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Average densities of the image and zero set of stable processes

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Abstract

The 'order-two' or 'average' density of a measure μ at a point x is defined as $\lim_{T\to\infty}(1/T)\int_0^T \mu(B(x, e^{-s}))e^{\alpha s}ds$ for appropriate α. We show that, with probability one, the order-two density of the natural measure μ on the image set or zero set of a wide class of stable processes exists and takes the same value almost everywhere in the support of μ. We calculate this value in certain cases.

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1. Introduction

Bedford and Fisher (1992) introduced the concept of the 'order-two' or 'average' density of fractal sets and measures, which give a finer local description than that provided by Hausdorff dimension alone. Unlike the classical densities, the order-two densities have the advantage that they exist for a wide variety of sets. Such sets include many Cantor sets, self-similar and quasi-self-similar sets, mixing repellers, see Bedford and Fisher (1992), Falconer (1992), Patzschke and Zähle (1993), and the zero set of Brownian motion, see Bedford and Fisher (1992). Order-two densities also contain information relating to the geometric regularity of sets, see Falconer and Springer (1994). Parallel ideas have been used, Falconer (1992), Patzschke and Zähle (1992, 1993) to study the local behaviour of certain fractal functions and sample paths of self-affine stochastic processes.

In this paper we study the existence of order-two densities of the image and zero sets of the sample paths of a wide class of stable processes. We prove the existence of a constant η (which depends on the process) such that, with probability one, the order-two density of the natural measure on the image or zero set exists and equals η almost everywhere on the set.

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Section 2 contains the basic definitions and properties of order-two densities and stable processes that we shall require. In Section 3, we obtain some geometrical properties of stable processes and then we extend the method of Bedford and Fisher (1992) to prove the existence of order-two densities of transient stable processes in $\mathbb{R}^d$. By using the relationship between the zero set of a stable process of index $\alpha > 1$ and the image of a stable subordinator of index $(1 - 1/\alpha)$ we deduce the existence of order-two densities of the zero set of stable processes of index $\alpha > 1$ in $\mathbb{R}^1$. We end with a discussion of the actual value of the order-two densities of these sets.

2. Preliminaries

First we recall the definition of order-two density, due to Bedford and Fisher (1992). Let μ be a locally finite Borel measure on $\mathbb{R}^d$ and $0 < \alpha < \infty$. The *α-dimensional upper* and *lower order-two* or *average densities* of μ at $x \in \mathbb{R}^d$ are, respectively,

$$\bar{D}_2(\mu, x) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \frac{\mu(B(x, e^{-s}))}{e^{-\alpha s}} ds,$$

$$\underline{D}_2(\mu, x) = \liminf_{T \to \infty} \frac{1}{T} \int_0^T \frac{\mu(B(x, e^{-s}))}{e^{-\alpha s}} ds,$$

where $B(x, r)$ is the closed ball of centre x and radius r. If $\bar{D}_2(\mu, x) = \underline{D}_2(\mu, x)$, the common value is called the *α-dimensional order-two* or *average density* of μ at x and is denoted by $D_2(\mu, x)$. In this case, μ is said to be *order-two regular* at x. If μ is the restriction of a Hausdorff measure H^ϕ to a set E, we refer to the order-two density of the set E and write $D_2(E, x) = D_2(\mu, x)$. There is at most one value of α for which the order-two density exists and is positive and finite, namely the Hausdorff dimension of E or μ, and this is the value of α for which order-two densities are usually of interest.

If $d = 1$, the *right (left) order-two density* $D_2^r(\mu, x)$ (respectively, $D_2^l(\mu, x)$) of μ at $x \in \mathbb{R}$ is defined as above by replacing $\mu(B(x, e^{-s}))/e^{-\alpha s}$ by $\mu(x, x + e^{-s}))/e^{-\alpha s}$ (respectively, $\mu((x - e^{-s}, x))/e^{-\alpha s}$), and in this case, the order-two density is also referred to as the *symmetric order-two density*.

A *stable process of index* α $(0 < \alpha \le 2)$ in $\mathbb{R}^d$ is a stochastic process $X(t)$ $(t \in \mathbb{R})$, defined on a probability space $(\Omega, \mathscr{A}, P)$, with $X(0) = 0$, with stationary and independent increments and with characteristic function given by

$$E(\exp i\langle z, X(t) - X(s)\rangle) = \exp(-(t - s)\psi(z)) \quad (\forall t > s), \tag{2.1}$$

where $z \in \mathbb{R}^d$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$, and with

$$\psi(z) = \lambda |z|^\alpha [1 - ih \operatorname{sgn}(z)\, W(\alpha, z)], \tag{2.2}$$

