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Symmetry groups in d-space

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Abstract

We show that any finite-dimensional compact Lie group is isomorphic to the symmetry group of a full probability measure. The novelty of our proof is that an explicit formula for the measure and its support is given in terms of the Lie group. We also construct a full operator stable probability measure whose symmetry group has as its tangent space the tangent space of a given group. This provides a method for constructing an operator stable probability measure having a specified collection of exponents. A characterization of the compact groups of operators on a finite-dimensional space which can be the symmetry group of a full probability measure on that same space is given.

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1. Introduction

Probability measures on $\mathbb{R}^d$ which are invariant under a specified group of linear operators play an important role in modern statistical theory. The group of operators leaving a given probability measure invariant is called the symmetry group of the measure. We will show that any compact group of linear operators is isomorphic to the symmetry group of a full probability measure. Although this result can be deduced from the work of previous authors (for example, Bedford and Dadok, 1987, Theorem 3), our method of proof gives a formula for the probability measure in terms of the group of operators. Also, the construction explicitly displays the support of the measure, which is a subset of the unit sphere. This explicit representation of the measure and its support can be extremely useful in applications. Indeed, in our application at the end of the paper we make heavy use of the explicit form of the measure constructed. We also provide a characterization of the compact groups of linear operators on $\mathbb{R}^d$ which can occur as symmetry groups of full probability measures on that same space.

If μ is a probability measure on $\mathbb{R}^d$, the symmetries of μ are those linear operators h on $\mathbb{R}^d$ for which $h\mu = \mu$. Here the measure $h\mu$ is defined by $h\mu(A) = \mu(h^{-1}A)$. A probability measure on $\mathbb{R}^d$ is

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said to be full if it is not supported in any proper hyperplane of $\mathbb{R}^d$. Billingsley (1966) showed that the collection of symmetries $\mathscr{S}(\mu)$ of a full probability measure μ on $\mathbb{R}^d$ is a compact group. The symmetry group inherits the Lie group structure of the set $GL(\mathbb{R}^d)$ of invertible linear operators on $\mathbb{R}^d$.

Billingsley pointed out that not every compact group of operators on $\mathbb{R}^d$ can be the symmetry group of a full probability measure on that same space. For example, if every rotation is a symmetry then so is any reflection. Therefore the group of rotations cannot be the symmetry group of a full probability measure. Paradoxically, while the group of rotations on $\mathbb{R}^d$ cannot be the symmetry group of any full probability measure on $\mathbb{R}^d$, there is a full probability measure on $\mathbb{R}^{d^2}$ whose symmetry group is isomorphic to the group of rotations on $\mathbb{R}^d$.

The study of the construction of probability measures with a specified symmetry group was motivated by the authors' continuing study of operator stable probability measures. Operator stable probability measures are the higher-dimensional analogs of the classical stable measures on the real line. In the last section of this paper the techniques developed here are used to partially solve the problem of constructing a full operator stable probability measure with a specified symmetry group. We also provide there a brief summary of the theory of operator stable probability measures.

2. Results

The first objective of this paper is to show that, up to isomorphism, any compact Lie group is the symmetry group of *some* probability measure.

Theorem 1. *Let G be a finite-dimensional compact Lie group. Then there exists a full probability measure on $\mathbb{R}^n$ for some n whose symmetry group is isomorphic to G.*

If the group G is a group of operators on $\mathbb{R}^d$, the proof of Theorem 1 yields a full measure on $\mathbb{R}^n$ where $n = d^2$. The proof leans heavily on ideas contained in Bedford and Dadok (1987), and in fact shows how to construct such a measure. The measure constructed has compact support which is a subset of the unit sphere in $\mathbb{R}^n$.

The second objective of this paper is to characterize those compact groups of operators on $\mathbb{R}^d$ which are the symmetry group of some full probability measure on the same space $\mathbb{R}^d$. The characterization is as follows. Begin by defining an equivalence relation $\sim$ on the class of all subgroups of $GL(\mathbb{R}^d)$ by saying that $G_1 \sim G_2$ if $G_1 x = G_2 x$ for all $x \in \mathbb{R}^d$. Let $[G]$ denote the equivalence class of G. We also partial order the subsets of $GL(\mathbb{R}^d)$ by set inclusion.

