}} - e^{-\lambda x}\left(1+\frac{x}{t}\right)^{H-\frac{1}{\alpha}}\right| \\ &\leq e^{-\lambda x}\Big[\Big|e^{-\lambda}\big(1+\frac{1}{t}+\frac{x}{t}\big)^{H-\frac{1}{\alpha}}\Big| + \Big|\big(1+\frac{x}{t}\big)^{H-\frac{1}{\alpha}}\Big|\Big] \\ &\leq e^{-\lambda x}\Big[e^{-\lambda}(2+x)^{H-\frac{1}{\alpha}} + (1+x)^{H-\frac{1}{\alpha}}\Big] \\ &\leq e^{-\lambda x}(2+x)^{H-\frac{1}{\alpha}}(e^{-\lambda}+1). \end{aligned}$$

Then,

$$\begin{aligned} &\sup_{t>1}\left|e^{\lambda t}\, t^{-(H-\frac{1}{\alpha})} f_{t+1,t}(x)\right| = \sup_{t>1}\left|a_t \cdot b_t\right| = \left|\theta_1\right|^\alpha \sup_{t>1}\left|h(t)(d(t))^\alpha\right| \\ &\quad\leq \left|\theta_1\right|^\alpha \sup_{t>1}\left|h(t)\right| \sup_{t>1}\left|(d(t))^\alpha\right| \\ &\quad\leq \left|\theta_1\right|^\alpha e^{-\lambda\alpha x}(2+x)^{H\alpha-1}(e^{-\lambda}+1)^\alpha e^{-(H-\frac{1}{\alpha})(\alpha-1)}\Big[\frac{H-\frac{1}{\alpha}}{\lambda}\Big]^{(\alpha-1)(H-\frac{1}{\alpha})}, \end{aligned}$$

and so $f_{t+1,t}(x)$ is bounded by an $L^1(0,\infty)$ function. Therefore, the dominated convergence theorem implies that

$$\int_0^\infty f_{t+1,t}(x)\, \mathrm{d}x \to 0 \tag{2.29}$$

as $t \to \infty$. Consider now, $e^{\lambda t} t^{-(H-\frac{1}{\alpha})} g_{t,t+1,0,1}$ where $g_{t,t+1,0,1}$ is given by (2.20). Then,

$$e^{\lambda t} t^{-(H-\frac{1}{\alpha})} g_{t,t+1,0,1} = \left|a_t + b_t\right|^\alpha - \left|b_t\right|^\alpha$$

where

$$a_t := \theta_1\left[e^{-\lambda t(1-\frac{1}{\alpha})} e^{-\lambda(1+x)}\left(\frac{t+1+x}{t^{\frac{1}{\alpha}}}\right)^{(H-\frac{1}{\alpha})} - e^{-\lambda t(1-\frac{1}{\alpha})} e^{-\lambda x}\left(\frac{t+x}{t^{\frac{1}{\alpha}}}\right)^{(H-\frac{1}{\alpha})}\right]$$

and

$$b_t := \theta_2\left[e^{\frac{\lambda t}{\alpha}} t^{\frac{-(H-\frac{1}{\alpha})}{\alpha}}\left[e^{-\lambda(1+x)}(1+x)^{(H-\frac{1}{\alpha})} - e^{-\lambda x} x^{(H-\frac{1}{\alpha})}\right]\right]$$

Observe that $\lim_{t\to\infty} b_t = -\infty$ and $\lim_{t\to\infty} a_t = 0$. Since $|a_t + b_t|^\alpha - |b_t|^\alpha \sim \alpha|a_t||b_t|^{\alpha-1}$, as $t \to \infty$, we get,

$$
\begin{aligned}
& e^{\lambda t} t^{-(H-\frac{1}{\alpha})} g_{t,t+1,0,1} \sim \alpha \Big|\theta_1\Big| \\
& \quad \times \Big| e^{-\lambda t(1-\frac{1}{\alpha})} e^{-\lambda(1+x)} \Big(\frac{t+1+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})} - e^{-\lambda t(1-\frac{1}{\alpha})} e^{-\lambda x} \Big(\frac{t+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})} \Big| \\
& \quad \times \Big|\theta_2\Big|^{\alpha-1} e^{\lambda t(1-\frac{1}{\alpha})} t^{-(H-\frac{1}{\alpha})(1-\frac{1}{\alpha})} \Big| e^{-\lambda(1+x)} (1+x)^{(H-\frac{1}{\alpha})} - e^{-\lambda x} x^{(H-\frac{1}{\alpha})} \Big|^{\alpha-1}
\end{aligned}
$$

consequently,

$$
\begin{aligned}
e^{\lambda t} t^{-(H-\frac{1}{\alpha})} g_{t,t+1,0,1} \to & \ \alpha \Big|\theta_1\Big| \Big| e^{-\lambda(1+x)} - e^{-\lambda x} \Big| \\
& \times \Big|\theta_2\Big|^{\alpha-1} \Big| e^{-\lambda(1+x)} (1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda x} x^{H-\frac{1}{\alpha}} \Big|^{\alpha-1}.
\end{aligned}
$$

Moreover,

$$
\sup_{t\ge 1} \Big| e^{\lambda t} t^{-(H-\frac{1}{\alpha})} g_{t,t+1,0,1} \Big| = \sup_{t\ge 1} \Big| \Big|a_t + b_t\Big|^{\alpha} - \Big|b_t\Big|^{\alpha} \Big| \le \sup_{t\ge 1} \Big|a_t\Big|^{\alpha} + \alpha \sup_{t\ge 1} \Big|a_t\Big| \Big|b_t\Big|^{\alpha-1} \tag{2.30}
$$

where we have used the following inequalities (see for example Magdziarz [6, Lemma 2]): $|a-b|^{\alpha} \le a^{\alpha} + b^{\alpha}$ and $\Big| |a+b|^{\alpha} - |b|^{\alpha} \Big| \le \Big|a\Big|^{\alpha} + \alpha \Big|a\Big| \Big|b\Big|^{\alpha-1}$ valid for $a \ge 0$ and $b \ge 0$ and $\alpha \in (1,2)$. In order to find an upper bound for $\sup_{t\ge 1} |a_t|^{\alpha}$, write

$$
\begin{aligned}
& \Big|a_t\Big|^{\alpha} \\
& \quad = \Big|\theta_1\Big|^{\alpha} \Big| e^{-\lambda t(1-\frac{1}{\alpha})} e^{-\lambda(1+x)} \Big(\frac{t+1+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})} - e^{-\lambda t(1-\frac{1}{\alpha})} e^{-\lambda x} \Big(\frac{t+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})} \Big|^{\alpha} \\
& \quad = \Big|\theta_1\Big|^{\alpha} e^{-\lambda \alpha x} e^{-\lambda t(\alpha-1)} \Big| e^{-\lambda} \Big(\frac{t+1+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})} - \Big(\frac{t+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})} \Big|^{\alpha} \\
& \quad \le \Big|\theta_1\Big|^{\alpha} e^{-\lambda \alpha x} \Big| e^{-\lambda} (1+1+x)^{H-\frac{1}{\alpha}} - (1+x)^{H-\frac{1}{\alpha}} \Big|^{\alpha} \\
& \quad \le \Big|\theta_1\Big|^{\alpha} e^{-\lambda \alpha x} \Big[e^{-\lambda \alpha} (2+x)^{H\alpha-1} + (1+x)^{H\alpha-1} \Big] \\
& \quad \le 2 \Big|\theta_1\Big|^{\alpha} e^{-\lambda \alpha x} (2+x)^{H\alpha-1}.
\end{aligned} \tag{2.31}
$$