where λ and h are real constants satisfying $|h| \le 1$ and $\lambda > 0$, and

$$W(\alpha, z) = \tan \frac{\pi\alpha}{2} \quad \text{if } \alpha \ne 1,$$

$$W(1, z) = -\frac{2 \log |z|}{\pi}.$$

If $b = 0$ then $X(t)$ is called a *symmetric* stable process. If $\alpha = 2$ and $\lambda = \frac{1}{2}$, then $X(t)$ ($t \in \mathbb{R}$) is the usual d-dimensional Brownian motion. If $d = 1, 0 < \alpha < 1$ and $b = 1$, the corresponding real-valued process $X(t)(t \in \mathbb{R})$ has increasing sample paths and is called a *stable subordinator of index* α. In this case, the Laplace transform of the distribution of $X(t)(t \in \mathbb{R}_+)$ is given by

$$E(\exp(-sX(t))) = \exp(-bts^\alpha), \tag{2.3}$$

where b is a positive constant.

Let $X_+(t) = X(t)$ ($t \in \mathbb{R}_+$), and $X_-(t) = -X(-t)(t \in \mathbb{R}_+)$; then the two processes X_+ and X_- are independent and

$$X_+(\cdot) \stackrel{d}{=} X_-(\cdot),$$

where "$X \stackrel{d}{=} Y$" means that the two stochastic processes X and Y have the same finite dimensional distributions. Hence we need only to consider the process $X(t)$ ($t \in \mathbb{R}_+$).

We assume throughout this paper that $X(t)$ ($t \in \mathbb{R}_+$) satisfies Hunt's hypothesis (A), see Hunt (1957). In particular, this requires the sample paths to be right continuous and have left limits everywhere, and $X(t)$ ($t \in \mathbb{R}_+$) to have the strong Markov property.

It follows from (2.1) and (2.2) that the stable process of index $\alpha \neq 1$ (or $\alpha = 1, b = 0$) satisfies the following scaling property

$$\forall a > 0, \quad X(a\,\cdot) \stackrel{d}{=} a^{1/\alpha} X(\cdot). \tag{2.4}$$

Stochastic processes satisfying (2.4) are called $1/\alpha$ *self-similar processes*. There has been considerable interest in the sample path properties of self-similar processes, see Kôno (1991) and references therein.

In studying the sample path properties of transient stable processes, Taylor (1967) distinguished stable processes of types A and B. Let $p(t, x)$ be the density function of $X(t)$: a stable process $X(t)$ of index α in $\mathbb{R}^d$ is said to be of *type A* if $p(1, 0) > 0$. Otherwise, it is said to be of *type B*. Note that if $\alpha \geq 1$, all stable processes of index α are of type A and all stable subordinators are of type B. The following theorem is from Boylan (1964).

Theorem A. *Suppose $X(t)(t \in \mathbb{R}_+)$ is a nondegenerate stable process in $\mathbb{R}^d$ of index $\alpha < d$, $\alpha \neq 1$ and*

$$\phi(s) = \begin{cases} s^\alpha \log\log \dfrac{1}{s} & \text{if } X(t) \text{ is of type } A, \\ s^\alpha \left(\log\log \dfrac{1}{s}\right)^{1-\alpha} & \text{if } X(t) \text{ is of type } B. \end{cases}$$

Then there is a finite constant $\xi > 0$, depending only on parameters of the process, such that with probability one,

$$H^{\phi}(X(E)) = \xi m(E) \tag{2.5}$$

for all Borel sets $E \subset \mathbb{R}_+$, where $X(E) = \{X(t): t \in E\}$, where m is Lebesgue measure and where H^{ϕ} is ϕ-Hausdorff measure.

In the case of stable subordinators, Hawkes (1973) calculated the constant ξ explicitly: let $\tau(t)(t \in \mathbb{R}_+)$ be a stable subordinator of index α with Laplace transform given by (2.3) normalised so that $b = 1$, then with probability one,

$$H^{\phi}(\tau(E)) = \alpha^{\alpha}(1-\alpha)^{1-\alpha} m(E) \tag{2.6}$$

for all Borel sets $E \subset \mathbb{R}_+$.

It is easy to verify that for the functions ϕ defined above, the Hausdorff measure H^{ϕ} has the following scaling property: for any $a > 0$ and any H^{ϕ}-measurable set E,

$$H^{\phi}(aE) = a^{\alpha} H^{\phi}(E). \tag{2.7}$$

(See Falconer (1990) or Taylor (1986) for the definition and properties of Hausdorff measures.)

Let $\operatorname{Im} X = X((-\infty, +\infty))$ be the image of the process $X(t)$. Denote the restriction of H^{ϕ} to $\operatorname{Im} X$ by μ; then μ is a σ-finite Borel measure on $\mathbb{R}^d$. The order-two density $D_2(\mu, x)$ is also called the order-two density of $\operatorname{Im} X$ at x and is denoted by $D_2(\operatorname{Im} X, x)$.

