Measuring the Range of an Additive Lévy Process

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May 24, 2002

Abstract

The primary goal of this paper is to study the range of the random field $X(t) = \sum_{j=1}^{N} X_j(t_j)$, where $X_1, \ldots, X_N$ are independent Lévy processes in $\mathbb{R}^d$.

To cite a typical result of this paper, let us suppose that Ψ_i denotes the Lévy exponent of X_i for each $i = 1, \ldots, N$. Then, under certain mild conditions, we show that a necessary and sufficient condition for $X(\mathbb{R}_+^N)$ to have positive d-dimensional Lebesgue measure is the integrability of the function $\mathbb{R}^d \ni \xi \mapsto \prod_{j=1}^{N} \operatorname{Re}\{1 + \Psi_j(\xi)\}^{-1}$. This extends a celebrated result of (Kesten 1969; Bretagnolle 1971) in the one-parameter setting. Furthermore, we show that the existence of square integrable local times is yet another equivalent condition for the mentioned integrability criterion. This extends a theorem of Hawkes (1986) to the present random fields setting, and completes the analysis of local times for additive Lévy processes initiated in the companion paper Khoshnevisan, Xiao and Zhong (2002).

Running Title The Range of an Additive Lévy Process
AMS2000 subject classifications. 60G60; 60J55; 60J45.
Key words and phrases. Additive Lévy processes, strictly stable processes, capacity, energy, local times, Hausdorff dimension.

Contents

*Research partially supported by grants from NSF and NATO.
†Supported by the National Natural Science Foundation of China.

1 Introduction

An N-parameter d-dimensional random field $X = \{X(t);\ t \in \mathbb{R}_+^N\}$ is an *additive Lévy process* if X has the following pathwise decomposition:

$$X(t) = X_1(t_1) + \cdots + X_N(t_N), \qquad \forall t \in \mathbb{R}_+^N,$$

where $X_1, \cdots, X_N$ are independent classical Lévy processes on $\mathbb{R}^d$. Using tensor notation, we will often write $X = X_1 \oplus \cdots \oplus X_N$ for brevity, and will always assume that $X_j(0) = 0$ for all $j = 1, \ldots, N$. Finally, if $\Psi_1, \ldots, \Psi_N$ denote the Lévy exponents of $X_1, \ldots, X_N$, respectively, we define the *Lévy exponent* of X to be $\Psi = (\Psi_1, \ldots, \Psi_N)$. See the companion paper Khoshnevisan, Xiao and Zhong (2002) for more detailed historical information, as well as a number of collected facts about additive Lévy processes.

The following question is the starting point of our investigation:

$$\text{“}\textit{When can the range of } X \textit{ have positive Lebesgue measure?}\text{”} \tag{1.1}$$

In the one-parameter setting, i.e., when $N = 1$, this question has a long history as well as the following remarkable answer, discovered in (Bretagnolle 1971; Kesten 1969): If λ_d denotes Lebesgue measure in $\mathbb{R}^d$,

$$\mathbb{E}\{\lambda_d(X(\mathbb{R}_+))\} > 0 \iff \int_{\mathbb{R}^d} \mathrm{Re}\left(\frac{1}{1+\Psi(\xi)}\right) d\xi < +\infty, \tag{1.2}$$

where Ψ denotes the Lévy exponent of X, and $\mathrm{Re}\, z$ denotes the real part of $z \in \mathbb{C}$. In the sequel, the imaginary part and the conjugate of z will be denoted by $\mathrm{Im}\, z$ and $\overline{z}$, respectively.

The primary objective of this paper is to answer Question (1.1) for the range of an additive Lévy process $X = \{X(t);\, t \in \mathbb{R}_+^N\}$. It is quite standard to show that $\prod_{j=1}^N \mathrm{Re}\,\{1+\Psi_j\}^{-1} \in L^1(\mathbb{R}^d)$ is a sufficient condition for $\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} > 0$. The converse is much more difficult to prove, and we have succeeded in doing so as long as there exists a positive constant $\vartheta > 0$ such that

$$\mathrm{Re}\left(\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\right) \geq \vartheta \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right); \tag{1.3}$$

see Theorem 1.1 below. We note that when $N = 1$, Condition (1.3) holds vacuously with $\vartheta = 1$.

Among other things, we will show in this paper that when (1.3) holds, the proper setting for the analysis of question (1.1) is potential theory and its various connections to the random field X, as well as energy that we will describe below. Various aspects of the potential theory of multiparameter processes have been treated in Evans (1987a, b), Fitzsimmons and Salisbury (1989), Hirsch (1995), Hirsch and Song (1995a, 1995b), and Khoshnevisan and Xiao (2002a).

As we mentioned earlier, we propose to derive the following multiparameter version of (1.2). It will be a consequence of some of the later results of this article.

Theorem 1.1 *Let X be an additive Lévy process in $\mathbb{R}^d$ with Lévy exponent $(\Psi_1, \ldots, \Psi_N)$, and suppose that Condition (1.3) holds. Then,*

$$\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} > 0 \iff \int_{\mathbb{R}^d} \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi < +\infty.$$

Remark 1.2 We record the fact that Condition (1.3) is only needed for proving the direction "$\Longrightarrow$." We also mention the fact that under the conditions of Proposition 6.5 below, $\lambda_d(X(\mathbb{R}^N_+))$ is almost surely equal to $+\infty$ (respectively, 0), if $\int_{\mathbb{R}^d} \prod_{j=1}^N \mathrm{Re}\,\{1+\Psi_j(\xi)\}^{-1}\, d\xi$ is finite (respectively, infinite). □

Theorem 1.1 has the following equivalent formulation which addresses existence questions for the local times of the companion paper Khoshnevisan, Xiao and Zhong (2002). When $N = 1$, it is a well known theorem of Hawkes (1986).

Theorem 1.3 *Let X be an additive Lévy process in $\mathbb{R}^d$ that satisfies Condition (1.3). Then X has square integrable local times if and only if $\prod_{j=1}^N \mathrm{Re}\,\{1+\Psi_j\}^{-1}$ is integrable in $\mathbb{R}^d$, where $(\Psi_1, \ldots, \Psi_N)$ is the Lévy exponent of X.*

We have already mentioned that Theorem 1.3 is an equivalent probabilistic interpretation of Theorem 1.1. But, in fact, our formulation of Theorem 1.3 lies at the heart of our proof of Theorem 1.1 and its further refinements; cf. Theorem 2.1 below.

Remark 1.4 When $N = 1$, Condition (1.3) always holds with $\vartheta = 1$. Hence, our theorems extend those of Bretagnolle (1971), Kesten (1969) and Hawkes (1986). □

In general, any additive Lévy process X with Lévy exponent $\Psi = (\Psi_1, \ldots, \Psi_N)$ induces an *energy form* $\mathcal{E}_\Psi$ that can be described as follows: For all finite measures μ on $\mathbb{R}^d$, and/or all integrable functions $\mu : \mathbb{R}^d \to \mathbb{R}$,

$$\mathcal{E}_\Psi(\mu) = (2\pi)^{-d} \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi, \tag{1.4}$$

where $\widehat{\ }$ denotes the Fourier transform normalized as $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(x)\, dx$ $[f \in L^1(\mathbb{R}^d)]$.

Frequently, we may refer to "the energy" of a measure (or function) in the context of an additive Lévy process X without explicitly mentioning its dependence on the Lévy exponent of X. This makes for a simpler presentation and should not cause ambiguities.

Having introduced energies, we can present a key result of this paper. When $N = 1$, it can be found in Bertoin (1996, p. 60).