On the other hand,

$$
\begin{aligned}
\alpha \Big|a_t\Big| \Big|b_t\Big|^{\alpha-1} = & \ \alpha \Big|\theta_1\Big| \Big|\theta_2\Big|^{\alpha-1} \\
& \times \Big| \underbrace{e^{-\lambda(1+x)} \Big(\frac{t+1+x}{t}\Big)^{(H-\frac{1}{\alpha})} - e^{-\lambda x} \Big(\frac{t+x}{t}\Big)^{(H-\frac{1}{\alpha})}}_{:=S(t)} \Big| \times K(x)
\end{aligned}
$$

where

$$K(x) = \left|e^{-\lambda(1+x)}(1+x)^{(H-\frac{1}{\alpha})} - e^{-\lambda x}(x)^{(H-\frac{1}{\alpha})}\right|^{\alpha-1}. \tag{2.32}$$

Note that $S(t)$ is a decreasing function and hence

$$\sup_{t\geq 1} \alpha \left|a_t\right|\left|b_t\right|^{\alpha-1} = \alpha\left|\theta_1\right|\left|\theta_2\right|^{\alpha-1} \left|e^{-\lambda(1+x)}(2+x)^{(H-\frac{1}{\alpha})} - e^{-\lambda x}(1+x)^{(H-\frac{1}{\alpha})}\right| \times K(x) \tag{2.33}$$

where $K(x)$ is given by (2.32). From (2.30), (2.31) and (2.33)

$$\sup_{t\geq 1}\left|e^{\lambda t}t^{-(H-\frac{1}{\alpha})}g_{t,t+1,0,1}\right| \leq 2\left|\theta_1\right|^{\alpha}e^{-\lambda\alpha x}(2+x)^{H\alpha-1} + \alpha\left|\theta_1\right|\left|\theta_2\right|^{\alpha-1} \left|e^{-\lambda(1+x)}(2+x)^{(H-\frac{1}{\alpha})} - e^{-\lambda x}(1+x)^{(H-\frac{1}{\alpha})}\right| \times K(x) \tag{2.34}$$

which belongs to $L^1(0,\infty)$, since $H\alpha > 1$. Then, the dominated convergence theorem implies that

$$\begin{aligned}\int_0^\infty g_{t,t+1,0,1}(x)\,\mathrm{d}x &\to \alpha\theta_1\left|\theta_2\right|^{\alpha-1}e^{-\lambda t}t^{(H-\frac{1}{\alpha})} \\ &\times \int_0^\infty \left|e^{-\lambda(1+x)} - e^{-\lambda x}\right|\left|e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda x}x^{H-\frac{1}{\alpha}}\right|^{\alpha-1}\mathrm{d}x \\ &= C_2(\alpha,\lambda,\theta_1,\theta_2)e^{-\lambda t}t^{H-\frac{1}{\alpha}}\end{aligned} \tag{2.35}$$

as $t \to \infty$, where

$$C_2(\alpha,\lambda,\theta_1,\theta_2) = \alpha\theta_1\left|\theta_2\right|^{\alpha-1} \int_0^\infty \left|e^{-\lambda(1+x)} - e^{-\lambda x}\right|\left|e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda x}x^{H-\frac{1}{\alpha}}\right|^{\alpha-1}\mathrm{d}x \tag{2.36}$$

is a constant independent of t. Therefore, from (2.29) and (2.35) we have

$$I_1(t) \sim C_2(\alpha,\lambda,\theta_1,\theta_2)e^{-\lambda t}t^{H-\frac{1}{\alpha}} \tag{2.37}$$

as $t \to \infty$.

Finally, recall that

$$I_2(t) = \int_0^1 \left|\theta_1[g_{t+1}(x) - g_t(x)] + \theta_2 g_1(x)\right|^\alpha \mathrm{d}x - \int_0^1 \left|\theta_1[g_{t+1}(x) - g_t(x)]\right|^\alpha \mathrm{d}x - \int_0^1 \left|\theta_2\, g_1(x)\right|^\alpha \mathrm{d}x,$$

and that $u_t(x)$ and $v(x)$ are given by (2.23) and (2.24), respectively. Then,

$$I_2(t) = \int_0^1 \xi(u_t(x) + v(x)) - \xi(u_t(x)) - \xi(v(x))\,\mathrm{d}x$$

where $\xi(x)$ is given by (2.25).

To finish the proof, we need an upper bound for $u_t(x)$. Applying an argument similar to (2.27), using the mean value theorem, and recalling that $H - \frac{1}{\alpha} > 0$, for any fixed $0 < x < 1$ and any $t \geq 2$, for some $u \in (t, t+1)$, we have

$$\begin{aligned}
\left|u_t(x)\right| &\leq \left|\theta_1\right|\left| - \lambda e^{-\lambda(u-x)}(u-x)^{H-\frac{1}{\alpha}} + (H - \frac{1}{\alpha})e^{-\lambda(u-x)}(u-x)^{H-\frac{1}{\alpha}-1}\right| \\
&\leq \left|\theta_1\right| e^{-\lambda(t-1)}\Big[(H - \frac{1}{\alpha})\left|t+1\right|^{H-\frac{1}{\alpha}-1} + \lambda\,|t+1|^{H-\frac{1}{\alpha}}\Big] \\
&\leq \left|\theta_1\right| e^{-\lambda(t-1)}\Big[H - \frac{1}{\alpha} + \lambda\Big]\left|t+1\right|^{H-\frac{1}{\alpha}}.
\end{aligned}$$