To prove the existence of the order-two density of $\operatorname{Im} X$, we consider stable processes in the following setting. Let

$$D^d = \{f: \mathbb{R}_+ \to \mathbb{R}^d \mid f \text{ is right continuous and has left limits everywhere, and } f(0) = 0\}$$

with the Skorohod metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min(1, d_n(f, g)).$$

Here for any $T > 0$, we define $d_T(f, g)$ to be the infimum of all those values of δ for which there exist grids $0 = t_0 < t_1 < \cdots < t_k$, with $t_k \geq T$, and $0 = u_0 < u_1 < \cdots < u_k$, with $u_k \geq T$, such that $|t_i - u_i| \leq \delta$, $i = 0, 1, \ldots, k$, and

$$|f(t) - g(u)| \leq \delta \quad \text{if } t_i \leq t < t_{i+1}, \text{ and } u_i \leq u < u_{i+1}$$

for $i = 0, 1, \ldots, k-1$.

It is shown in Pollard (1984) that under the Skorohod metric, D^d is a separable space and the Borel σ-algebra $\mathscr{D}^d$ coincides with the σ-algebra generated by cylinder sets. Let P_X be the distribution of the stable process $X(t)(t \in \mathbb{R}_+)$ and $\mathscr{D}^d_X$ be the P_X-completion of $\mathscr{D}^d$. From now on, we also refer to the triple $(D^d, \mathscr{D}^d_X, P_X)$ as a stable process in $\mathbb{R}^d$.

Now let C be the space of all continuous real valued functions on $\mathbb{R}_+$ with value 0 at time 0, i.e.

$$C = \{g: \mathbb{R}_+ \to \mathbb{R} | g \text{ is continuous and } g(0) = 0\}$$

with the topology $\mathscr{T}$ of uniform convergence on compact subsets of $\mathbb{R}_+$. Let $\mathscr{C}$ be the Borel σ-algebra generated by $\mathscr{T}$. Specializing from the Skorohod metric we see that $\mathscr{T}$ admits a metric, with respect to which C is a complete separable metric space (see also Fisher (1987)) and $\mathscr{C}$ coincides with the σ-algebra generated by the cylinder sets in C (for a proof see Lemma 72 of Freedman (1983)).

We end this section by recalling from Matheron (1975) some properties of the topology $\mathscr{T}_f$ on the family $\mathscr{F}$ of all closed subsets in $\mathbb{R}^d$. The topology $\mathscr{T}_f$ is generated by the two families $\mathscr{F}^K (K \in \mathscr{K})$ and $\mathscr{F}_G (G \in \mathscr{G})$, where $\mathscr{K}$ and $\mathscr{G}$ are the families of compact subsets and open subsets in $\mathbb{R}^d$ respectively, and

$$\mathscr{F}^K = \{F \in \mathscr{F},\ F \cap K = \phi\}$$

$$\mathscr{F}_G = \{F \in \mathscr{F},\ F \cap G \neq \phi\}.$$

We denote by σ_f the σ-algebra generated by $\mathscr{T}_f$. Note that σ_f is generated by the single class $\mathscr{F}^K (K \in \mathscr{K})$, as well as by the single class $\mathscr{F}_G$ $(G \in \mathscr{G})$.

Let $\{F_n\}$ be a sequence in $\mathscr{F}$; by definition, lim inf F_n and lim sup F_n are, respectively, the intersection and the union of the limits of all the subsequences of $\{F_n\}$ that are convergent in $\mathscr{F}$.

Let Σ be a topological space, and ψ a mapping from Σ into $\mathscr{F}$. Then ψ is called upper semicontinuous (u.s.c.) if, for any $K \in \mathscr{K}$, the set $\psi^{-1}(\mathscr{F}^K)$ is open in Σ, and ψ is called lower semicontinuous (l.s.c.) if, for any $G \in \mathscr{G}$, the set $\psi^{-1}(\mathscr{F}_G)$ is open in Σ. Clearly, if ψ is u.s.c. or l.s.c., then ψ is measurable with respect to the Borel σ-algebra in Σ.

We need the following lemma from Matheron (1975).

Lemma 1. *Let Σ be a separable space and ψ a mapping from Σ into $\mathscr{F}$. Then*

(a) *ψ is u.s.c. if and only if $\psi(\omega) \supset \limsup \psi(\omega_n)$ for any $\omega \in \Sigma$ and any sequence $\{\omega_n\}$ converging to ω in Σ.*

(b) *ψ is l.s.c. if and only if $\psi(\omega) \subset \liminf \psi(\omega_n)$ for any $\omega \in \Sigma$ and any sequence $\{\omega_n\}$ converging to ω in Σ.*

3. Main Results

In this section, we prove the existence of order-two densities of the image and zero set of stable processes.