Theorem 1.5 *Consider any d-dimensional additive Lévy process X, whose Lévy exponent Ψ satisfies (1.3). Then, given any nonrandom compact set $F \subset \mathbb{R}^d$, $\mathbb{E}\{\lambda_d(X(\mathbb{R}^N_+) \oplus F)\} > 0$ if and only if F carries a finite measure of finite energy.*

We have adopted the notation that for all sets A and B, $A \oplus B = \{a + b;\ a \in A,\ b \in B\}$. This should not be confused with our tensor notation for $X = X_1 \oplus \cdots \oplus X_N$.

Note, in particular, that if we choose F to be a singleton in Theorem 1.5, we immediately obtain Theorem 1.1.

Next, we apply Theorem 1.5 to compute the Hausdorff dimension of the range of an arbitrary additive Lévy process.

Theorem 1.6 *Given an additive Lévy process X in $\mathbb{R}^d$ with Lévy exponent $(\Psi_1, \ldots, \Psi_N)$ that satisfies (1.3),*

$$\dim\left(X(\mathbb{R}^N_+)\right) = d - \eta, \qquad \mathbb{P}\text{-}a.s.,$$

where

$$\eta = \sup\left\{\alpha > 0 : \int_{\xi\in\mathbb{R}^d:\ \|\xi\|>1} \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) \frac{d\xi}{\|\xi\|^\alpha} = +\infty\right\}.$$

Here, $\dim(\bullet)$ *denotes Hausdorff dimension,* $\sup\varnothing = 0$, *and* $\|\bullet\|$ *denotes the Euclidean ℓ^2-norm.*

Remark 1.7 It can be checked directly that

$$\eta = \inf\left\{\alpha > 0 : \int_{\xi\in\mathbb{R}^d:\ \|\xi\|>1} \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) \frac{d\xi}{\|\xi\|^\alpha} < +\infty\right\}.$$

Furthermore, one always has $\eta \le d$. □

When $N = 1$, i.e., when X is an ordinary Lévy process in $\mathbb{R}^d$, Pruitt (1969) has shown that the Hausdorff dimension of the range of X is

$$\gamma = \sup\left\{\alpha \ge 0 : \limsup_{r\to 0}\ r^{-\alpha}\int_0^1 \mathbb{P}\left\{|X(t)| \le r\right\} dt < +\infty\right\}.$$

In general, this computation is not satisfying, since the above lim sup is not easy to evaluate. Pruitt (1969, Theorem 5) addresses this issue by verifying the following estimate for γ in terms of the Lévy exponent Ψ of X:

$$\gamma \ge \sup\left\{\alpha < d : \int_{\|\xi\|\ge 1} \frac{1}{|\Psi(\xi)|}\frac{d\xi}{\|\xi\|^{d-\alpha}} < +\infty\right\}.$$

Moreover, it is shown there that if, in addition, $\mathrm{Re}\,\Psi(\xi) \ge 2\log\|\xi\|$ (for all $\|\xi\|$ large), then,

$$\gamma = \sup\left\{\alpha < d : \int_{\mathbb{R}^d} \mathrm{Re}\left(\frac{1-e^{-\Psi(\xi)}}{\Psi(\xi)}\right)\frac{d\xi}{\|\xi\|^{d-\alpha}} < +\infty\right\}.$$

See Fristedt (1974, 377–378) for further discussions on Pruitt's work in this area.

Our Theorem 1.6 readily implies the following representation for the index γ that holds under no restrictions. To the best of our knowledge, it is new.

Corollary 1.8 *If X denotes a Lévy process in $\mathbb{R}^d$ with Lévy exponent Ψ,*

$$\gamma = \sup\left\{\alpha < d : \int_{\xi\in\mathbb{R}^d:\ \|\xi\|>1} \mathrm{Re}\left(\frac{1}{1+\Psi(\xi)}\right)\frac{d\xi}{\|\xi\|^{d-\alpha}} < +\infty\right\}.$$

Remark 1.9 To paraphrase Kesten (1969, p.7), Eq. (1.2) has the somewhat unexpected consequence that the range of a Lévy process $\{X(t);\, t \ge 0\}$ has a better chance of having positive Lebesgue's measure than the range of its symmetrization $\{X(t) - X'(t);\, t \ge 0\}$, where X' is an independent copy of X. Thanks to Corollary 1.8, this qualitative statement has a *quantitative* version. Namely, with probability one, $\dim(X(\mathbb{R}_+)) \ge \dim(Y(\mathbb{R}_+))$, where $Y(t) = X(t) - X'(t)$ is the symmetrization of X. To prove this, one need only note that $\mathrm{Re}\,\{1+\Psi\}^{-1} \le \{1 + \mathrm{Re}\,\Psi\}^{-1}$, pointwise. □

Remark 1.10 The preceding Remark can be adapted to show that for any additive Lévy process $\{X(t);\, t \in \mathbb{R}_+^N\}$ that satisfies Eq. (1.3),

$$\dim(X(\mathbb{R}_+^N)) \ge \dim(Y(\mathbb{R}_+^N)), \qquad \text{a.s.} \tag{1.5}$$

Here, Y is the symmetrization of X defined by $Y(t) = X(t) - X'(t)$, where X' is an independent copy of X. To verify the displayed inequality, we first note that Y also satisfies Eq. (1.3); cf. Example 1.16 below. Thus, our claim follows from Theorem 1.6 and the elementary pointwise inequality: $\prod_{j=1}^N \mathrm{Re}\,\{1+\Psi_j\}^{-1} \le \prod_{j=1}^N\{1+\mathrm{Re}\,\Psi_j\}^{-1}$. Furthermore, we note that the strict inequality in (1.5) may hold even for $N = 1$; see Pruitt (1969, §4) for an example. This example was also noticed by Taylor (1973, p.401), but there was a minor error in his statement on line -3: "smaller" should be "larger". It is worthwhile to point out that Hawkes (1974), by modifying the construction of Pruitt (1969), defined another Lévy process X in $\mathbb{R}$ such that its range has positive 1-dimensional Lebesgue measure, while the Hausdorff dimension of the range of its symmetrization is strictly smaller than 1. □

Remark 1.11 There are several interesting "indices" for Lévy processes, one of which is the index γ mentioned earlier. These indices arise when studying various properties of the sample paths of Lévy processes, and include the *upper index* β, the *lower indices* β' and β'' (Blumenthal and Getoor 1961), and the index γ' (Hendricks 1983). Rather than reintroducing these indices, we only mention that

$$0 \le \beta' \wedge d \le \gamma \le \gamma' \le \beta \wedge d.$$

Pruitt and Taylor (1996) discuss some open problems regarding these indices. □

The following is an outline of the paper: In Section 2 (Theorem 2.1) we state a complete characterization of all compact sets $E \subset \mathbb{R}_+^N$ for which the stochastic image $X(E)$ can have positive Lebesgue measure. After establishing a number of preparatory lemmas about the semigroup and the resolvent of an additive Lévy process in Section 3, we complete our proof of Theorem 2.1 in Section 4. Our proofs of Theorems 1.1, 1.3, and 1.5 can be found in Section 5. In Section 6 we briefly discuss some of the existing connections between the energy $\mathcal{E}_\Psi(\mu)$—introduced in (1.4)—and classical convolution-based energy forms. In Section 7 we utilize additive Lévy processes to describe a probabilistic interpretation of all sets of positive α-dimensional Bessel–Riesz capacity where $\alpha \ge 0$ is arbitrary. This probabilistic representation is used in Section 8 where Theorem 1.6 derived. In a final Section 9, we have stated some remaining open problems.

Since Condition (1.3) will play an important role in our arguments, we end this section with some examples of additive Lévy processes that satisfy (1.3).