Now, using [1, Eq. (3.9)] and the above upper bound for $u_t(x)$ we have

$$\begin{aligned}
|I_2(t)| &\leq \int_0^1 |\xi(u_t(x) + v(x)) - \xi(u_t(x)) - \xi(v(x))|\,\mathrm{d}x \\
&\leq \int_0^1 \alpha\left|u_t(x)\right|\left|v(x)\right|^{\alpha-1}\mathrm{d}x + (\alpha+1)\int_0^1 \left|u_t(x)\right|^\alpha \mathrm{d}x \\
&\leq \alpha\left|\theta_1\right|\int_0^1 \Big[H - \frac{1}{\alpha} + \lambda\Big]\left|t+1\right|^{H-\frac{1}{\alpha}} e^{-\lambda(t-1)}\left|v(x)\right|^{\alpha-1}\mathrm{d}x \\
&\quad + (\alpha+1)\left|\theta_1\right|^\alpha\Big[H - \frac{1}{\alpha} + \lambda\Big]^\alpha\left|t+1\right|^{H\alpha-1} e^{-\lambda\alpha(t-1)} \\
&= \alpha\left|\theta_1\right|\Big|\Big[H - \frac{1}{\alpha} + \lambda\Big]\Big|\left|t+1\right|^{H-\frac{1}{\alpha}} e^{-\lambda(t-1)} \\
&\quad \times \int_0^1 \left|\theta_2 e^{-\lambda(1-x)}(1-x)^{H-\frac{1}{\alpha}}\right|^{\alpha-1}\mathrm{d}x \\
&\quad + (\alpha+1)\left|\theta_1\right|^\alpha\Big[H - \frac{1}{\alpha} + \lambda\Big]^\alpha\left|t+1\right|^{H\alpha-1} e^{-\lambda\alpha(t-1)} \\
&= C_3(\alpha, \lambda, \theta_1)\left|t+1\right|^{H-\frac{1}{\alpha}} e^{-\lambda(t-1)} \\
&\quad + (\alpha+1)\left|\theta_1\right|^\alpha\Big[H - \frac{1}{\alpha} + \lambda\Big]^\alpha\left|t+1\right|^{H\alpha-1} e^{-\lambda\alpha(t-1)},
\end{aligned} \tag{2.38}$$

where

$$C_3(\alpha,\lambda,\theta_1) := \alpha\left|\theta_1\right|\left[H - \frac{1}{\alpha} + \lambda\right]\int_0^1 \left|\theta_2 e^{-\lambda(1-x)}(1-x)^{H-\frac{1}{\alpha}}\right|^{\alpha-1} \mathrm{d}x$$

is a constant. Note that the upper bound in (2.38) is of the same order as the upper bound for $I_1(t)$, given by (2.37). Hence,

$$r(t) \sim -I(t) \asymp e^{-\lambda t} t^{(H-\frac{1}{\alpha})}$$

as $t \to \infty$. □

Remark 2.7 We say that a stationary SαS process $\{Y_t\}$ exhibits *long-range dependence* if

$$\sum_{n=0}^{\infty} \left| r(\theta_1, \theta_2, n) \right| = \infty, \tag{2.39}$$

where $r(\theta_1, \theta_2, t)$ was defined in (2.10). LTFSM is not long-range dependent, but it does exhibit *semi-long-range dependence* under the assumptions of Theorems 2.5 and 2.6. That is, for $\lambda > 0$ sufficiently small, the sum in (2.39) is large, since it tends to infinity as $\lambda \to 0$. TFSN therefore provides a useful alternative model for data that exhibit strong dependence, which is in some sense more tractable. In applications to turbulence with heavy tails, it can also provide a useful model extension that more closely fits the observed dependence structure outside the inertial range [10,14].

3 Harmonizable Process

Let $X = X_1 + iX_2$ be a complex-valued random variable. We say X is isotropic SαS if the vector (X_1, X_2) is SαS and for any $\theta = \theta_1 + i\theta_2$ we have

$$\mathbb{E}\left[e^{i(\theta_1 X_1 + \theta_2 X_2)}\right] = e^{-c|\theta|^\alpha}$$

for some constant $c > 0$ [16, Section 2.6]. A complex-valued stochastic process $\{\widetilde{X}(t)\}$ is called isotropic SαS if all complex linear combinations $\sum_{j=1}^{n} \theta_j \widetilde{X}(t_j)$ are complex-valued isotropic SαS random variables. We say that $\widetilde{Z}_\alpha(\mathrm{d}k)$ is a complex-valued isotropic SαS random measure with Lebesgue control measure $\mathrm{d}k$ if

$$\mathbb{E}\left[e^{i\mathrm{Re}(\overline{\theta}\widetilde{Z_\alpha}(B))}\right] = e^{-|B||\theta|^\alpha},$$

where $|B|$ denotes the Lebesgue measure of the set $B \in \mathcal{B}(\mathbb{R})$ [16, Section 6.1] and $\theta \in \mathbb{C}$. For any $f \in L^\alpha(\mathbb{R})$, the stochastic integral

$$\tilde{I}(f) := \mathrm{Re}\int_{-\infty}^{+\infty} f(k)\tilde{Z}_\alpha(\mathrm{d}k)$$

is a complex-valued SαS random variable with characteristic function

$$\mathbb{E}\left[e^{i\theta \widetilde{I}(f)}\right] = \exp\left\{|\theta|^\alpha \int_{-\infty}^{+\infty} \left|f(k)\right|^\alpha \,\mathrm{d}k\right\} \tag{3.1}$$

hence,

$$\left\|\widetilde{I}(f)\right\|_\alpha^\alpha := -\log \mathbb{E}\left[e^{i\widetilde{I}(f)}\right] = \int_{-\infty}^{+\infty} \left|f(k)\right|^\alpha \,\mathrm{d}k \tag{3.2}$$

for any $0 < \alpha < 2$.

Definition 3.1 Given a complex isotropic SαS random measure $\widetilde{Z}_\alpha$ with Lebesgue control measure, the stochastic integral

$$\widetilde{X}_{H,\alpha,\lambda}(t) = \mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{-ikt}-1}{(\lambda - ik)^{H+\frac{1}{\alpha}}}\widetilde{Z}_\alpha(\mathrm{d}k) \tag{3.3}$$

with $0 < \alpha < 2$, $H > 0$, and $\lambda > 0$ will be called a *real harmonizable tempered fractional stable motion* (HTFSM).

If we define

$$\tilde{g}_{\alpha,\lambda,t}(k) := \frac{e^{-ikt}-1}{(\lambda - ik)^{H+\frac{1}{\alpha}}} \tag{3.4}$$

then $|\tilde{g}_{\alpha,\lambda,t}(k)|^\alpha$ is $O(|k|^{-H\alpha-1})$ as $|k| \to \infty$, and tends to zero as $|k| \to 0$. Hence, $\tilde{g}_{\alpha,\lambda,t} \in L^\alpha(\mathbb{R})$, so that HTFSM is well defined. The term $(\lambda - ik)^{-H-\frac{1}{\alpha}}$ in (3.3) is the Fourier symbol of tempered fractional integral [9, Lemma 2.6]. Hence, HTFSM is also constructed from the tempered fractional integral of a stable noise.

Proposition 3.2 *The* HTFSM (3.3) *is an isotropic SαS process with stationary increments, such that*

$$\left\{\widetilde{X}_{H,\alpha,\lambda}(ct)\right\}_{t\in\mathbb{R}} \triangleq \left\{c^H \widetilde{X}_{H,\alpha,c\lambda}(t)\right\}_{t\in\mathbb{R}} \tag{3.5}$$

for any scale factor $c > 0$.