Theorem 2. *A compact subgroup G of $GL(\mathbb{R}^d)$ is the symmetry group of some full probability measure on $\mathbb{R}^d$ if and only if G is a maximal element of $[G]$.*

3. Proofs

We begin with a lemma which establishes the invariance of the support of a measure under the action of its symmetries.

Lemma 1. *Let μ be a full probability measure on $\mathbb{R}^d$ with symmetry group $\mathscr{S}(\mu)$. Each $h \in \mathscr{S}(\mu)$ maps the support of μ onto itself.*

Proof. Denote the support of μ by S. Note that if $h \in \mathscr{S}(\mu)$ then $h^{-1} \in \mathscr{S}(\mu)$ as well, and both h and h^{-1} are continuous. Suppose that $x \in S$ and $h^{-1}(x) \notin S$. Since S is closed, by continuity of h^{-1} there is an open set $A \subset \mathbb{R}^d$ with $x \in A$ and $h^{-1}(A) \subset S^c$. Since S is the support of μ, we also have $\mu(A) > 0$. Since $h\mu = \mu$ we also have $\mu(h^{-1}A) = \mu(A)$. But the left-hand side of this equality is zero, while the right-hand side is strictly positive. This contradiction shows that there is no such x. We have thus proved that $h^{-1}S \subset S$. Since this inclusion holds for all symmetries h of μ, it also holds when h is replaced by h^{-1}. Thus $hS \subset S$, i.e., $S \subset h^{-1}S$. □

Suppose G is a finite-dimensional compact Lie group. By Theorem 6.1.1 of Price (1977), G is isomorphic to a subgroup of the orthogonal group on $\mathbb{R}^d$ for some d. We can therefore imbed G as a subgroup of the orthogonal group on $\mathbb{R}^{d^2}$ by letting G 'act diagonally'. This means the following. First we identify an element h of G with its matrix relative to the standard basis for $\mathbb{R}^d$. Then we construct a $d^2 \times d^2$ block diagonal matrix with d identical blocks, each of which is the $d \times d$ matrix of h. This block diagonal matrix is the matrix of an orthogonal linear operator on $\mathbb{R}^{d^2}$ with respect to the standard basis for that space. The embedding maps h onto this operator.

Lemma 2. *There exists a finite collection of unit vectors $x_1, \ldots, x_N$ in $\mathbb{R}^{d^2}$ with the property that if h is a linear operator on $\mathbb{R}^{d^2}$ and $hx_i \in Gx_i$ for all $1 \leqslant i \leqslant N$ then $h \in G$.*

Proof. Let $x_1, \ldots, x_{d^2}$ be the standard basis for $\mathbb{R}^{d^2}$. Then for $i = 1, \ldots, d$ define $x_{d^2+i} = (x_i + x_{d+i} + \cdots + x_{(d-1)d+i})/\sqrt{d}$. Finally let $x_N = (x_1 + x_{d+2} + \cdots + x_{d^2})/\sqrt{d}$, where $N = d^2 + d + 1$. Suppose h is a linear operator on $\mathbb{R}^{d^2}$ and that $hx_i \in Gx_i$ for $1 \leqslant i \leqslant N$. This means that there are elements $g_1, \ldots, g_N$ of G so that $hx_i = g_i x_i$ for $1 \leqslant i \leqslant N$. Since all elements of G act diagonally, it is easy to use the fact that $hx_i = g_i x_i$ for $1 \leqslant i \leqslant d^2$ to show that the matrix of h relative to the standard basis for $\mathbb{R}^{d^2}$ is block diagonal, with d possibly unequal $d \times d$ block elements. Then by using the vectors x_i for $d^2 + 1 \leqslant i < N$, one easily sees that the block elements of h are in fact equal. Finally, by using the vectors x_N one sees that there is a $g \in G$ which is equal to the given h. □

Lemma 3. *There is no proper hyperplane of $\mathbb{R}^{d^2}$ which contains the vectors $x_1, \ldots, x_N$ of Lemma 2.*