Example 1.12 Consider the following condition:

$$\text{At least } N-1 \text{ of the Lévy processes } X_1, \ldots, X_N \text{ are symmetric.} \tag{1.6}$$

By using induction, we can see that the above implies that

$$\operatorname{Re}\left(\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\right) = \prod_{j=1}^N \operatorname{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right). \tag{1.7}$$

In particular, Condition (1.6) implies (1.3) with $\vartheta = 1$. [It may help to recall that an ordinary Lévy process Y is *symmetric* if $Y(1)$ and $-Y(1)$ have the same distribution.] □

Example 1.13 Consider a two-parameter additive Lévy process $\mathbb{R}_+^2 \ni t \mapsto X_1(t_1) + X_2(t_2)$, where X_1 and X_2 are i.i.d. Lévy processes on $\mathbb{R}^d$ with exponent Ψ_1 each. Then, it is possible to directly check that Condition (1.3) holds if and only if

$$\exists \delta \in (0,1): \ \forall \xi \in \mathbb{R}^d, \qquad |\operatorname{Im}\Psi_1(\xi)| \le \delta(1 + \operatorname{Re}\Psi_1(\xi)). \tag{1.8}$$

This is a kind of sector condition on Ψ_1. □

Example 1.14 Suppose X_1 and X_2 are independent Lévy processes on $\mathbb{R}^d$, and with Lévy exponents Ψ_1 and Ψ_2, respectively. Then, one can checks directly that the two-parameter additive Lévy process $\mathbb{R}_+^2 \ni t \mapsto X_1(t_1) - X_2(t_2)$ satisfies Condition (1.3) as long as

$$\operatorname{Im}\Psi_1(\xi) \cdot \operatorname{Im}\Psi_2(\xi) \ge 0, \qquad \forall \xi \in \mathbb{R}^d.$$

In particular, if X_1 and X_2 are i.i.d., Condition (1.3) *always* holds for the process $t \mapsto X_1(t_1) - X_2(t_2)$. This process arises in studying the self-intersections of Lévy processes. □

Example 1.15 Suppose $N > 2$, and consider the N-parameter additive Lévy process $X = X_1 \oplus \cdots \oplus X_N$, where $X_1, \ldots, X_N$ are i.i.d. Lévy processes on $\mathbb{R}^d$, all with the same Lévy exponent Ψ_1. Writing in polar coordinates, we have $\{1 + \Psi_1(\xi)\}^{-1} = re^{i\theta}$ where $r = |1 + \Psi_1(\xi)|^{-1}$, and $\cos(\theta) = |1 + \Psi_1(\xi)|^{-1}\{1 + \mathrm{Re}\,\Psi_1(\xi)\}$. According to Taylor's formula,

$$\cos(N\theta) \geq 1 - \frac{1}{2}N^2\theta^2 \geq \frac{1}{2} \geq \frac{1}{2}(\cos\theta)^N,$$

as long as $|\theta| \leq N^{-1}$. Consequently, $|\theta| \leq N^{-1}$ implies Condition (1.3). Equivalently,

$$|\mathrm{Im}\,\Psi_1(\xi)| \leq \tan(\tfrac{1}{N})\,|1 + \mathrm{Re}\,\Psi_1(\xi)|, \qquad \forall \xi \in \mathbb{R}^d$$

implies Condition (1.3). □

Example 1.16 (Symmetrization) Suppose $X_1, \ldots, X_k; \bar{X}_1, \ldots, \bar{X}_k$ are independent Lévy processes on $\mathbb{R}^d$ with Lévy exponents $\Psi_1, \ldots, \Psi_k; \overline{\Psi_1}, \ldots, \overline{\Psi_k}$, respectively, where $\overline{\Psi_\ell}$ denotes the complex conjugate of Ψ_ℓ ($\ell = 1, \ldots, k$). Then, the additive Lévy processes V satisfies Condition (1.3) with $\vartheta = 1$, where

$$V = X_1 \oplus \cdots \oplus X_k \oplus \bar{X}_1 \oplus \cdots \oplus \bar{X}_k.$$

If, in addition, Y is an arbitrary $\mathbb{R}^d$-valued Lévy process that is independent of V, the additive Lévy process $V \oplus Y$ also satisfies (1.3) with $\vartheta = 1$. □

2 Images and Local Times

Throughout, we let $X = X_1 \oplus \cdots \oplus X_N$ denote a d-dimensional additive Lévy process with Lévy exponent $\Psi = (\Psi_1, \ldots, \Psi_N)$. In this section we seek to find a general condition that guarantees that the image $X(E)$ of a compact set $E \subset \mathbb{R}^N_+$ can have positive Lebesgue measure. Under regularity conditions on X, this was achieved in Khoshnevisan and Xiao (2002a,b). Our goal, here, is to find conditions for the positivity of the image $\lambda_d(X(E))$ that hold quite generally.

Any finite measure μ on $\mathbb{R}^N_+$ defines an *occupation measure* $\mathbb{O}_\mu$ on $\mathbb{R}^d$ via the prescription

$$\mathbb{O}_\mu(A) = \int_{\mathbb{R}^N_+} \mathbb{1}_A(X(s))\,\mu(ds), \qquad A \in \mathcal{B}(\mathbb{R}^d), \tag{2.1}$$

where $\mathbb{1}_A(\cdot)$ is the indicator function of A and $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field of $\mathbb{R}^d$.

We take the distribution approach to measures. In particular, we tacitly identify the preceding random measure with the random linear operator $\mathbb{O}_\mu$ defined as

$$\mathbb{O}_\mu(f) = \int_{\mathbb{R}^N_+} f(X(s))\,\mu(ds).$$

The following result is the main inequality of this section, where $\mathcal{P}(E)$ is the collection of all probability measures on E.

Theorem 2.1 *For all compact sets $E \subset \mathbb{R}^N_+$,*

$$\left[(2\pi)^{-d} \inf_{\mu \in \mathcal{P}(E)} \mathbb{E}\left\{\|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)}\right\}\right]^{-1} \leq \mathbb{E}\left\{\lambda_d(X(E))\right\} \leq 16^N \left[(2\pi)^{-d} \inf_{\mu \in \mathcal{P}(E)} \mathbb{E}\left\{\|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)}\right\}\right]^{-1}.$$

Remark 2.2 Condition (1.3) is *not* needed here. □

Consequently, in order for $\lambda_d(X(E))$ to have positive expectation, it is necessary, as well as sufficient, that for some probability measure μ on E, the $L^2(\mathbb{R}^d)$-norm of the Fourier transform of $\mathbb{O}_\mu$ be square integrable with respect to $\mathbb{P}$.

Suppose, then, that $\mathbb{E}\{\lambda_d(X(E))\} > 0$. Thanks to the foregoing discussion, there exists $\mu \in \mathcal{P}(E)$ such that $\|\widehat{\mathbb{O}_\mu}\|_{L^2(\mathbb{R}^d)}$ is in $L^2(\mathbb{P})$; in particular, it is finite, a.s. By Plancherel's theorem, $\mathbb{O}_\mu$ is absolutely continuous with respect to λ_d, a.s. Let L_μ^x denote this density. In other words, $L_\mu = \{L_\mu^x;\, x \in \mathbb{R}^d\}$ is the process defined by the following: For all measurable functions $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\mathbb{O}_\mu(f) = \int_{\mathbb{R}^d} f(x) L_\mu^x \, dx; \tag{2.2}$$

cf. (2.1). We can always choose a measurable version of $\mathbb{R}^d \times \Omega \ni (x, \omega) \mapsto L_\mu^x(\omega)$, which we take for granted. Of course, Ω denotes the underlying probability space. Furthermore, we can apply Plancherel's formula, once again, to deduce that

$$\|L_\mu^\bullet\|^2_{L^2(\mathbb{R}^d)} = (2\pi)^{-d} \, \|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)}, \qquad \mathbb{P}\text{-a.s.} \tag{2.3}$$

The process L_μ is the *local times of* X, under the measure μ. The above, together with Theorem 2.1, shows the following.