Proof The proof is similar to Proposition 2.3. Since $\widetilde{Z}_\alpha(\mathrm{d}k)$ has control measure $\mathrm{d}k$, $\widetilde{Z}_\alpha(c\,\mathrm{d}x)$ has control measure $c^{\frac{1}{\alpha}}\mathrm{d}k$. Then, a simple change of variables in Definition (3.3) shows that $\widetilde{X}_{H,\alpha,\lambda}(ct) \simeq c^H \widetilde{X}_{H,\alpha,c\lambda}(t)$. For any $s, t \in \mathbb{R}$, write

$$\widetilde{X}_{H,\alpha,\lambda}(t+s) - \widetilde{X}_{H,\alpha,\lambda}(s) = \mathrm{Re}\int_{-\infty}^{+\infty} e^{-iks}\frac{e^{-ikt}-1}{(\lambda - ik)^{H+\frac{1}{\alpha}}}\widetilde{Z}_\alpha(\mathrm{d}k).$$

Since $|e^{-iks}| = 1$, it follows immediately from (3.1) that $\widetilde{X}_{H,\alpha,\lambda}(t+s) - \widetilde{X}_{H,\alpha,\lambda}(s) \simeq \widetilde{X}_{H,\alpha,\lambda}(t)$. The same arguments extend easily to finite dimensional distributions. □

Definition 3.3 Given an HTFSM (3.3), we define the *tempered fractional harmonizable stable noise* (TFHSN)

$$\widetilde{Y}_{H,\alpha,\lambda}(t) := \widetilde{X}_{H,\alpha,\lambda}(t+1) - \widetilde{X}_{H,\alpha,\lambda}(t) \quad \text{for integers } -\infty < t < \infty. \tag{3.6}$$

Theorem 3.4 *The tempered fractional stable motion (LTFSM) defined in (2.4) and tempered fractional harmonizable stable motion (HTFSM) defined in (3.3) are different processes.*

Proof Theorems 2.5 and 2.6 imply that

$$\lim_{t\to\infty} r_{Y_{H,\alpha,\lambda}}(\theta_1, \theta_2, t) = 0, \tag{3.7}$$

for $0 < \alpha \le 1, 0 < H < 1$ and $1 < \alpha < 2, \frac{1}{\alpha} < H < 1$, respectively (in fact, according to Theorem 2.1 in [4], $\lim_{t\to\infty} r_X = 0$ for any α-stable moving average representation). It follows easily from (3.3) that

$$\widetilde{Y}_{H,\alpha,\lambda}(t) = \mathrm{Re}\int_{-\infty}^{+\infty} e^{-ikt}\, \Psi(\mathrm{d}k) \tag{3.8}$$

where

$$\Psi(\mathrm{d}k) = \frac{e^{-ik}-1}{(\lambda - ik)^{H+\frac{1}{\alpha}}}\widetilde{Z}_\alpha(\mathrm{d}k)$$

is a complex symmetric α-stable ($S\alpha S$) random measure with control measure

$$m(\mathrm{d}k) = \frac{|e^{-ik}-1|^\alpha}{|\lambda - ik|^{H\alpha+1}}\,\mathrm{d}k.$$

Then, it follows from Levy and Taqqu [4, Theorem 3.1] that

$$\begin{aligned}&\liminf_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} r_{\widetilde{Y}_{H,\alpha,\lambda}}(\theta_1, \theta_2, t)\, dt\\ &\quad\ge K(\theta_1, \theta_2)c_0\left(m(\{0\})F_0 + \frac{1}{2\pi}m(\mathbb{R}-\{0\})F_1\right) > 0\end{aligned}$$

where $F_0 \in \mathbb{R}$ and $F_1 > 0$ are constants depending on $\alpha,\ m,\ \theta_1$ and θ_2. Then, we have

$$\lim_{t\to\infty} r_{\widetilde{Y}_{H,\alpha,\lambda}}(\theta_1, \theta_2, t) > 0, \tag{3.9}$$

and the theorem follows. □

Remark 3.5 A simpler proof of (3.7) follows from Kokoszka and Taqqu [13, Lemma 6.1], but Theorem 2.5 gives more information on the dependence structure.

4 Sample Path Properties

In this section, we develop sample path properties of tempered fractional stable motions. The path behavior of a linear tempered fractional stable motion $X_{H,\alpha,\lambda}$ depends on the structure of the kernel (2.5). When $H - \frac{1}{\alpha} < 0$, the function $g_{\alpha,\lambda,t}(x)$

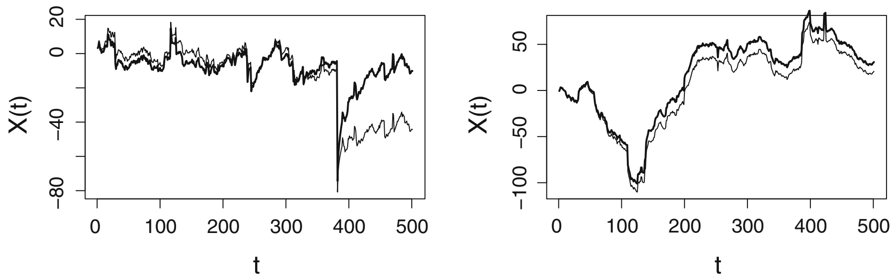

Fig. 1 *Left panel* Sample paths of LTFSM with $\alpha = 1.5$ and $H = 0.3$ for $\lambda = 0.03$ (*thick line*) and $\lambda = 0$ (*thin line*). Both graphs use the same noise realization $Z_\alpha(t)$. The *right panel* shows the same plots for $H = 0.7$, comparing $\lambda = 0.001$ (*thick line*) and $\lambda = 0$ (*thin line*)

has singularities at $x = 0$ and $x = t$. These singularities, together with the heavy tails of the stable noise process $Z_\alpha(\mathrm{d}x)$, induce path irregularity, see Stoev and Taqqu [17] for the case $\lambda = 0$. The left panel in Fig. 1 compares a typical sample path of tempered and untempered linear fractional stable motion, using the same noise realization $Z_\alpha(t)$, in the case $H - \frac{1}{\alpha} < 0$. In the case $H - \frac{1}{\alpha} > 0$ (since $0 < H < 1$, it follows that $\alpha > 1$), the paths of a linear (tempered) fractional stable motion can be made continuous with probability one (see [16, Chapter 10] for the untempered case), since its kernel is bounded and positive for all $t > 0$. The right panel in Figure 1 shows a typical sample path in the case. These simulations use a simple discretized version of the moving average representation (2.4). The remainder of this section develops these ideas in detail and provides smoothness (Hölder continuity) estimates in the case $H > \frac{1}{\alpha}$.

Recall that a stochastic process $\{X(t), t \in T\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called separable if there is a countable set $T^* \subset T$ and an event $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 0$ such that for any closed set $F \subset \mathbb{R}$ we have

$$\left\{\omega : X(t) \in F, \forall t \in T^*\right\} \setminus \{\omega : X(t) \in F, \forall t \in T\} \subset \Omega_0.$$

See [16, Chapter 9] for more details.

Theorem 4.1 *Suppose that* $0 < H < \frac{1}{\alpha}$ *for some* $0 < \alpha < 2$. *Then, for any separable version of the LTFSM process defined in* (2.4), *for any* $\lambda > 0$, *we have that*

$$\mathbb{P}\Big(\{\omega : \sup_{t \in (a,b)} \left|X_{H,\alpha,\lambda}(t,\omega)\right| = \infty\}\Big) = 1,$$

Hence, every separable version of the LTFSM process has unbounded paths in this case.