Proof. A hyperplane in $\mathbb{R}^{d^2}$ consists of the vectors x which are solutions of one or more equations of the form $\langle x, v \rangle = c$ for some vector v and some constant c. If the vectors $x_1, \ldots, x_N$ of Lemma 2 solve such an equation we see by using the first d^2 of the x's that the vector v has all of its components equal to c. Using the vector x_N shows that $c = 0$ and hence $v = 0$. Thus any hyperplane containing $x_1, \ldots, x_N$ coincides with $\mathbb{R}^{d^2}$. □

Proof of Theorem 1. Following the discussion preceding Lemma 2 we will construct a full probability measure μ on $\mathbb{R}^{d^2}$ with the property that $\mathscr{S}(\mu) = G$. Let $x_1, \ldots, x_N$ be as in Lemma 2 and let $\mathscr{H}$ be the Haar probability measure on G. Because of the compactness of G and the continuity of the group action each of the orbits Gx_i is a compact set. Also the number of connected components of an orbit is no more than the (finite) number of connected components of G. Note that two orbits are either disjoint or coincide. After relabeling if necessary we suppose that $Gx_1, \ldots, Gx_J$ is a list of the distinct orbits among $Gx_1, \ldots, Gx_N$. Suppose the orbit Gx_i has n_i connected components. Define

$$v(A) = \sum_{i=1}^{J} \int_G (in_i) \cdot \delta_{x_i}(gA) \, \mathrm{d}\mathscr{H}(g).$$

Then $\mu(A) = v(A)/v(\mathbb{R}^{d^2})$ is a probability measure on $\mathbb{R}^{d^2}$, and from our construction it is immediate that every $g \in G$ is a symmetry of μ. Since the support of μ contains the vectors $x_1, \ldots, x_N$ of Lemma 2, Lemma 3 shows that μ is full. Thus it only remains to show that every $h \in \mathscr{S}(\mu)$ is an element of G.

Suppose that $h \in \mathscr{S}(\mu)$. Then Lemma 1 implies that h maps the support of μ onto itself. It is easy to see that the support of μ is the union of the orbits, $\bigcup_{i=1}^{J} Gx_i$. Each connected component of the support of μ consists of one of the connected components of some orbit Gx_i. Note also that all connected components of the orbit Gx_i have equal mass, namely $i/\sum_{k=1}^{J} kn_k$. Since h is continuous, the image of one of the connected components of the support of μ under h must be another connected component. Since h is a symmetry and the connected components of different orbits have different measure we must have $hGx_i = Gx_i$ for all $1 \leqslant i \leqslant J$. If $j > J$ then $Gx_j = Gx_i$ for some $1 \leqslant i \leqslant J$, and so we have $hGx_i = Gx_i$ for all $1 \leqslant i \leqslant N$. By Lemma 2, $h \in G$. □

We have shown that every compact subgroup of $GL(\mathbb{R}^d)$ is isomorphic to the symmetry group of some full probability measure on $\mathbb{R}^{d^2}$. There remains the question of which compact subgroups are actually the symmetry groups of some full probability measure on $\mathbb{R}^d$. To address this question we need to review some further facts from Lie theory. Some details are presented for the benefit of those who are not familiar with Lie theory.

We first recall that if L is a Lie group, then the connected component of the identity, C, is a normal subgroup of L. This is easily seen from the fact that the map $x \to gxg^{-1}$ is continuous and maps the identity to itself. This implies that $gCg^{-1} \subset C$ for all $g \in L$. Hence C is normal. (Similar arguments show that C is a subgroup). We also recall that if L is compact then L/C is a finite group and the order of this quotient group is equal to the number of connected components of L. This is easily seen by noting that C is open and thus $\{gC: g \in L\}$ forms an open cover of L which by compactness must have a finite subcover.