For $a > 0$, define the map $\Delta_a : D^d \to D^d$ by

$$(\Delta_a f)(t) = \frac{f(at)}{a^{1/\alpha}}$$

and define the *scaling flow* τ_s on D^d by $\tau_s = \Delta_{\exp s} (s \in \mathbb{R})$. Clearly $\tau_a \circ \tau_b = \tau_{a+b}$, and the scaling property (2.4) of the stable process implies that $\tau_s (s \in \mathbb{R})$ is measure preserving, that is, τ_s is a flow. The following lemma is an immediate consequence of Proposition 4 in Patzschke and Zähle (1993).

Lemma 2. *$\tau_s (s \in \mathbb{R})$ is an ergodic flow on $(D^d, \mathscr{D}_X^d, P_X)$.*

We require the following proposition which may be of independent interest.

Proposition 1. *Let $X(t)$ $(t \in \mathbb{R}_+)$ be a stable process of index α in $\mathbb{R}^d$ with $\alpha \neq 1$ or $\alpha = 1$, $h = 0$. Then with probability one,*

$$m(t > 0 : |X(t)| = r) = 0 \quad \textit{for all } r > 0.$$

In particular, this implies that with probability 1 $m(t > 0: X(t) \in B(0, r))$ is continuous for $r > 0$.

To prove Proposition 1, we need the following lemma.

Lemma 3. *Let $0 < \beta < 1$ and $M > 0$ be given and let $B \subset \mathbb{R}^d$ be any ball of radius M. Then there exists a constant $c > 0$, depending only on d, M and β, such that for all $y, z \in B$,*

$$\int_B \frac{\mathrm{d}x}{||z - x| - |y - x||^\beta} \leq \frac{c}{|z - y|^\beta}.$$

Proof. Let $x, y, z \in B$. Choose as origin the mid-point of the line segment $\overline{yz}$. Let x have cylindrical coordinates $(\boldsymbol{r}, h)$ with respect to a co-ordinate system with cylindrical axis as the line through y and z. Then

$$|y - x|^2 = |\boldsymbol{r}|^2 + (h - \tfrac{1}{2}|y - z|)^2,$$

$$|z - x|^2 = |\boldsymbol{r}|^2 + (h + \tfrac{1}{2}|y - z|)^2.$$

Hence

$$||z - x| - |y - x|| = \frac{||z - x|^2 - |y - x|^2|}{|z - x| + |y - x|} \geq \frac{|h||z - y|}{2M}. \tag{3.1}$$

Let B_1 be the ball centre the mid-point of $\overline{yz}$ and radius $2M$ (note that $B \subset B_1$), then by (3.1)

$$\begin{aligned} \int_B \frac{\mathrm{d}x}{||z - x| - |y - x||^\beta} &\leq \int_{B_1} \frac{\mathrm{d}x}{||z - x| - |y - x||^\beta} \\ &\leq (2M)^\beta \int_{B_1} \frac{\mathrm{d}x}{|z - y|^\beta |h|^\beta} \quad (\text{where } x = (\boldsymbol{r}, h)) \\ &\leq \frac{(2M)^\beta}{|z - y|^\beta} \cdot (4M)^{d-1} \int_{-2M}^{2M} \frac{\mathrm{d}h}{|h|^\beta} \\ &= \frac{c}{|z - y|^\beta}. \quad \square \end{aligned}$$

Proof of Proposition 1. It is sufficient to prove that for all $\varepsilon > 0$, $N > 0$ and $M > 0$, with probability one,

$$m(t \in [\varepsilon, N]\colon |X(t)| = r) = 0 \quad \text{for all } 0 < r \le M. \tag{3.2}$$

Define a random Radon measure μ on $[0, \infty)$ by

$$\int_0^\infty f(u)\mathrm{d}\mu(u) = \int_\varepsilon^N f(|X(t)|)1_{[0,M]}(|X(t)|)\mathrm{d}t$$

for any $f \in C[0, \infty)$; then $\operatorname{spt}(\mu) \subseteq [0, M]$ and for $0 < \beta < 1$

$$\begin{aligned} I_\beta(\mu) &\equiv \int\int \frac{\mathrm{d}\mu(u)\mathrm{d}\mu(v)}{|u-v|^\beta} \\ &= \int_\varepsilon^N \int_\varepsilon^N \frac{1_{[0,M]}(|X(t)|)1_{[0,M]}(|X(s)|)}{||X(t)| - |X(s)||^\beta}\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_\varepsilon^N \int_\varepsilon^N \frac{1_{[0,M]}(|X(t)|)1_{[0,M]}(|X(s)|)}{||X(t) - X(\varepsilon) + X(\varepsilon)| - |X(s) - X(\varepsilon) + X(\varepsilon)||^\beta}\,\mathrm{d}t\,\mathrm{d}s. \end{aligned}$$