Corollary 2.3 *In order for $\mathbb{E}\{\lambda_d(X(E))\}$ to be positive, it is necessary and sufficient that there exists a probability measure μ on E, under which there are local times L_μ such that $\mathbb{E}\{\|L_\mu^\bullet\|^2_{L^2(\mathbb{R}^d)}\}$ is finite.*

We have developed the requisite material for the lower bound (i.e., the "*easy half*") in Theorem 2.1.

Proof of Theorem 2.1: Lower Bound Without loss of generality, we may assume that there is a probability measure μ on the Borel set E such that $\mathbb{E}\{\|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)}\} < +\infty$. Let L_μ denote the corresponding local times. It follows from (2.2) with $f(x) = \mathbb{1}_{X(E)}(x)$ that $\mathbb{P}$-a.s.,

$$\begin{aligned} 1 &= \left|\mathbb{O}_\mu(\mathbb{R}^d)\right|^2 \\ &= \left|\int_{X(E)} L_\mu^x \, dx\right|^2 \\ &\leq (2\pi)^{-d} \, \|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)} \cdot \lambda_d(X(E)), \end{aligned}$$

thanks to (2.3) and to the Cauchy–Schwarz inequality. Next, recall that for all positive random variables Z, $\mathbb{E}\{Z^{-1}\} \cdot \mathbb{E}\{Z\} \geq 1$. This also follows from the Cauchy–Schwarz inequality, or from Jensen's inequality. Thus, we obtain

$$\begin{aligned} \mathbb{E}\left\{\lambda_d(X(E))\right\} &\geq (2\pi)^d \, \mathbb{E}\left\{\|\widehat{\mathbb{O}_\mu}\|^{-2}_{L^2(\mathbb{R}^d)}\right\} \\ &\geq (2\pi)^d \left[\mathbb{E}\left\{\|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)}\right\}\right]^{-1}. \end{aligned}$$

This proves the lower bound in Theorem 2.1. □

We conclude this section with the following analytical description of $\mathbb{E}\{\|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)}\}$. Its derivation is simple, but we include it as a natural way to introduce the *associated process* $\widetilde{X}$ in (2.4) below. The remainder of Theorem 2.1 is proved in Section 4 after our presentation of Section 3 that is concerned with some calculations.

Lemma 2.4 *For any finite measure μ on $\mathbb{R}^N_+$,*

$$\mathbb{E}\left\{\|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)}\right\} = \int_{\mathbb{R}^d} \iint_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} e^{-\sum_{j=1}^N |s_j - t_j| \Psi_j(\mathrm{sgn}(s_j - t_j)\xi)}\, \mu(ds)\, \mu(dt)\, d\xi.$$

Proof This is an exercise in Fubini's theorem. Indeed, by (2.1),

$$\mathbb{E}\left\{\|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)}\right\} = \int_{\mathbb{R}^d} \iint_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} \mathbb{E}\left\{e^{i\xi\cdot[X(s)-X(t)]}\right\}\, \mu(ds)\, \mu(dt)\, d\xi.$$

Define the N-parameter process $\widetilde{X} = \{\widetilde{X}(t);\, t \in \mathbb{R}^N\}$ by

$$\widetilde{X}(t) = \sum_{j=1}^N \mathrm{sgn}(t_j) X_j(|t_j|), \qquad \forall t \in \mathbb{R}^N. \tag{2.4}$$

We emphasize that $\widetilde{X}$ is a process indexed by all of $\mathbb{R}^N$, and that

$$\forall s, t \in \mathbb{R}^N_+ :\ X(t) - X(s) \text{ has the same distribution as } \widetilde{X}(t-s). \tag{2.5}$$

Moreover,

$$\mathbb{E}\left\{e^{i\xi\cdot\widetilde{X}(t)}\right\} = e^{-\sum_{j=1}^N |t_j| \Psi_j(\mathrm{sgn}(t_j)\xi)}, \qquad \forall \xi \in \mathbb{R}^d,\ t \in \mathbb{R}^N. \tag{2.6}$$

Our lemma follows. □

3 Some Calculations

Recall the associated process $\widetilde{X}$ from (2.4), and let $P = \{P_t;\, t \in \mathbb{R}^N\}$ be the family of operators on $L^\infty(\mathbb{R}^d)$ defined by

$$P_t f(x) = \mathbb{E}\left\{f(\widetilde{X}(t) + x)\right\}, \qquad \forall t \in \mathbb{R}^N,\ f \in L^\infty(\mathbb{R}^d) \text{ and } x \in \mathbb{R}^d. \tag{3.1}$$

This is not an N-parameter semigroup; i.e., it is not true that $P_{t+s} = P_t P_s$ for all $s, t \in \mathbb{R}^N$. However, each of its 2^N restrictions $\{P_t;\, t \in (\pm\mathbb{R})^N\}$ *is* an N-parameter semigroup. Let U denote the 1-*potential* of the family P. That is, for all $f \in L^\infty(\mathbb{R}^d)$

$$Uf(x) = \int_{\mathbb{R}^N} e^{-\sum_{j=1}^N |s_j|} P_s f(x)\, ds, \qquad \forall x \in \mathbb{R}^d. \tag{3.2}$$

We will also need the following potential operator

$$U_+ f\,(x) = \int_{\mathbb{R}^N_+} e^{-\sum_{j=1}^N s_j} P_s f(x)\, ds, \qquad \forall x \in \mathbb{R}^d. \tag{3.3}$$

Our next lemma computes $P_t f(x)$, $Uf(x)$ and $U_+ f(x)$ in terms of the Lévy exponent of X. In light of Theorem 1.1, it shows that $\widetilde{X}$ and its 1-potential U are the "right" objects to consider. Indeed, the integral of Theorem 1.1 is nothing but $U\delta_0(0)$, where δ_a is point mass at $a \in \mathbb{R}^d$, while Condition (1.3) allows us to compare $U\delta_0(0)$ and $U_+\delta_0(0)$.

Lemma 3.1 *The operators P_t, U and U_+, are convolution operators. Moreover, if f, $\widehat{f} \in L^1(\mathbb{R}^d)$, then for all $t \in \mathbb{R}^N$ and all $x \in \mathbb{R}^d$,*

$$P_t f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(-\xi)\, e^{-\sum_{j=1}^N |t_j| \Psi_j(\mathrm{sgn}(t_j)\xi)}\, d\xi,$$

$$Uf(x) = 2^N (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(-\xi) \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi,$$

$$U_+ f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(-\xi) \prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\, d\xi.$$

Proof Temporarily let μ_t denote the distribution of $-\widetilde{X}(t)$ to see that $P_t f(x) = \mu_t \star f(x)$ and $Uf(x) = \int_{\mathbb{R}^N} e^{-\sum_{j=1}^N |t_j|} \mu_t \star f(x)\, dt$, where $\star$ denotes convolution. Since $f \in L^1(\mathbb{R}^d)$, it follows from Fubini's theorem and (2.6) that

$$\begin{aligned} \widehat{P_t f}(\xi) &= \widehat{f}(\xi)\widehat{\mu_t}(\xi) \\ &= \widehat{f}(\xi) e^{-\sum_{j=1}^N |t_j| \Psi_j(-\mathrm{sgn}(t_j)\xi)}. \end{aligned} \tag{3.4}$$