Suppose $G \subset H$ and both are finite-dimensional compact Lie groups. We claim that if the inclusion is proper then either the dimension of G is strictly less than the dimension of H or the number of components of G is strictly less than the number of components of H. Let us recall that the dimension of a Lie group G is the dimension of the corresponding Lie algebra (or tangent space) TG. (When G is a set of linear operators, the tangent space is the collection of all operators which can be written as $\lim_{n\to\infty}(G_n - I)/g_n$ where $\{G_n\}$ is a sequence of operators from G and $\{g_n\}$ is a sequence of real numbers which converges to 0.) Since $G \subset H$ we have $TG \subset TH$. If the dimension did not decrease, then the Lie algebras are equal. Since the exponential map sends the Lie algebra onto the connected component of the identity, we see that in this case the connected component of the identity, C say, of the two groups is the same. Now we have $G/C \subset H/C$ where both are finite groups. If the inclusion were not proper, we would conclude that $G = H$. Hence if the dimension did not decrease, the number of components must decrease. It follows that any decreasing sequence of properly nested compact finite dimensional Lie groups must eventually terminate.

Proof of Theorem 2. We begin by showing that if G is a symmetry group of a full probability measure, $G = \mathscr{S}(\mu)$ say, then G is maximal. Before doing this we note that since G is a symmetry group of a full measure G is compact. If G is not a subgroup of the orthogonal group there is a positive-definite operator W so that WGW^{-1} is a subgroup of the orthogonal group (see Billingsley, 1966). Since $WGW^{-1} = \mathscr{S}(W\mu)$ and $[WGW^{-1}] = W[G]W^{-1}$ we see that the assumption that G is a subgroup of the orthogonal group causes no loss of generality. To continue the proof, we suppose that $H \in [G]$ with $G \subset H$ and we will show that $H = G$. Since $H \in [G]$ we have $Hx = Gx$ for all $x \in \mathbb{R}^d$. It is easily seen that since G is a subgroup of the orthogonal group then so is H. Let $\hat{\mu}(x)$ denote the characteristic function of μ. A linear operator A belongs to the symmetry group of μ if and only if $\hat{\mu}(A^*x) = \hat{\mu}(x)$ for all $x \in \mathbb{R}^d$. For any $h \in H$ and any $x \in \mathbb{R}^d$ there is some $g \in G$ such that $g^*x = g^{-1}x = h^{-1}x = h^*x$, so $\hat{\mu}(h^*x) = \hat{\mu}(g^*x) = \hat{\mu}(x)$. Hence $G \subset H \subset \mathscr{S}(\mu) = G$. This shows that G is maximal.

It remains to show that if G is maximal then there is a full probability measure μ such that $\mathscr{S}(\mu) = G$. If X is a finite set of elements in $\mathbb{R}^d$, use the construction of Theorem 1 to define a probability measure μ_X which is supported on the union of the disjoint orbits Gx for $x \in X$, with the property that $G \subset \mathscr{S}(\mu_X)$. Now observe that if $X \subset Y$, then $\mathscr{S}(\mu_Y) \subset \mathscr{S}(\mu_X)$ since the restriction of μ_Y to the support of μ_X is just a constant multiple of μ_X. Let X_0 be an arbitrary finite set of vectors in $\mathbb{R}^d$ for which μ_{X_0} is full. If $\mathscr{S}(\mu_{X_0})x = Gx$ for all $x \in \mathbb{R}^d$ then $\mathscr{S}(\mu_{X_0}) \sim G$, and since G is maximal we must have $G = \mathscr{S}(\mu_{X_0})$. Otherwise, there exists some nonzero vector $x \in \mathbb{R}^d$ such that Gx is strictly contained in $\mathscr{S}(\mu_{X_0})x$. Set $X_1 = X_0 \cup \{x\}$ and consider the probability measure μ_{X_1}. Since $\mathscr{S}(\mu_{X_1})x = Gx$, the containment $\mathscr{S}(\mu_{X_1}) \subset \mathscr{S}(\mu_{X_0})$ must be strict. Continue in this manner to obtain a strictly decreasing nested sequence of symmetry groups $\{\mathscr{S}(\mu_{X_i})\}$ all of which contain G. By the above discussion, we must have $G = \mathscr{S}(\mu_{X_i})$ for some i. □

4. Application

As mentioned in the introduction, the original motivation for studying symmetry groups was as part of the authors' ongoing research into the structure of operator stable probability measures. We present a brief review of the theory of operator stable probability measures and then use the techniques developed in this paper to construct a full operator stable probability measure whose symmetry group has a specified tangent space. An open question is whether a similar construction can yield an operator stable probability measure with a specified symmetry group. A brute force construction of one example of this sort was carried out in Meerschaert and Veeh (1992).