Proof We apply Theorem 10.2.3 in [16]. Indeed, consider the countable set $T^* := \mathbb{Q} \cap [a, b]$, where $\mathbb{Q}$ denotes the set of rational numbers. Since T^* is dense in $[a, b]$, there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \in T^*$ such that $t_n \to x$ as $n \to \infty$, for any $x \in [a, b]$.

Therefore,

$$f^*(T^*;x) := \sup_{t\in T^*}\left|g_{\alpha,\lambda,t}(x)\right| \geq \sup_{t_n\in T^*}\left|g_{\alpha,\lambda,t_n}(x)\right| =: f_n^*(T^*;x) = \infty,$$

as $n\to\infty$; hence, $\int_a^b f^*(T^*;x)\,\mathrm{d}x=\infty$, and this contradicts Condition (10.2.14) of Theorem 10.2.3 in [16]. Therefore, the stochastic process $\{X_{H,\alpha,\lambda}\}$ does not have a version with bounded paths on the interval (a,b), and this completes the proof. □

Lemma 4.2 *Suppose that* $\frac{1}{\alpha}<H<1$ *for some* $1<\alpha<2$. *Then, there exist positive constants* C_1 *and* C_2 *such that the LTFSM* (2.4) *satisfies*

$$C_1\left|t-s\right|^{H\alpha} \leq \left\|X_{H,\alpha,\lambda}(t)-X_{H,\alpha,\lambda}(s)\right\|_\alpha^\alpha \leq C_2\left|t-s\right|^{H\alpha}$$

locally uniformly in $s,t\in[0,1]$, *for any* $\lambda>0$.

Proof Assume $s<t$, and write

$$\begin{aligned}\left\|X_{H,\alpha,\lambda}(t)-X_{H,\alpha,\lambda}(s)\right\|_\alpha^\alpha &\geq \int_s^t |t-x|^{\alpha(H-\frac{1}{\alpha})}e^{-\lambda\alpha|t-x|}\,\mathrm{d}x\\ &\geq e^{-\lambda\alpha|t-s|}\int_s^t |t-x|^{H\alpha-1}\,\mathrm{d}x\\ &= \frac{e^{-\lambda\alpha|t-s|}}{H\alpha}|t-s|^{H\alpha}\\ &\geq \frac{e^{-\lambda\alpha}}{H\alpha}|t-s|^{H\alpha}\end{aligned}$$

for any $0\leq s<t\leq 1$, which establishes the lower bound.

It follows from (2.6) that $\|X_{H,\alpha,\lambda}(t)-X_{H,\alpha,\lambda}(s)\|_\alpha^\alpha=(I_1+I_2)$ where

$$\begin{aligned}I_1&=\int_{-\infty}^s\left|e^{-\lambda(t-x)}(t-x)^{H-\frac{1}{\alpha}}-e^{-\lambda(s-x)}(s-x)^{H-\frac{1}{\alpha}}\right|^\alpha\mathrm{d}x,\\ I_2&=\int_s^t\left|e^{-\lambda(t-x)}(t-x)^{H-\frac{1}{\alpha}}\right|^\alpha\mathrm{d}x\leq\int_s^t\left|(t-x)^{H-\frac{1}{\alpha}}\right|^\alpha\mathrm{d}x=\frac{1}{H\alpha}|t-s|^{H\alpha}.\end{aligned}$$

Using the inequality $|x+y|^\alpha\leq 2^\alpha(|x|^\alpha+|y|^\alpha)$ for $x,y\in\mathbb{R}$ and $\alpha>0$ we have $I_1\leq 2^\alpha(I_{11}+I_{12})$ where

$$\begin{aligned}I_{11}&=\int_{-\infty}^s\left|e^{-\lambda(t-x)}(t-x)^{H-\frac{1}{\alpha}}-e^{-\lambda(t-x)}(s-x)^{H-\frac{1}{\alpha}}\right|^\alpha\mathrm{d}x,\\ I_{12}&=\int_{-\infty}^s\left|e^{-\lambda(t-x)}(s-x)^{H-\frac{1}{\alpha}}-e^{-\lambda(s-x)}(s-x)^{H-\frac{1}{\alpha}}\right|^\alpha\mathrm{d}x.\end{aligned}$$

Use the inequality $|e^{-x} - e^{-y}| \le |x - y|$ for $x, y > 0$, substitute $u = s - x$ and then $w = \lambda\alpha u$ to see that

$$\begin{aligned} I_{12} &= \int_{-\infty}^{s} (s-x)^{H\alpha-1} \left| e^{-\lambda(t-x)} - e^{-\lambda(s-x)} \right|^{\alpha} \mathrm{d}x \\ &= \left| e^{-\lambda(t-s)} - 1 \right|^{\alpha} \int_0^{\infty} u^{H\alpha-1} e^{-\lambda\alpha u} \,\mathrm{d}x \\ &\le \lambda^{\alpha} |t-s|^{\alpha} (\lambda\alpha)^{H\alpha} \int_0^{\infty} e^{-w} w^{H\alpha-1}\, dw \\ &= \lambda^{\alpha} (\lambda\alpha)^{-H\alpha} \Gamma(H\alpha) |t-s|^{\alpha} \\ &\le \lambda^{\alpha} (\lambda\alpha)^{-H\alpha} \Gamma(H\alpha) |t-s|^{H\alpha} \end{aligned}$$

for $0 \le s < t \le 1$, since $\alpha > H\alpha > 0$. Here, $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$ is the gamma function. Let $h = t - s > 0$ and write

$$\begin{aligned} I_{11} &= \int_{-\infty}^{s} e^{-\lambda(t-x)} \left| (t-x)^{H-\frac{1}{\alpha}} - (s-x)^{H-\frac{1}{\alpha}} \right|^{\alpha} \mathrm{d}x \\ &\le \int_{-\infty}^{s} \left| (s+h-x)^{H-\frac{1}{\alpha}} - (s-x)^{H-\frac{1}{\alpha}} \right|^{\alpha} \mathrm{d}x \\ &= h^{H\alpha-1} \int_{-\infty}^{s} \left| \left(1 + \frac{s-x}{h}\right)^{H-\frac{1}{\alpha}} - \left(\frac{s-x}{h}\right)^{H-\frac{1}{\alpha}} \right|^{\alpha} \mathrm{d}x \\ &= h^{H\alpha} \int_0^{\infty} \left| (1+u)^{H-\frac{1}{\alpha}} - (u)^{H-\frac{1}{\alpha}} \right|^{\alpha} \mathrm{d}x = C_{11} |t-s|^{H\alpha} \end{aligned}$$

which concludes the proof. □

Lemma 4.3 *Suppose that* $\frac{1}{\alpha} < H < 1$ *for some* $1 < \alpha < 2$. *Then, there exist positive constants* C_1 *and* C_2 *such that the HTFSM* (3.3) *satisfies*

$$C_1 \left| t - s \right|^{H\alpha} \le \left\| \widetilde{X}_{H,\alpha,\lambda}(t) - \widetilde{X}_{H,\alpha,\lambda}(s) \right\|_{\alpha}^{\alpha} \le C_2 \left| t - s \right|^{H\alpha} \tag{4.1}$$

locally uniformly in $s, t \in [0, 1]$, *for any* $\lambda > 0$.