Since for any $t > \varepsilon$, $X(t) - X(\varepsilon)$ and $X(\varepsilon)$ are independent, we have

$$\begin{aligned} E(I_\beta(\mu)) &= \int E(I_\beta(\mu)|X(\varepsilon) = x)\,p(\varepsilon, x)\,\mathrm{d}x \\ &= \int_\varepsilon^N \int_\varepsilon^N E\int \frac{p(\varepsilon, x)}{||X(t) - X(\varepsilon) + x| - |X(s) - X(\varepsilon) + x||^\beta}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s \end{aligned}$$

where $p(\varepsilon, x)$ is the density function of $X(\varepsilon)$ which is bounded and continuous, and where the inner integral is over those x such that $|X(t) - X(\varepsilon) + x| \le M$ and $|X(s) - X(\varepsilon) + x| \le M$. Since this inner domain of integration is contained in a ball B of radius $2M$ with $X(t) - X(\varepsilon) \in B$ and $X(s) - X(\varepsilon) \in B$, it follows from Lemma 3 that

$$E(I_\beta(\mu)) \le c \int_\varepsilon^N \int_\varepsilon^N E\left(\frac{1}{|X(t) - X(s)|^\beta}\right)\mathrm{d}t\,\mathrm{d}s < +\infty$$

if $0 < \beta < \alpha$. Thus almost surely $I_\beta(\mu) < +\infty$ and in particular μ has no atoms, which implies (3.2). □

It is well known that if $\alpha < d$, then $X(t)$ is a.s. transient in the sense that $|X(t)| \to +\infty$ as $t \to +\infty$. Combining this with Theorem A and Proposition 1, we deduce the following proposition.

Proposition 2. *If $\alpha < d$, there is a τ_s invariant set D_1^d (which we may take to be Borel) contained in D^d such that $P_X(D_1^d) = 1$, and for any $f \in D_1^d$, $|f(t)| \to \infty$ as $t \to +\infty$ and $H^\phi(f(t)\colon |f(t)| = r) = 0$ for all $r > 0$.*

We define a map $H: D_1^d \to C$ by

$$H: f \to H^\phi(\overline{\mathrm{Im} f} \cap B(0,\cdot)),$$

where $\overline{\mathrm{Im} f}$ is the closure of $\mathrm{Im} f = f([0, \infty))$. Note that, for every $f \in D_1^d$, $\overline{\mathrm{Im} f}$ differs from $\mathrm{Im} f$ only in the countable set of left limits $f(t-0)$ at points of discontinuity. The continuity of the function $H^\phi(\overline{\mathrm{Im} f} \cap B(0,\cdot))$ follows from Proposition 2.

Proposition 3. *H is $(\mathscr{D}_X^d, \mathscr{C})$ measurable.*

Proof. This may be proved in the four steps listed below. We omit the routine details.

(i) The map from D_1^d into $\mathscr{F}$ defined by $f \to \overline{\mathrm{Im} f}$ in l.s.c. and hence is $(\mathscr{D}_X^d, \sigma_f)$ measurable. This may be shown from the definition of the Skorohod metric.

(ii) For each fixed $t \geq 0$, the map $F \to F \cap B(0,t)$ from $\mathscr{F}$ into itself is u.s.c. In particular, it is σ_f measurable. This follows easily from Lemma 1(a).

(iii) For any positive Borel measure μ on $\mathbb{R}^d$, the map $F \to \mu(F)$ from $\mathscr{F}$ into $\mathbb{R}_+$ is $(\sigma_f, \mathscr{B}(\mathbb{R}^1))$ measurable, where $\mathscr{B}(\mathbb{R}^k)$ is the Borel σ-algebra in $\mathbb{R}^k$.

To see this define a two-valued map $\lambda: \mathbb{R}^d \times \mathscr{F} \to \{0, 1\}$ by

$$\lambda(x, F) = 1_F(x);$$

then it is easy to verify that λ is measurable with respect to $\mathscr{B}(\mathbb{R}^d) \times \sigma_f$. Hence by Fubini's theorem, the map $F \mapsto \mu(F) = \int \lambda(x, F)\mu(\mathrm{d}x)$ is σ_f measurable.

(iv) The map H is $(\mathscr{D}_X^d, \mathscr{C})$ measurable.

It follows from (i)–(iii) that for each $t \geq 0$, the map H_t from D_1^d into $\mathbb{R}_+$ defined by

$$H_t: f \mapsto H^\phi(\overline{\mathrm{Im} f} \cap B(0,t))$$

is $(\mathscr{D}_X^d, \mathscr{B}(\mathbb{R}^1))$ measurable, which implies that for any finite cylinder set $B \subset C$, we have $H^{-1}(B) \in \mathscr{D}_X^d$. This proves the $(\mathscr{D}_X^d, \mathscr{C})$ measurability of H. □