Hence, the asserted formula for $P_t f(x)$ follows from the inversion theorem of Fourier transforms, after making a change of variables. To obtain the second equation, we integrate $P_t f$, viz.,

$$Uf(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(-\xi) \int_{\mathbb{R}^N} \exp\left\{-\sum_{j=1}^N |s_j|\, [1+\Psi_j(\mathrm{sgn}(s_j)\xi)]\right\} ds\, d\xi.$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}^N} e^{-\sum_{j=1}^N |s_j| [1+\Psi_j(\mathrm{sgn}(s_j)\xi)]}\, ds &= \prod_{j=1}^N \left\{ \int_0^\infty e^{-s[1+\Psi_j(\xi)]}\, ds + \int_0^\infty e^{-s[1+\overline{\Psi_j(\xi)}]}\, ds \right\} \\ &= 2^N \prod_{j=1}^N \mathrm{Re}\, \{1+\Psi_j(\xi)\}^{-1}. \end{aligned}$$

The mentioned computation of $Uf(x)$ follows readily from this. Our computation of $U_+ f$ is made in like fashion, and we omit the details. □

Throughout, we assume that the underlying sample space Ω is the collection of all paths $\omega : \mathbb{R}_+^N \to \mathbb{R}^d$ that have the form $\omega(t) = \sum_{j=1}^N \omega_j(t_j)$ for $t \in \mathbb{R}_+^N$, where ω_j is in $D_{\mathbb{R}^d}[0,\infty)$—the usual space of $\mathbb{R}^d$-valued cadlag functions—for every $j = 1, \ldots, N$. The space Ω inherits its Borel field from the Skorohod topology on $D_{\mathbb{R}^d}[0,\infty)$ in a standard way. The additive Lévy process X of the Introduction is in *canonical form* if

$$X(t)(\omega) = \omega(t), \qquad \forall \omega \in \Omega,\ t \in \mathbb{R}_+^N.$$

Since we are only interested in distributional results about X, we can assume, with no loss in generality, that it is in canonical form under a fixed probability measure, denoted by $\mathbb{P}$. This is a standard result and we will not dwell on it here. Henceforth, the canonical form of X is tacitly assumed. We will also assume, with no further mention, that $\Psi = (\Psi_1, \ldots, \Psi_N)$ is the Lévy exponent of the additive process X. We note for future reference that this is equivalent to

$$\mathbb{E}\{e^{i\xi\cdot X(t)}\} = e^{-t\cdot\Psi(\xi)}, \qquad \forall t \in \mathbb{R}_+^N,\ \xi \in \mathbb{R}^d. \tag{3.5}$$

In agreement with the notation of classical Lévy processes, we define $\mathbb{P}_x$ to be the law of $x + X$ for any $x \in \mathbb{R}^d$, and let $\mathbb{E}_x$ be the corresponding expectation operator. To be precise, we define for all Borel sets $A \subset \Omega$,

$$\mathbb{P}_x\{\omega \in \Omega : \omega \in A\} = \mathbb{P}\{\omega \in \Omega : x + \omega \in A\},$$

where $(x+\omega)(t) = x + \omega(t)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}^N_+$. This allows us to also define a sigma-finite measure $\mathbb{P}_{\lambda_d}$, and a corresponding integration (or expectation) operator, $\mathbb{E}_{\lambda_d}$, via

$$\mathbb{P}_{\lambda_d}\{A\} = \int_{\mathbb{R}^d} \mathbb{P}_x\{A\}\, dx, \qquad \forall A \subset \mathbb{R}^d \text{ Borel, and}$$
$$\mathbb{E}_{\lambda_d}\{Z\} = \int_{\mathbb{R}^d} \mathbb{E}_x\{Z\}\, dx, \qquad \forall Z : \Omega \to \mathbb{R}_+, \text{ measurable.}$$

The last line holds for a larger class of random variables Z by standard monotone class arguments.

Let $\Pi = \{1, \ldots, N\}$, and for all $A \subseteq \Pi$ define the partial order $\preccurlyeq_{(A)}$ on $\mathbb{R}^N$ by

$$s \preccurlyeq_{(A)} t \iff \begin{cases} s_i \le t_i, & \text{for all } i \in A, \\ s_i \ge t_i, & \text{for all } i \in A^{\complement}. \end{cases}$$

We may also write $t \succcurlyeq_{(A)} s$ for $s \preccurlyeq_{(A)} t$. We note that, used in conjunction, the partial orders $\{\preccurlyeq_{(A)};\, A \subseteq \Pi\}$ totally order $\mathbb{R}^N$ in the sense that for all $s, t \in \mathbb{R}^N$, we can find $A \subseteq \Pi$ such that $s \preccurlyeq_{(A)} t$. We will use this simple fact several times. The final piece of notation is that of filtrations in each partial order $\preccurlyeq_{(A)}$. Namely, we define $\mathcal{F}^A(t)$ to be the sigma-field generated by $\{X(r);\, r \preccurlyeq_{(A)} t\}$. We can, and will, assume that each $\mathcal{F}^A(t)$ is $\mathbb{P}_x$-complete for all $x \in \mathbb{R}^d$, and each $\mathcal{F}^A$ is $\preccurlyeq_{(A)}$-right continuous. The latter means that for all $t \in \mathbb{R}^N_+$, $\mathcal{F}^A(t) = \cap_{s \succcurlyeq_{(A)} t}\, \mathcal{F}^A(s)$.

The following key fact is borrowed from Khoshnevisan and Xiao (2002a), which we reproduce for the sake of completeness.

Proposition 3.2 (The Markov property) *Suppose that $A \subseteq \Pi$, and that $s \preccurlyeq_{(A)} t$ where s and t are both in $\mathbb{R}^N_+$. Then, for any measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$,*

$$\mathbb{E}_{\lambda_d}\big\{ f(X(t)) \bigm| \mathcal{F}^A(s) \big\} = P_{t-s} f(X(s)),$$

$\mathbb{P}_{\lambda_d}$-a.s., where P_t is defined in (3.1).

Remark 3.3 This is not generally true under $\mathbb{P}_x$. Also note that conditional expectations under the sigma-finite measure $\mathbb{P}_{\lambda_d}$ are defined in exactly the same manner as those with respect to probability measures. □

Proof Consider measurable functions $f, g, h_1, \ldots, h_m : \mathbb{R}^d \to \mathbb{R}_+$ and times $t, s, s^1, \ldots, s^m \in \mathbb{R}^N_+$ such that $t \succcurlyeq_{(A)} s \succcurlyeq_{(A)} s^j$ for all $j = 1, \ldots, m$. Then, since the X_j's are independent from one another, and by appealing to the independent-increments property of each X_j, we deduce

$$\begin{aligned}
&\mathbb{E}_{\lambda_d}\left\{ f(X(t)) \cdot g(X(s)) \cdot \prod_{j=1}^m h_j(X(s^j)) \right\} \\
&= \int_{\mathbb{R}^d} \mathbb{E}\left\{ f(X(t)+x) \cdot g(X(s)+x) \cdot \prod_{j=1}^m h_j(X(s^j)+x) \right\} dx \\
&= \int_{\mathbb{R}^d} \mathbb{E}\left\{ f\left(X(t) - X(s) + y\right) \right\} \cdot \mathbb{E}\left\{ \prod_{j=1}^m h_j\left(X(s^j) - X(s) + y\right) \right\} g(y)\, dy.
\end{aligned}$$

Thanks to (2.4), (2.5), and (3.1), the first term under the integral equals $P_{t-s}f(y)$. Noting that, under the measure $\mathbb{P}_{\lambda_d}$, the distribution of $X(s)$ is λ_d, we see that the desired result follows. $\square$

Lemma 3.4 *Suppose $f, g : \mathbb{R}^d \to \mathbb{R}$ are in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and their Fourier transforms are in $L^1(\mathbb{R}^d)$. Then, for all $s, t \in \mathbb{R}^N_+$,*

$$\mathbb{E}_{\lambda_d}\left\{f(X(s))\, g(X(t))\right\} = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi)\, \overline{\widehat{g}(\xi)}\, e^{-\sum_{j=1}^N |t_j - s_j| \Psi_j(\mathrm{sgn}(t_j - s_j)\xi)}\, d\xi.$$