The theory of operator stable probability measures was begun by Sharpe (1969). A probability measure μ on $\mathbb{R}^d$ which is full is said to be operator stable if there is a linear operator B on $\mathbb{R}^d$ (called an exponent of μ) and a vector valued function b_t so that, for all $t > 0$,

$$\mu^t = t^B \mu * \delta(b_t).$$

Here μ is known to be infinitely divisible, so μ^t, the tth convolutory power, can be defined by the characteristic function. The operator t^B is defined via power series as $\exp\{B \ln t\}$. A linear operator A is said to be a symmetry of the full operator stable measure μ if there exists a vector a so that $A\mu = \mu * \delta(a)$. We use this alternate definition of symmetry only for operator stable measures. Under this alternate definition the collection of all symmetries of a full operator stable probability measure form a compact group which we continue to denote $\mathscr{S}(\mu)$.

In the case $d = 1$ the operator stable measures are exactly the classical stable measures and the exponent is uniquely determined as the reciprocal of the classical index of stability. In general, operator stable probability measures do not have unique exponents. If $\mathscr{E}(\mu)$ denotes the collection of exponents of the operator stable measure μ then Holmes, et al. (1982) established that

$$\mathscr{E}(\mu) = B + T\mathscr{S}(\mu),$$

where $B \in \mathscr{E}(\mu)$. Since the exponents of an operator stable measure play a role analogous to the role of the index of stability in the theory of the classical stable measures on $\mathbb{R}^1$ it is desirable to be able to construct operator stable measures with a specified symmetry group. One can then use the above relation to see what the collection of exponents is.

A key element in our construction is the fact that an operator stable measure is infinitely divisible. Hence if the operator stable measure has no Gaussian component it is determined by its Lévy measure. Furthermore, the Lévy measure M of an operator stable measure can always be represented as

$$M(E) = \int_S \int_0^\infty \mathbf{1}_E(t^B x) t^{-2} \, dt \, dK(x),$$

where K is a Borel measure on a Borel set S which is intersected exactly once by each orbit $\{t^B v : t > 0\}$. For more details of this representation, see Hudson et al. (1986). The measure K is called the mixing measure. In our construction we will always take S to be the surface of the unit sphere.

There is an intimate connection between the symmetries of the operator stable measure μ, the symmetries of its Lévy measure M, and the symmetries of its mixing measure K. From the uniqueness of the Lévy–Khinchine representation it follows that if $A \in \mathscr{S}(\mu)$ then $AM = M$. Since for Borel subsets E of S we have the formula $K(E) = M(\{t^B x : t > 1, x \in E\})$ it is easy to see that a symmetry of the operator stable measure which commutes with B will be a symmetry of the mixing measure *provided* that the symmetry leaves the set S invariant. It should be remarked that fullness of μ does not imply fullness of K, so that the collection of symmetries of K is neither a group nor compact in general. This will not pose a problem for our construction since we will construct a full mixing measure K and use it to construct M and μ which will then necessarily be full.

Suppose we are given a compact subgroup G of the orthogonal group on $\mathbb{R}^d$ and we wish to construct an operator stable probability measure having G as its symmetry group. Of course, this will not be possible in general if the operator stable measure is required to be supported in $\mathbb{R}^d$. As in the first part of this paper we let G act diagonally on $\mathbb{R}^{d^2}$ and we will continue to denote this imbedded group by G. We now construct an operator stable measure μ on $\mathbb{R}^{d^2}$ for which $T\mathscr{S}(\mu) = TG$.