Let ν_H be the Borel measure on C which is the image of P_X under the map H and let $\mathscr{C}_H$ denote the completion of $\mathscr{C}$. Then by the scaling property (2.7) of the Hausdorff measure H^ϕ, it follows that $(C, \mathscr{C}_H, \nu_H, \tilde{\tau}_s)$ is a flow, where $\tilde{\tau}_s = \tilde{\Delta}_{\exp s}$ ($s \in \mathbb{R}$) and, for $a > 0$, we define $\tilde{\Delta}_a: C \to C$ by

$$(\tilde{\Delta}_a f)(t) = \frac{f(a^{1/\alpha} t)}{a}.$$

Recall from Fisher (1987) the definition of homeomorphism of flows. Let $(X, \mathscr{X}, \mu)$, $(Y, \mathscr{Y}, \nu)$ be measure spaces with probability measures μ, ν and complete σ-algebras $\mathscr{X}$, $\mathscr{Y}$. By a *homeomorphism* $\gamma: X \to Y$ we mean a function $\gamma: X \backslash N \to Y$ with N a null set, such that γ is $(\mathscr{X}, \mathscr{Y})$ measurable and measure preserving. A *homeomorphism of flows* (X, τ_s) and $(Y, \tilde{\tau}_s)$ is a homeomorphism γ from $X \backslash N$ to Y such that

$$\tilde{\tau}_s \gamma = \gamma \tau_s, \quad \mu \text{ a.e.}$$

such that the null set can be taken to be the same for all $s \in \mathbb{R}$.

Proposition 4. *H is a homeomorphism of flows. In particular, $\tilde{\tau}_s$ is ergodic.*

Proof. By the definition of ν_H, we know that the map H is measure preserving. It remains to verify that for any $s \in \mathbb{R}$,

$$H \circ \tau_s = \tilde{\tau}_s \circ H$$

on D_1^d, or equivalently, for any $a > 0$,

$$H \circ \Delta_a = \tilde{\Delta}_a \circ H \tag{3.3}$$

on D_1^d. Now given $f \in D_1^d$ and $t \geq 0$, it follows from (2.7) that

$$\begin{aligned}(H \circ \Delta_a f)(t) &= H^{\phi}(\overline{\mathrm{Im}\, \Delta_a f} \cap B(0, t)) \\ &= H^{\phi}(a^{-1/\alpha}\, \overline{\mathrm{Im}\, f} \cap B(0, a^{1/\alpha} t)) \\ &= a^{-1} H^{\phi}(\overline{\mathrm{Im}\, f} \cap B(0, a^{1/\alpha} t)) \\ &= (\tilde{\Delta}_a \circ Hf)(t),\end{aligned}$$

proving (3.3). □

Now we are in a position to prove the main result of this section.

Theorem 1. *Let $X(t)$ ($t \in \mathbb{R}$) be a stable process of index α in $\mathbb{R}^d$, $\alpha < d$, $\alpha \neq 1$. Then there exists a constant $\eta > 0$, depending only on the parameters of the process, such that with probability one, the order-two density $D_2(\mathrm{Im}\, X, x)$ exists and equals η at H^{ϕ}-almost all $x \in \mathrm{Im}\, X$ where ϕ is defined in Theorem A. In fact $\eta = 2\xi E(T(1))$, where ξ is the almost sure value of $H^{\phi}(X([0, 1]))$ and $T(1)$ is the sojourn time of $X(t)(t \in \mathbb{R}_+)$ in $B(0, 1)$.*

Proof. We first consider the point $x = 0$ and show that there is a finite constant $\eta > 0$ such that for P_X-almost all $f \in D^d$,

$$D_2(\mathrm{Im}\, f, 0) = \frac{\eta}{2}. \tag{3.4}$$

Define a map $F: C \to \mathbb{R}$ by

$$F(g) = g(1);$$

then $F \in L^1(C, \nu_H)$ and by Theorem A

$$\begin{aligned}\int F(g)\nu_H(\mathrm{d}g) &= \int H^{\phi}(\mathrm{Im}\, f \cap B(0, 1)) P_X(\mathrm{d}f) \\ &= \xi E(T(1)) < +\infty,\end{aligned}$$

where $T(1) = \int_0^{\infty} 1_{B(0,1)}(X(s))\mathrm{d}s$ is the sojourn time of $X(t)$ ($t \in \mathbb{R}_+$) in the closed unit ball $B(0, 1)$ and ξ is the constant of (2.5). The finiteness of $E(T(1))$ for types A and B stable processes follows from Lemma 5 and Lemma 6 in Taylor (1967). Then

Proposition 4 and the Birkhoff ergodic theorem for flows imply that for ν_H almost all $g \in C$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T F(\tilde{\tau}_s g)\,\mathrm{d}s = \xi E(T(1)).$$

Hence for P_X almost all $f \in D^d$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{H^{\phi}(\mathrm{Im} f \cap B(0, \mathrm{e}^{-s}))}{\mathrm{e}^{-\alpha s}}\,\mathrm{d}s = \xi E(T(1)).$$

Letting

$$\eta = 2\xi E(T(1)) \tag{3.5}$$

gives (3.4).