Proof Find $A \subseteq \Pi$ such that $s \preccurlyeq_{(A)} t$. Then,

$$\begin{aligned}\mathbb{E}_{\lambda_d}\left\{f(X(s))\, g(X(t))\right\} &= \mathbb{E}_{\lambda_d}\left\{f(X(s))\, \mathbb{E}_{\lambda_d}\left\{g(X(t)) \,\middle|\, \mathcal{F}^A(s)\right\}\right\} \\ &= \mathbb{E}_{\lambda_d}\left\{f(X(s))\, P_{t-s}g(X(s))\right\},\end{aligned}$$

thanks to Proposition 3.2. Since the $\mathbb{P}_{\lambda_d}$-distribution of $X(s)$ is λ_d for any $s \in \mathbb{R}^N_+$ and both $f, g \in L^2(\mathbb{R}^d)$, we can deduce the following from Plancherel's formula:

$$\begin{aligned}\mathbb{E}_{\lambda_d}\left\{f(X(s))\, g(X(t))\right\} &= \int_{\mathbb{R}^d} f(x)\, P_{t-s}g(x)\, dx \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi)\, \overline{\widehat{P_{t-s}g}(\xi)}\, d\xi.\end{aligned}$$

Our lemma follows from (3.4). $\square$

Next, we recall the occupation measures $\mathbb{O}_\mu$ from (2.1). The following is a function analogue of Lemma 2.4.

Lemma 3.5 *For all $f : \mathbb{R}^d \to \mathbb{R}_+$ in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $\widehat{f} \in L^1(\mathbb{R}^d)$, and for all probability measures μ on $\mathbb{R}^N_+$,*

$$\mathbb{E}_{\lambda_d}\left\{\left|\mathbb{O}_\mu(f)\right|^2\right\} = (2\pi)^{-d} \int_{\mathbb{R}^d} \left|\widehat{f}(\xi)\right|^2 Q_\mu(\xi)\, d\xi,$$

where

$$Q_\mu(\xi) = \iint_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} e^{-\sum_{j=1}^N |t_j - s_j| \Psi_j(\mathrm{sgn}(t_j - s_j)\xi)}\, \mu(ds)\, \mu(dt). \tag{3.6}$$

Proof This follows from Lemma 3.4 and Fubini's theorem, once we verify that the function Q_μ is nonnegative. On the other hand,

$$Q_\mu(\xi) = \mathbb{E}\left\{\left|\int_{\mathbb{R}^N_+} e^{i\xi\cdot X(t)}\, \mu(dt)\right|^2\right\}, \qquad \forall \xi \in \mathbb{R}^d, \tag{3.7}$$

thanks to (2.5) and (2.6). This implies the pointwise positivity of Q_μ, thus concluding our proof. $\square$

4 Proof of Theorem 2.1: Upper bound

In Section 2 we proved the easy half (i.e., the lower bound) in Theorem 2.1. We now use the results of the previous section to derive the hard half of Theorem 2.1.

For all measurable $f : \mathbb{R}^d \to \mathbb{R}_+$, all probability measures μ on $\mathbb{R}^N_+$, and all $A \subseteq \Pi$, define the process $\mathcal{M}^A_\mu f$ by

$$\mathcal{M}^A_\mu f(t) = \mathbb{E}_{\lambda_d}\left\{\mathbb{O}_\mu(f) \,\middle|\, \mathcal{F}^A(t)\right\}, \qquad \forall t \in \mathbb{R}^N_+. \tag{4.1}$$

Lemma 4.1 *Suppose $f : \mathbb{R}^d \to \mathbb{R}_+$ is in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and $\widehat{f} \in L^1(\mathbb{R}^d)$, $A \subseteq \Pi$, and μ is a probability measure on $\mathbb{R}^N_+$. Then, recalling (3.6), we have*

$$\mathbb{E}_{\lambda_d}\{\mathcal{M}^A_\mu f(t)\} = \int_{\mathbb{R}^d} f(x)\,dx, \qquad \forall \xi \in \mathbb{R}^d, \text{ and}$$

$$\sup_{t\in\mathbb{R}^N_+} \mathbb{E}_{\lambda_d}\left\{\left|\mathcal{M}^A_\mu f(t)\right|^2\right\} \le (2\pi)^{-d}\int_{\mathbb{R}^d}\left|\widehat{f}(\xi)\right|^2 Q_\mu(\xi)\,d\xi.$$

Proof The $\mathbb{P}_{\lambda_d}$-expectation of $\mathcal{M}^A_\mu f(t)$ follows immediately from Fubini's theorem, and the elementary fact that $\mathbb{E}_{\lambda_d}\{f(X(t))\} = \int_{\mathbb{R}^d} f(x)\,dx$. For the second identity, we note that, by the Cauchy–Schwarz inequality for conditional expectation under $\mathbb{P}_{\lambda_d}$, for all $t \in \mathbb{R}^N_+$, $\mathbb{E}_{\lambda_d}\{|\mathcal{M}^A_\mu f(t)|^2\} \le \mathbb{E}_{\lambda_d}\{|\mathbb{O}_\mu(f)|^2\}$. The lemma follows from Lemma 3.5. □

Lemma 4.2 *For all $s \in \mathbb{R}^N_+$, all measurable functions $f : \mathbb{R}^d \to \mathbb{R}_+$ in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ with $\widehat{f} \in L^1(\mathbb{R}^d)$, and for any probability measure μ on $\mathbb{R}^N_+$, $\mathbb{P}_{\lambda_d}$-almost surely, the following holds:*

$$\sum_{A\subseteq\Pi} \mathcal{M}^A_\mu f(s) \ge \int_{\mathbb{R}^N_+} P_{t-s}f(X(s))\,\mu(dt).$$

Moreover,

$$\mathbb{E}_{\lambda_d}\left\{\sup_{s\in\mathbb{Q}^N_+}\left|\mathcal{M}^A_\mu f(s)\right|^2\right\} \le 4^N(2\pi)^{-d}\int_{\mathbb{R}^d}\left|\widehat{f}(\xi)\right|^2 Q_\mu(\xi)\,d\xi.$$

Remark 4.3 Since our filtrations satisfy the "usual conditions," one can show that when f is bounded, say, $\mathcal{M}^A_\mu f$ has a $\preccurlyeq_{(A)}$-right continuous modification. Consequently, for this modification, the former inequality holds almost surely, where the null set in question is independent of $t \in \mathbb{R}^N_+$. See Bakry (1979) for a version of such a regularity result. □

Proof For the first expression, we note, from (4.1), that since $f \ge 0$,

$$\begin{aligned}\mathcal{M}^A_\mu f(s) &\ge \mathbb{E}_{\lambda_d}\left\{\int_{t\succcurlyeq_{(A)}s} f(X(t))\,\mu(dt)\,\middle|\,\mathcal{F}^A(s)\right\}\\ &= \int_{t\succcurlyeq_{(A)}s} P_{t-s}f(X(s))\,\mu(dt),\end{aligned}$$

$\mathbb{P}_{\lambda_d}$-a.s. for any probability measure μ on $\mathbb{R}^d$, thanks to Proposition 3.2. Summing this over all $A \subseteq \Pi$ and recalling that, together, the $\preccurlyeq_{(A)}$'s order $\mathbb{R}^N_+$, we obtain the first inequality. The second inequality requires a little measure theory, and Cairoli's maximal inequality; cf. Walsh (1986) for the latter. We provide a brief, but self-contained proof below.