We have seen that an operator stable probability measure with no Gaussian component can be specified by specifying an exponent and a mixing measure. We begin the construction of the mixing measure by letting $v_1, \ldots, v_N$ be the vectors of the proof of Lemma 2. After relabeling if necessary we let $C_1, \ldots, C_J$ be the distinct orbits among $Gv_1, \ldots, Gv_N$. Define the mixing measure K on the

Borel subsets of the unit sphere in $\mathbb{R}^{d^2}$ by the formula

$$K(A) = \sum_{i=1}^{J} \int_G (in_i)\delta_{v_i}(gA)\,\mathrm{d}\mathscr{H}(g).$$

Here the notation is as in the proof of Theorem 1. We use K to construct a full operator stable probability measure μ on $\mathbb{R}^{d^2}$ with exponent $B = I$. We now claim that $T\mathscr{S}(\mu) = TG$. To prove the claim, it suffices to show that G is both open and closed in $\mathscr{S}(\mu)$. This is sufficient because if G is both open and closed, then G must contain the connected component of the identity in $\mathscr{S}(\mu)$ since G is a group. It is the connected component of the identity which determines the tangent space. Since G is a compact group it is closed in $\mathscr{S}(\mu)$. It remains to show that G is open in $\mathscr{S}(\mu)$. It would be easy to establish this if it were known that $\mathscr{S}(\mu) = \mathscr{S}(K)$ (in fact, this would imply $\mathscr{S}(\mu) = G$). But it is not known that $\mathscr{S}(\mu)$ consists only of orthogonal operators, so the elements of $\mathscr{S}(\mu)$ need not leave the set S invariant. Thus there is no immediate connection between elements of $\mathscr{S}(\mu)$ and symmetries of K.

To proceed, first note that since μ is full, $\mathscr{S}(\mu)$ is compact so there is a positive definite operator W so that $W\mathscr{S}(\mu)W^{-1} = \mathscr{S}(W\mu)$ is contained in the orthogonal group (see Billingsley, 1966).

We claim that if $g \in G$ then $Wg = gW$. To see this, we observe that since WgW^{-1} is orthogonal we have $\langle WgW^{-1}x, WgW^{-1}y\rangle = \langle x, y\rangle$ for all x and y. Hence $\langle gx, W^2gy\rangle = \langle x, W^2y\rangle$ for all x and y. Similarly, since g is orthogonal we have $\langle gx, gW^2y\rangle = \langle x, W^2y\rangle$ for all x and y. By equating these two expressions and using the fact that x and y are arbitrary we obtain $gW^2 = W^2g$. Since W is positive-definite we have $gW = Wg$, as desired.

We now pause to understand the structure of the mixing measure of $W\mu$. Since W commutes with each element of G we see that $WC_i = GWv_i$. If we denote $C_i' = (1/\|Wv_i\|)WC_i$, then the set C_i' is the G orbit of $Wv_i/\|Wv_i\|$ so all elements of C_i' lie on the surface of the unit sphere. It is not difficult to see that the mixing measure K' of $W\mu$ has as its support $\bigcup_{i=1}^{J} C_i'$. Define

$$\delta = \inf\{\|x - y\| : x \in C_i',\ y \in C_j',\ i \in J,\ j \in J,\ i \neq j\}.$$

Note that $\delta > 0$ since the C_i' are disjoint and compact.

Suppose $g \in G$. We claim that if $h \in \mathscr{S}(\mu)$ with $\|WhW^{-1} - g\| < \delta$ then $h \in G$. This will show G is open. Now WhW^{-1} is a symmetry of $W\mu$ and is orthogonal. Hence it is a symmetry of K'. Thus by Lemma 1 WhW^{-1} permutes the C_i'. Since $gC_i' = C_i'$ and $\|WhW^{-1} - g\| < \delta$ we see that

$$WhW^{-1}C_i' \subset C_i' = GWv_i/\|Wv_i\|.$$

From the definition of C_i' this means $hv_i \in Gv_i$ for $1 \leqslant i \leqslant J$, and hence for all $1 \leqslant i \leqslant N$. As in the proof of Lemma 2 we now see that $h \in G$. This shows that G is open in $\mathscr{S}(\mu)$ and completes the construction.

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