Now let D_0^d be the subset of D^d with $P_X(D_0^d) = 1$ such that (3.4) holds for every $f \in D_0^d$. We may further assume D_0^d is a Borel set. For any fixed $t \in \mathbb{R}_+$, let $X_t(s) = X(s+t) - X(t)$, then $X_t(s)\,(s \in \mathbb{R}_+)$ is also a stable process in $\mathbb{R}^d$ of index $\alpha < d$ with $X_t(0) = 0$. By the above, we have

$$D_2(X_t[0, +\infty), 0) = \frac{\eta}{2} \quad \text{a.s.} \tag{3.6}$$

For any $T > 0$, let

$$A_T = \{(t, X) \in [0, T] \times D^d \colon X_t(\cdot) \in D_0^d\}.$$

Since for any $s \in \mathbb{R}_+$, the map $\beta \colon \mathbb{R}_+ \times D^d \to \mathbb{R}^d$ defined by

$$(t, X) \to X_t(s)$$

is right continuous in t, we have that β is $\mathscr{B}(\mathbb{R}) \times \mathscr{D}_X^d$ measurable. Hence the map $(t, X) \mapsto X_t(\cdot)$ is also $\mathscr{B}(\mathbb{R}) \times \mathscr{D}_X^d$ measurable, implying that A_T is $\mathscr{B}(\mathbb{R}) \times \mathscr{D}_X^d$ measurable. By (3.6), for each fixed $t \in [0, T]$, the set $\{X \colon (t, X) \in A_T)$ has P_X measure 1. Therefore by Fubini's Theorem, we deduce that, for P_X almost all $X \in D^d$,

$$D_2(X_t[0, +\infty), 0) = \frac{\eta}{2}$$

for m-almost all $t \in \mathbb{R}_+$. Returning to the probability space $(\Omega, \mathscr{A}, P)$, we have with probability one

$$D_2(X(t, +\infty), X(t)) = \frac{\eta}{2}, \tag{3.7}$$

$$D_2(X(-\infty, t], X(t)) = \frac{\eta}{2} \tag{3.8}$$

for m-almost all $t \in \mathbb{R}$. It follows from Theorem 3 in Taylor (1967) that with probability one, for all $t \in \mathbb{R}$,

$$\dim(X(-\infty, t] \cap X[t, +\infty)) \begin{cases} \le 2\alpha - d < \alpha, & \text{if } X \text{ is of type } A \text{ and } 2\alpha - d > 0, \\ = 0 & \text{otherwise.} \end{cases}$$

Hence

$$H^{\phi}(\operatorname{Im} X \cap B(X(t), e^{-s})) = H^{\phi}(X(-\infty, t] \cap B(X(t), e^{-s})) + H^{\phi}(X[t, +\infty) \cap B(X(t), e^{-s})).$$

By (3.7) and (3.8) we have, with probability one, that

$$D_2(\operatorname{Im} X, X(t)) = \eta$$

for m-almost all $t \in \mathbb{R}$. Again using (2.4), we conclude that with probability one,

$$D_2(\operatorname{Im} X, x) = \eta$$

for H^{ϕ}-almost all $x \in \operatorname{Im} X$. □

The following result is clear from the proof of Theorem 1.

Corollary 1. Let $X(t)$ $(t \in \mathbb{R})$ be a stable subordinator of index α. Then there exists a constant $\eta > 0$, depending only on the parameters of the process, such that with probability one, the right, left and symmetric α-dimensional order-two densities exist and equal to $\eta/2$, $\eta/2$ and η, respectively at H^{ϕ}-almost all $x \in \operatorname{Im} X$, where $\phi(s) = s^{\alpha}(\log\log 1/s)^{1-\alpha}$.

Remarks 1. The conclusion of Theorem 1 is also true if $X(t)$ is a symmetric Cauchy process ($\alpha = 1$, $h = 0$) in $\mathbb{R}^d$ with $d > 1$, the same proof working using the results in Taylor (1967).

2. In the cases of symmetric Cauchy process in $\mathbb{R}^1$(i.e. $\alpha = d = 1$, $h = 0$) or planar Brownian motion ($\alpha = d = 2$, $h = 0$), the exact Hausdorff measure functions for the image sets were found in Ray (1963), Taylor (1964, 1967). In these cases the sojourn time $T(1)$ cannot be defined as above so the method of Theorem 1 is not applicable. It would be interesting to study the existence of order-two densities of the image sets of these processes.