Proof To get the upper bound, note that

$$\begin{aligned} \left\| \widetilde{X}_{H,\alpha,\lambda}(t) - \widetilde{X}_{H,\alpha,\lambda}(s) \right\|_{\alpha}^{\alpha} &= \int_{-\infty}^{+\infty} \frac{|e^{-ikt} - e^{-iks}|^{\alpha}}{|\lambda - ik|^{H\alpha+1}} \,\mathrm{d}k \\ &\le C \int_{-\infty}^{+\infty} \left(1 \wedge |t-s|^{\alpha} |k|^{\alpha}\right) |\lambda - ik|^{-H\alpha-1} \,\mathrm{d}k \\ &= C \Big[|t-s|^{\alpha} \int_{|k| < \frac{1}{|t-s|}} |k|^{\alpha} ||\lambda - ik|^{-H\alpha-1} \,\mathrm{d}k \\ &\quad + \int_{|k| > \frac{1}{|t-s|}} |\lambda - ik|^{-H\alpha-1} \,\mathrm{d}k \Big] \end{aligned}$$

$$\leq C\Big[|t-s|^{\alpha} I_1 + I_2\Big] \tag{4.2}$$

for some constant $C > 0$, where

$$I_1 := \int_{|k|<\frac{1}{|t-s|}} \left|k\right|^{\alpha} \left|\lambda - ik\right|^{-H\alpha-1} \mathrm{d}k \quad \text{and} \quad I_2 := \int_{|k|>\frac{1}{|t-s|}} \left|\lambda - ik\right|^{-H\alpha-1} \mathrm{d}k.$$

Observe that

$$\begin{aligned} I_1 &= \int_{|k|<\frac{1}{|t-s|}} \left|k\right|^{\alpha} \left|\lambda^2 + k^2\right|^{\frac{-H\alpha-1}{2}} \mathrm{d}k \\ &\leq \int_{|k|<\frac{1}{|t-s|}} \left|k\right|^{\alpha} \left|k\right|^{-H\alpha-1} \mathrm{d}k = \int_{|k|<\frac{1}{|t-s|}} \left|k\right|^{-H\alpha-1+\alpha} \mathrm{d}k \\ &\leq \left|t-s\right|^{H\alpha-\alpha} \cdot \frac{2}{\alpha(1-H)} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} I_2 &= \int_{|k|>\frac{1}{|t-s|}} \left|\lambda^2 + k^2\right|^{\frac{-H\alpha-1}{2}} \mathrm{d}k \\ &\leq \int_{|k|>\frac{1}{|t-s|}} \left|k^2\right|^{\frac{-H\alpha-1}{2}} \mathrm{d}k = \int_{|k|>\frac{1}{|t-s|}} \left|k\right|^{-H\alpha-1} \mathrm{d}k \\ &\leq \left|t-s\right|^{H\alpha} \cdot \frac{2}{H\alpha}. \end{aligned} \tag{4.4}$$

Finally, from (4.2), (4.3) and (4.4) we get

$$\begin{aligned} \left\|\widetilde{X}_{H,\alpha,\lambda}(t) - \widetilde{X}_{H,\alpha,\lambda}(s)\right\|_{\alpha}^{\alpha} &\leq C\left[\left|t-s\right|^{\alpha} I_1 + I_2\right] \\ &\leq C\left[\frac{2}{\alpha(1-H)} + \frac{2}{H\alpha}\right]\left|t-s\right|^{H\alpha} \\ &= C_2\left|t-s\right|^{H\alpha} \end{aligned}$$

which gives the upper bound in (4.1). In order to get the lower bound, we use the fact that there exist positive constants c_1, c_2 such that $|e^{-iy} - 1| > c_1|y|$ for $|y| < c_2$. Therefore,

$$\begin{aligned} \left\|\widetilde{X}_{H,\alpha,\lambda}(t) - \widetilde{X}_{H,\alpha,\lambda}(s)\right\|_{\alpha}^{\alpha} &= \int_{-\infty}^{+\infty} \left|e^{-ikt} - e^{-iks}\right|^{\alpha} \left|\lambda - ik\right|^{-(H\alpha+1)} \mathrm{d}k \\ &= \int_{-\infty}^{+\infty} \left|e^{-ik(t-s)} - 1\right|^{\alpha} \left|\lambda - ik\right|^{-(H\alpha+1)} \mathrm{d}k \end{aligned}$$

$$\geq c_1^{\alpha} \int_{|k|<\frac{c_2}{|t-s|}} \left|k\right|^{\alpha} \left|t-s\right|^{\alpha} \left|\lambda - ik\right|^{-(H\alpha+1)} \mathrm{d}k$$
$$= c_1^{\alpha} |t-s|^{\alpha} \int_{|k|<\frac{c_2}{|t-s|}} \left|k\right|^{\alpha} (\lambda^2 + k^2)^{\frac{-(H\alpha+1)}{2}} \mathrm{d}k.$$

We now use the fact that

$$\left(\lambda^2 + k^2\right)^{\frac{-(H\alpha+1)}{2}} \geq \left(1 + c_2^2\right)^{\frac{-(H\alpha+1)}{2}} \left|t-s\right|^{H\alpha+1}$$

for $\lambda < \frac{1}{|t-s|}$ and $|k| < \frac{c_2}{|t-s|}$ to continue the rest of the proof as follows:

$$c_1^{\alpha} \left|t-s\right|^{\alpha} \int_{|k|<\frac{c_2}{|t-s|}} \left|k\right|^{\alpha} \left(\lambda^2 + k^2\right)^{\frac{-(H\alpha+1)}{2}} \mathrm{d}k$$
$$\geq 2c_1^{\alpha} \left(1 + c_2^2\right)^{\frac{-(H\alpha+1)}{2}} \left|t-s\right|^{\alpha} \left|t-s\right|^{H\alpha+1} \int_0^{\frac{c_2}{|t-s|}} k^{\alpha} \,\mathrm{d}k$$
$$= C_1 \left|t-s\right|^{H\alpha+\alpha+1} \left|t-s\right|^{-\alpha-1} = C_1 \left|t-s\right|^{H\alpha}$$

and this gives the lower bound. □

5 Local Times and Local Nondeterminism

In this section, we prove the existence of local times for LTFSM and HTFSM for $1 < \alpha < 2$ and $\frac{1}{\alpha} < H < 1$. In this case, we will also show that LTFSM and HTFSM are locally nondeterministic on every compact interval. Suppose $X = \{X(t)\}_{t \geq 0}$ is a real-valued separable random process with Borel sample functions. The random Borel measure

$$\mu_B(A) = \int_{s \in B} I\{X(s) \in A\} \,\mathrm{d}s$$

defined for Borel sets $A \subseteq B \subseteq \mathbb{R}^+$ is called the occupation measure of X on B. If μ_B is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^+$, then the Radon-Nikodym derivative of μ_B with respect to Lebesgue measure is called the local time of X on B, denoted by $L(B, x)$. See Boufoussi et al. [2] for more details. For brevity, we will also write $L(t, x)$ for the local time $L([0, t], x)$.