We define one-parameter "filtrations" $\mathcal{F}^A_1, \mathcal{F}^A_2, \ldots, \mathcal{F}^A_N$ for each $A \subseteq \Pi$, by insisting that $\mathcal{F}^A_j(t_j)$ is the sigma-field generated by $\{X_j(r);\, r \ge t_j\}$ if $j \in A$, whereas $\{X_j(r);\, 0 \le r \le t_j\}$ if $j \in A^\complement$. We add all $\mathbb{P}_x$-null sets for all $x \in \mathbb{R}^d$ to these without changing our notation. A little thought shows the following: *For any $t \in \mathbb{R}^N_+$, given $\mathcal{F}^A(t)$, the sigma-fields $\mathcal{F}^A_1(t_1), \ldots, \mathcal{F}^A_N(t_N)$ are conditionally independent under the sigma-finite measure $\mathbb{P}_{\lambda_d}$.* Consequently, by standard arguments from the theory of Markov random fields, applied to the sigma-finite measure $\mathbb{P}_{\lambda_d}$,

$$\mathbb{E}_{\lambda_d}\left\{Z \,\middle|\, \mathcal{F}^A(t)\right\} = \mathbb{E}_{\lambda_d}\left[\cdots\mathbb{E}_{\lambda_d}\{Z \,|\, \mathcal{F}^A_N(t_1)\}\cdots \,\middle|\, \mathcal{F}^A_N(t_N)\right],$$

$\mathbb{P}_{\lambda_d}$-a.s. for any nonnegative measurable function Z on Ω; cf. ROZANOV (1982) where this sort of result is systematically developed for probability measures. The same arguments work for $\mathbb{P}_{\lambda_d}$. Consequently, we apply Doob's maximal inequality, one parameter at a time, to obtain

$$\mathbb{E}_{\lambda_d}\left\{\sup_{t\in\mathbb{Q}_+^N}\left|\mathcal{M}_\mu^A f(t)\right|^2\right\} \le 4^N \sup_{t\in\mathbb{R}_+^N}\mathbb{E}_{\lambda_d}\left\{\left|\mathcal{M}_\mu^A f(t)\right|^2\right\},$$

all the time noting that by applying the method used to prove Kolmogorov's maximal inequality, one verifies that Doob's maximal inequality also works for $\mathbb{P}_{\lambda_d}$-martingales; we refer to DELLACHERIE-MEYER (1982, 40.2, p. 34) for the one-parameter discrete setting. One generalizes this development to the multiparameter setting by applying the arguments of R. Cairoli; cf. WALSH (1986). Thus, Lemma 4.1 concludes our proof. □

We are ready to begin our

Proof of Theorem 2.1: Upper bound Henceforth, we may assume that

$$\mathbb{E}\{\lambda_d(X(E))\} > 0, \tag{4.2}$$

for, otherwise, there is nothing left to prove.

For any $\delta > 0$, define E^δ to be the closed δ-enlargement of E, which is of course a compact set itself. Choose some point $\Delta \notin \mathbb{R}_+^N$, and let T^δ denote any measurable $(\mathbb{Q}_+^N \cap E^\delta) \cup \Delta$-valued function on Ω, such that $T^\delta \neq \Delta$ if and only if $|X(T^\delta)| \le \delta$. This can always be done, since the X_j's have cadlag paths, and since $\mathcal{B}(0,\delta) = \{x \in \mathbb{R}^d : |x| \le \delta\}$ has an open interior, where $|\bullet|$ denotes the ℓ^∞-norm in any Euclidean space. Informally, T^δ is any measurably selected point in E^δ such that $|X(T^\delta)| \le \delta$, as long as such a point exists. If not, T^δ is set to Δ. We now argue that for all $\delta > 0$, and all $k > 0$ large, $\mu^{\delta,k} \in \mathcal{P}(E^\delta)$, where

$$\mu^{\delta,k}(\bullet) = \frac{\mathbb{P}_{\lambda_d}\{T^\delta \in \bullet\,,\, T^\delta \neq \Delta\,,\, |X(0)| \le k\}}{\mathbb{P}_{\lambda_d}\{T^\delta \neq \Delta\,,\, |X(0)| \le k\}}.$$

Indeed, note that, by Fubini's theorem,

$$\begin{aligned}\mathbb{P}_{\lambda_d}\{T^\delta \neq \Delta\,,\, |X(0)| \le k\} &= \mathbb{P}_{\lambda_d}\{X(E^\delta) \cap \mathcal{B}(0,\delta) \neq \varnothing\,,\, |X(0)| \le k\} \\ &= \int_{[-k,\,k]^d} \mathbb{P}\{X(E^\delta) \cap \mathcal{B}(x,\delta) \neq \varnothing\}\, dx \\ &= \mathbb{E}\Big\{\lambda_d\big((X(E^\delta) \oplus \mathcal{B}(0,\delta)) \cap [-k,\,k]^d\big)\Big\}.\end{aligned} \tag{4.3}$$

In particular, $\mathbb{P}_{\lambda_d}\{T^\delta \neq \Delta\,,\, |X(0)| \le k\}$ is greater than $\mathbb{E}\{\lambda_d(X(E)\cap[-k,\,k]^d)\} > 0$ for all $k > 0$ large, thanks to (4.2). Thus, once we argue that $\mathbb{P}_{\lambda_d}\{T^\delta \neq \Delta\,,\, |X(0)| \le k\} < +\infty$, this development shows that $\mu^{\delta,k} \in \mathcal{P}(E^\delta)$, as asserted. But

$$\mathbb{P}_{\lambda_d}\{T^\delta \neq \Delta\,,\, |X(0)| \le k\} \le \mathbb{P}_{\lambda_d}\{|X(0)| \le k\} = (2k)^d,$$

which is finite. Thus, we indeed have $\mu^{\delta,k} \in \mathcal{P}(E^\delta)$.

We apply Lemma 4.2 to $\mu = \mu^{\delta,k}$ and $s = T^\delta$ on $\{T^\delta \neq \Delta\}$, and note that on the latter, $T^\delta \in \mathbb{Q}_+^N$, so there are no problems with null sets. In this way we obtain the following, where $f : \mathbb{R}^d \to \mathbb{R}_+$ is any measurable function:

$$\sum_{A\subseteq\Pi}\sup_{s\in\mathbb{Q}_+^N}\mathcal{M}_{\mu^{\delta,k}}^A f(s) \ge \int_{\mathbb{R}_+^N}\inf_{x\in\mathbb{R}^d:\,|x|\le\delta} P_{t-T^\delta} f(x)\,\mu^{\delta,k}(dt)\cdot\mathbb{1}_{\{T^\delta\neq\Delta\,,\,|X(0)|\le k\}},$$

$\mathbb{P}_{\lambda_d}$-a.s., where the null set is independent of the choice of $\delta > 0$. The special choice of $\mu^{\delta,k}$ yields the following upon squaring and taking $\mathbb{P}_{\lambda_d}$-expectations:

$$\begin{aligned}\mathbb{E}_{\lambda_d}\left\{\Big|\sum_{A\subseteq\Pi}\sup_{s\in\mathbb{Q}_+^N}\mathcal{M}^A_{\mu^{\delta,k}}f(s)\Big|^2\right\} &\geq \int_{\mathbb{R}_+^N}\left[\int_{\mathbb{R}_+^N}\inf_{x\in\mathbb{R}^d:\,|x|\leq\delta}P_{t-s}f(x)\,\mu^{\delta,k}(dt)\right]^2\mu^{\delta,k}(ds)\\ &\qquad\times\mathbb{P}_{\lambda_d}\{T^\delta\neq\Delta\,,\,|X(0)|\leq k\}\\ &\geq\left[\int_{\mathbb{R}_+^N}\int_{\mathbb{R}_+^N}\inf_{x\in\mathbb{R}^d:\,|x|\leq\delta}P_{t-s}f(x)\,\mu^{\delta,k}(dt)\,\mu^{\delta,k}(ds)\right]^2\\ &\qquad\times\mathbb{P}_{\lambda_d}\{T^\delta\neq\Delta\,,\,|X(0)|\leq k\},\end{aligned}$$