3. It seems difficult to calculate the value of the constant η in Theorem 1 explicitly. In some special cases, we can relate η to other constants. If $X(t)$ $(t \in \mathbb{R})$ is a Brownian motion in $\mathbb{R}^d$ $(d \geq 3)$, Ciesielski and Taylor (1962) obtained the distribution function of the sojourn time. Using this we can express η as

$$\eta = 2\xi \sum_{r=1}^{\infty} \frac{\psi_{d,r}}{p_{d,r}^2},$$

where $\{p_{d,r}\}$ $(r = 1, 2, \dots)$ are the positive zeros of the Bessel function $J_{\mu}(z)$ with $\mu = d/2 - 2$ and

$$\psi_{d,r} = \frac{1}{2^{\mu-1}\,\Gamma(\mu - 1)} \cdot \frac{p_{d,r}^{\mu-1}}{J_{\mu+1}(p_{d,r})}.$$

Now consider the case where $X(t)$ is a stable subordinator of index $\alpha(0 < \alpha < 1)$ with Laplace transform given by (2.3). Since $X(\cdot) \overset{d}{=} b^{1/\alpha}\tau(\cdot)$, by (2.6), (2.7) and (3.5) we have

$$\eta = 2b\alpha^{\alpha}(1-\alpha)^{1-\alpha}E(T(1)), \tag{3.9}$$

where $T(1)$ is the sojourn time in $B(0, 1)$ of $\tau(t)$ ($t \in \mathbb{R}_+$). By the formula for the inversion of a Laplace transform, the density function of $\tau(t)$ is

$$\begin{aligned} p(t,x) &= \mathscr{L}^{-1}(\exp(-ts^{\alpha})) \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \exp(-ts^{\alpha}+sx)\,ds \end{aligned}$$

for any $a > 0$, where the integration is along the straight line $\operatorname{Re} s = a$. If $\alpha = \frac{1}{2}$, then

$$p(1,x) = \frac{1}{2\sqrt{\pi}x^{3/2}}e^{-1/4x};$$

hence

$$\begin{aligned} E(T(1)) &= E\int_0^{\infty} 1_{B(0,1)}(\tau(t))\,dt \\ &= \int_0^{\infty} dt \int_0^{t^{-2}} \frac{1}{2\sqrt{\pi}x^{3/2}}e^{-1/4x}\,dx \\ &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} dt \int_t^{\infty} e^{-u^2/4}\,du \\ &= \frac{2}{\sqrt{\pi}}. \end{aligned}$$

Thus using (3.9) we get that for stable subordinators of index $\alpha = \frac{1}{2}$ and Laplace transform (2.3)

$$\eta = \frac{2b}{\sqrt{\pi}}. \tag{3.10}$$

Now let $X(t)(t \in \mathbb{R})$ be a stable process in $\mathbb{R}^1$ of index $\alpha > 1$; we consider

$$Z = \{t \in \mathbb{R};\ X(t) = 0\},$$

the zero set of $X(t)(t \in \mathbb{R})$.

It is known that the local time $L(t, x)$ of a stable process $X(t)$ ($t \in \mathbb{R}_+$) in $\mathbb{R}^1$ of index $\alpha > 1$ exists and is jointly continuous with probability 1 (Boylan (1964)). Denote the local time at $x = 0$ by $L(t)$. Stone (1963) proved that

$$Y(t) = \inf\{u\colon L(u) > t\} \quad t \geq 0$$

is a stable subordinator of index $\beta = 1 - 1/\alpha$ with the Laplace transform of $Y(t)$ given by (2.3), where

$$b = \alpha(\sin \pi\alpha^{-1})\lambda^{1/\alpha}\left[\operatorname{Re}\left(1 + ih\tan\left(\frac{\pi\alpha}{2}\right)\right)^{-1/\alpha}\right]^{-1}. \tag{3.11}$$

Further, he showed that the zero set Z of $X(t)$ $(t \in \mathbb{R}_+)$ is almost surely equal to $\overline{\text{Im}\, Y}$. Thus the exact Hausdorff measure function for Z obtained by Taylor and Wendel (1966) is a special case of Theorem A, and we have the following consequence of Corollary 1.

Corollary 2. *Let $X(t)$ $(t \in \mathbb{R})$ be a stable process in $\mathbb{R}^1$ of index $\alpha > 1$. Then there exists a constant $\eta > 0$, depending only on parameters of the process, such that with probability one, the right, left and symmetric $(1 - 1/\alpha)$-dimensional order-two densities of Z exist and equal to $\eta/2$, $\eta/2$ and η, respectively, at H^{ϕ}-almost all $x \in Z$, where $\phi(s) = s^{1-1/\alpha}(\log\log 1/s)^{1/\alpha}$.*

By (3.10) and (3.11), we have that if $\alpha = 2$, $\lambda = \frac{1}{2}$ then $X(t)$ $(t \in \mathbb{R})$ is Brownian motion in $\mathbb{R}^1$, so $\eta = 2\sqrt{2/\pi}$ and Corollary 2 agrees with Theorem 5.5 in Bedford and Fisher (1992).

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