Proposition 5.1 *If* $\frac{1}{\alpha} < H < 1$ *for some* $1 < \alpha < 2$*, then the LTFSM* (2.4) *has a square integrable local time* $L(t, x)$ *for any* $\lambda > 0$.

Proof It follows from Boufoussi et al. [2, Theorem 3.1] that a stochastic process $X = \{X(t)\}_{t \in [0,T]}$ has a local time $L(t, x)$ that is continuous in t for a.e. $x \in \mathbb{R}$, and square integrable with respect to x, if X satisfies:

Condition ($\mathcal{H}$): There exist positive numbers $(\rho_0, H) \in (0, \infty) \times (0, 1)$ and a positive function $\psi \in L^1(\mathbb{R})$ such that for all $\kappa \in \mathbb{R}$, $t, s \in [0, T]$, $0 < |t-s| < \rho_0$ we have

$$\left|\mathbb{E}\left[\exp\left(i\kappa \frac{X(t)-X(s)}{|t-s|^H}\right)\right]\right| \leq \psi(\kappa). \tag{5.1}$$

Apply (2.6) and Lemma 4.2 to get

$$\mathbb{E}\left[\exp\left(i\kappa \frac{X_{H,\alpha,\lambda}(t)-X_{H,\alpha,\lambda}(s)}{|t-s|^H}\right)\right] = \exp\left(-|\kappa|^\alpha \frac{\|X_{H,\alpha,\lambda}(t)-X_{H,\alpha,\lambda}(s)\|_\alpha^\alpha}{|t-s|^{\alpha H}}\right)$$
$$\leq \exp\left(-|\kappa|^\alpha C\right) := \psi(\kappa)$$

where the function $\psi(\kappa) \in L^1(\mathbb{R}, \mathrm{d}k)$. Hence, LTFSM satisfies Condition $\mathcal{H}$. □

Proposition 5.2 *If* $\frac{1}{\alpha} < H < 1$ *for some* $1 < \alpha < 2$*, then the HTFSM* (3.3) *has a square integrable local time* $L(t, x)$ *for any* $\lambda > 0$.

Proof Apply (3.2) and Lemma 4.3 to obtain

$$\mathbb{E}\left[\exp\left(i\kappa \frac{\widetilde{X}_{H,\alpha,\lambda}(t)-\widetilde{X}_{H,\alpha,\lambda}(s)}{|t-s|^H}\right)\right] = \exp\left(-|\kappa|^\alpha \frac{\|\widetilde{X}_{H,\alpha,\lambda}(t)-\widetilde{X}_{H,\alpha,\lambda}(s)\|_\alpha^\alpha}{|t-s|^{\alpha H}}\right)$$
$$\leq \exp\left(-|\kappa|^\alpha C\right) := \psi(\kappa).$$

Since $\psi(\kappa) \in L^1(\mathbb{R}, \mathrm{d}k)$, the HTFSM satisfies Condition $\mathcal{H}$. □

We next show that HTFSM is locally nondeterministic on every compact interval $[\epsilon, T]$, for any $0 < \epsilon < T < \infty$. Recall that a stochastic process $\{X(t)\}_{t \in T}$ is *locally nondeterministic* (LND) if:

(1) $\|X(t)\|_\alpha > 0$ for all $t \in T$
(2) $\|X(t) - X(s)\|_\alpha > 0$ for all $t, s \in T$ sufficiently close; and
(3) for any $m \geq 2$,

$$\liminf_{\epsilon \downarrow 0} \frac{\|X(t_m) - \mathrm{span}\{X(t_1), \ldots, X(t_{m-1})\}\|_\alpha}{\|X(t_m) - X(t_{m-1})\|_\alpha} > 0,$$

where $\mathrm{span}\{x_1, \ldots, x_m\}$ is the linear span of $x_1, \ldots, x_m$, the lim inf is taken over distinct, ordered $t_1 < t_2 < \cdots < t_m \in T$ with $|t_1 - t_m| < \epsilon$, $T \subset \mathbb{R}$, $1 < \alpha < 2$ and $\|X(t)\|_\alpha$ is the norm given by (2.1).

Remark 5.3 According to Nolan [12], the ratio in Condition (3) is a relative linear prediction error and is always between 0 and 1. If the ratio is bounded away from zero as $|t_1 - t_m| \to 0$, then we can approximate $X(t_m)$ in the $\|\cdot\|_\alpha$ norm by the most recent value $X(t_{m-1})$ with the same order of error as by the set of values $X(t_1), \ldots, X(t_{m-1})$.

Proposition 5.4 *The LTFSM* (2.4) *with* $1 < \alpha < 2$ *and* $\frac{1}{\alpha} < H < 1$ *is LND on every interval* $[\epsilon, \kappa]$ *for* $\epsilon < \kappa < \infty$.

Proof To prove LND for the LTFSM $\{X_{H,\alpha,\lambda}(t)\}$, we need to verify Conditions (1), (2) and (3) as described above (for $1 < \alpha < 2$). The first and second conditions follow from Lemma 4.2. That is,

$$\left\| X_{H,\alpha,\lambda}(t) - X_{H,\alpha,\lambda}(s) \right\|_\alpha^\alpha \geq C_1 \left| t - s \right|^{H\alpha}$$

where C_1 is a positive constant. It remains to show that the LTFSM $\{X_{H,\alpha,\lambda}(t)\}$ satisfies Condition (3):

$$\liminf_{\epsilon \downarrow 0} \frac{\left\| X_{H,\alpha,\lambda}(t_m) - \mathrm{span}\{X_{H,\alpha,\lambda}(t_1), \ldots, X_{H,\alpha,\lambda}(t_{m-1})\} \right\|_\alpha^\alpha}{\left\| X_{H,\alpha,\lambda}(t_m) - X_{H,\alpha,\lambda}(t_{m-1}) \right\|_\alpha^\alpha} > 0. \tag{5.2}$$

Observe that

$$\begin{aligned} &\left\| X_{H,\alpha,\lambda}(t_m) - \mathrm{span}(X_{H,\alpha,\lambda}(t_i), i = 1, \ldots, m-1) \right\|_\alpha^\alpha \\ &\geq \left\| X_{H,\alpha,\lambda}(t_m) - \mathrm{span}(X_{H,\alpha,\lambda}(u), u \leq t_{m-1}) \right\|_\alpha^\alpha \\ &= \int_{t_{m-1}}^{t_m} |t_m - u|^{\alpha(H - \frac{1}{\alpha})} e^{-\lambda\alpha|t_m - u|}\, du \\ &\geq e^{-\lambda\alpha|t_m - t_{m-1}|} \int_{t_{m-1}}^{t_m} |t_m - u|^{H\alpha - 1}\, du \\ &= \frac{e^{-\lambda\alpha|t_m - t_{m-1}|} \left| t_m - t_{m-1} \right|^{H\alpha}}{H\alpha} \end{aligned} \tag{5.3}$$

Now, apply Lemma 4.2 to see that

$$\left\| X_{H,\alpha,\lambda}(t_m) - X_{H,\alpha,\lambda}(t_{m-1}) \right\|_\alpha^\alpha \leq C_2 \left| t_m - t_{m-1} \right|^{H\alpha} \tag{5.4}$$

for $|t_m - t_{m-1}| < \epsilon$. Combining (5.3) and (5.4), we get that the ratio in (5.2) is bounded below by

$$\frac{e^{-\lambda\alpha|t_m - t_{m-1}|} \left| t_m - t_{m-1} \right|^{H\alpha}}{C_2 H\alpha \left| t_m - t_{m-1} \right|^{\alpha H}}.$$

Since $|t_m - t_{m-1}| < \epsilon$,

$$\liminf_{\epsilon \downarrow 0} \frac{e^{-\lambda\alpha|t_m - t_{m-1}|} \left| t_m - t_{m-1} \right|^{H\alpha}}{C_2 H\alpha \left| t_m - t_{m-1} \right|^{\alpha H}} \to \frac{1}{C_2\ H\alpha} = C > 0; \tag{5.5}$$

hence, (5.2) holds which means $\{X_{H,\alpha,\lambda}\}$ is LND. □

Proposition 5.5 *If* $\frac{1}{\alpha} < H < 1$ *for some* $1 < \alpha < 2$*, then the HTFSM* (3.3) *is LND on every interval* $[\epsilon, \kappa]$ *for any* $\epsilon < \kappa < \infty$ *and any* $\lambda > 0$.

Proof We follow the proof of Dozzi and Shevchenko [3, Theorem 3.3], who show that a harmonizable multifractional stable motion is LND on every interval $[\epsilon, \kappa]$ for $\epsilon < \kappa < \infty$. Conditions (1) and (2) follow from the lower bound in Lemma 4.3. Next, observe that the kernel

$$\tilde{g}_{\alpha,\lambda,t}(k) := \frac{e^{-ikt} - 1}{(\lambda - ik)^{H+\frac{1}{\alpha}}} \tag{5.6}$$

in Definition (3.3) of HTFSM is the Fourier transform of the function

$$\frac{\Gamma(H + \frac{1}{\alpha})}{\sqrt{2\pi}} \Big[e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{\alpha-1}{\alpha}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{\alpha-1}{\alpha}} \Big], \tag{5.7}$$

which is a constant multiple of the kernel in (2.4). Here, $\Gamma(x)$ is the gamma function. In order to verify Condition (3), we shall establish a lower bound for

$$\left\| \widetilde{X}_{H,\alpha,\lambda}(t_m) - \sum_{j=1}^{m-1} u_j \widetilde{X}_{H,\alpha,\lambda}(t_j) \right\|_\alpha = \left\| \tilde{g}_{\alpha,\lambda,t_m}(k) - \sum_{j=1}^{m-1} u_j \tilde{g}_{\alpha,\lambda,t_j}(k) \right\|_{L^\alpha(\mathbb{R})}$$

where $f_{H,\alpha,\lambda}(t, k)$ is defined in (5.6). Let $\beta = \frac{\alpha}{\alpha-1}$. Apply the Hausdorff–Young inequality [5, Theorem 5.7] to get

$$\begin{aligned}
&\left\| \tilde{g}_{\alpha,\lambda,t_m}(k) - \sum_{j=1}^{m-1} u_j \tilde{g}_{\alpha,\lambda,t_j}(k) \right\|_{L^\alpha(\mathbb{R})} \\
&\geq C \left\| \mathcal{F}^{-1}\tilde{g}_{\alpha,\lambda,t_m}(k) - \sum_{j=1}^{m-1} u_j \mathcal{F}^{-1}\tilde{g}_{\alpha,\lambda,t_j}(k) \right\|_{L^\beta(\mathbb{R})} \\
&= C \Bigg(\int_{-\infty}^{t_{m-1}} \left| \mathcal{F}^{-1}\tilde{g}_{\alpha,\lambda,t_m}(k) - \sum_{j=1}^{m-1} u_j \mathcal{F}^{-1}\tilde{g}_{\alpha,\lambda,t_j}(k) \right|^\beta \\
&\quad + \int_{t_{m-1}}^{t_m} \left| \mathcal{F}^{-1}\tilde{g}_{\alpha,\lambda,t_m}(k) \right|^\beta \mathrm{d}k \Bigg)^{\frac{1}{\beta}},
\end{aligned} \tag{5.8}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. From (5.7), we have

$$\mathcal{F}^{-1}\tilde{g}_{\alpha,\lambda,t_m}(k) = \frac{\Gamma(H + \frac{1}{\alpha})}{\sqrt{2\pi}} \Big[e^{-\lambda(t_m-x)_+}(t_m - x)_+^{H-\frac{\alpha-1}{\alpha}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{\alpha-1}{\alpha}} \Big]$$

and the second term, $e^{-\lambda(-x)_+}(-x)_+^{H-\frac{\alpha-1}{\alpha}}$, vanishes on the interval $[t_{m-1}, t_m]$. Hence, we can continue (5.8) as the following:

$$\geq C\Bigg[\int_{t_{m-1}}^{t_m} (t_m - x)^{\beta(H-\frac{1}{\beta})} e^{-\lambda\beta(t_m-x)}\, \mathrm{d}x\Bigg]^{\frac{1}{\beta}}$$
$$\geq C\, e^{-\lambda(t_m-t_{m-1})}\Big|t_m - t_{m-1}\Big|^H \geq C\, e^{-\lambda(\kappa-\epsilon)}\Big\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \widetilde{X}_{H,\alpha,\lambda}(t_{m-1})\Big\|_\alpha \quad (5.9)$$

for t_m and t_{m-1} close enough (and C is a constant). In the last line in (5.9), we used the fact that $|t_m - t_{m-1}| < \kappa - \epsilon$ and we also applied Lemma 4.3 to get the last inequality. Therefore,

$$\begin{aligned}
&\Big\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \mathrm{span}\{\widetilde{X}_{H,\alpha,\lambda}, \ldots, \widetilde{X}_{H,\alpha,\lambda}(t_{m-1})\}\Big\|_\alpha \\
&= \Big\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \sum_{j=1}^{m-1} u_j \widetilde{X}_{H,\alpha,\lambda}(t_j)\Big\|_\alpha \\
&\geq C\Big\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \widetilde{X}_{H,\alpha,\lambda}(t_{m-1})\Big\|_\alpha
\end{aligned}$$

and consequently

$$\liminf_{\epsilon\downarrow 0} \frac{\Big\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \mathrm{span}\{\widetilde{X}_{H,\alpha,\lambda}, \ldots, \widetilde{X}_{H,\alpha,\lambda}(t_{m-1})\}\Big\|_\alpha}{\Big\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \widetilde{X}_{H,\alpha,\lambda}(t_{m-1})\Big\|_\alpha} > C,$$

where C is a positive constant. □

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