thanks to the Cauchy–Schwarz inequality. Consequently, for all $\delta_0 > 0$, $\delta \in (0, \delta_0)$ and all $k > 0$ large,

$$\begin{aligned}&\mathbb{E}_{\lambda_d}\left\{\Big|\sum_{A\subseteq\Pi}\sup_{s\in\mathbb{Q}_+^N}\mathcal{M}^A_{\mu^{\delta,k}}f(s)\Big|^2\right\}\\ &\quad\geq\left[\int_{\mathbb{R}_+^N}\int_{\mathbb{R}_+^N}\inf_{x\in\mathbb{R}^d:\,|x|\leq\delta_0}P_{t-s}f(x)\,\mu^{\delta,k}(dt)\,\mu^{\delta,k}(ds)\right]^2\cdot\mathbb{P}_{\lambda_d}\{T^\delta\neq\Delta\,,\,|X(0)|\leq k\}.\end{aligned}\tag{4.4}$$

Another appeal to the Cauchy–Schwarz inequality reveals the following estimate for the left-hand side of the above:

$$\begin{aligned}\mathbb{E}_{\lambda_d}\left\{\Big|\sum_{A\subseteq\Pi}\sup_{s\in\mathbb{Q}_+^N}\mathcal{M}^A_{\mu^{\delta,k}}f(s)\Big|^2\right\}&\leq 2^N\sum_{A\subseteq\Pi}\mathbb{E}_{\lambda_d}\left\{\Big|\sup_{s\in\mathbb{Q}_+^N}\mathcal{M}^A_{\mu^{\delta,k}}f(s)\Big|^2\right\}\\ &\leq 16^N(2\pi)^{-d}\int_{\mathbb{R}^d}\big|\widehat{f}(\xi)\big|^2\,Q_{\mu^{\delta,k}}(\xi)\,d\xi,\end{aligned}\tag{4.5}$$

by Lemma 4.2. We now choose a "good" f in both (4.4) and (4.5). Namely, consider $f = f_\varepsilon$ for any $\varepsilon > 0$ such that f is of the form

$$f_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}}\exp\left(-\frac{\|x\|^2}{2\varepsilon}\right), \qquad \forall x\in\mathbb{R}^d.\tag{4.6}$$

Trivially, $f_\varepsilon \geq 0$, $\int f_\varepsilon\, d\lambda_d = 1$, $\widehat{f_\varepsilon}(\xi) = \exp(-\frac{1}{2}\varepsilon\|\xi\|^2)$, and both f_ε, $\widehat{f_\varepsilon} \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$.

We apply (4.4) and (4.5) for this choice of f_ε, and wish to take $\delta \to 0$. Since δ_0 is fixed, the explicit form of $P_t f$ shows that $(x, s, t) \mapsto P_{t-s}f(x)$ is continuous on $\mathcal{B}(0, \delta_0) \times E^\delta \times E^\delta$. Since $E^\delta \downarrow E$ are all compact, and since $\mu^{\delta,k} \in \mathcal{P}(E^\delta)$, Prohorov's theorem, together with the mentioned continuity fact about $\inf_x P_{t-s}f(x)$, implies the existence of a $\mu \in \mathcal{P}(E)$, such that along some subsequence $\delta' \to 0$ and $k' \to \infty$,

$$\iint_{\mathbb{R}_+^N\times\mathbb{R}_+^N}\inf_{|x|\leq\delta_0}P_{t-s}f_\varepsilon(x)\,\mu^{\delta',k'}(dt)\,\mu^{\delta',k'}(ds)\to\iint_{\mathbb{R}_+^N\times\mathbb{R}_+^N}\inf_{|x|\leq\delta_0}P_{t-s}f_\varepsilon(x)\,\mu(dt)\,\mu(ds).$$

Furthermore, $\mathbb{P}_{\lambda_d}\{T^\delta \neq \Delta\,,\,|X(0)| \leq k\} \to \mathbb{E}\{\lambda_d(\overline{X(E)})\}$, as $k \uparrow \infty$ and then $\delta \downarrow 0$; cf. (4.3). [Here, $\overline{X(E)})$ denotes the closure of $X(E)$.]

The preceding argument, used in conjunction with (4.4) and (4.5) (let $\delta_0 \downarrow 0$), yields

$$\begin{aligned}&16^N(2\pi)^{-d}\limsup_{\substack{\delta'\to0\\k'\to\infty}}\int_{\mathbb{R}^d}\big|\widehat{f_\varepsilon}(\xi)\big|^2Q_{\mu^{\delta',k'}}(\xi)\,d\xi\\ &\qquad\geq\left[\iint_{\mathbb{R}_+^N\times\mathbb{R}_+^N}P_{t-s}f_\varepsilon(0)\,\mu(dt)\,\mu(ds)\right]^2\cdot\mathbb{E}\big\{\lambda_d(X(E))\big\}.\end{aligned}$$

Now, $0 \le Q_{\mu^{\delta',k'}}(\xi) \le 1$ for all $\xi \in \mathbb{R}^d$ [cf. (3.7)], and $\widehat{f}_\varepsilon \in L^2(\mathbb{R}^d)$. Finally,

$$\lim_{\delta' \to 0, k' \to \infty} Q_{\mu^{\delta',k'}} = Q_\mu,$$

pointwise [cf. (3.6)]. Thus, by Lebesgue dominated convergence theorem,

$$\begin{aligned} 16^N (2\pi)^{-d} \int_{\mathbb{R}^d} \left|\widehat{f}_\varepsilon(\xi)\right|^2 Q_\mu(\xi)\, d\xi &\ge \left[\iint_{\mathbb{R}_+^N \times \mathbb{R}_+^N} P_{t-s} f_\varepsilon(0)\, \mu(dt)\, \mu(ds)\right]^2 \cdot \mathbb{E}\big\{\lambda_d(X(E))\big\} \\ &= (2\pi)^{-2d} \left[\int_{\mathbb{R}^d} \widehat{f}_\varepsilon(\xi) Q_\mu(\xi)\, d\xi\right]^2 \cdot \mathbb{E}\big\{\lambda_d(X(E))\big\}, \end{aligned}$$

thanks to Lemma 3.1. Our choice of f_ε guarantees that for all $\xi \in \mathbb{R}^d$, $\widehat{f}_\varepsilon(\xi) \ge |\widehat{f}_\varepsilon(\xi)|^2$. Thus, we use the square integrability of f_ε and the positivity and boundedness of Q_μ, once more, to obtain

$$\mathbb{E}\big\{\lambda_d(X(E))\big\} \le 16^N \left[(2\pi)^{-d} \int_{\mathbb{R}^d} \left|\widehat{f}_\varepsilon(\xi)\right|^2 Q_\mu(\xi)\, d\xi\right]^{-1},$$

since the right-hand side is obviously not zero; cf. (3.6) and (3.7). Let $\varepsilon \downarrow 0$ and use Lebesgue monotone convergence theorem to see that

$$\mathbb{E}\big\{\lambda_d(X(E))\big\} \le 16^N \left[(2\pi)^{-d} \int_{\mathbb{R}^d} Q_\mu(\xi)\, d\xi\right]^{-1} = 16^N \left[(2\pi)^{-d}\, \mathbb{E}\left\{\|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)}\right\}\right]^{-1},$$

by Lemma 2.4. This concludes our proof. □

5 Proofs of Theorems 1.1, 1.3 and 1.5

Theorem 1.3 follows from Theorem 1.1 by invoking the very argument that lead to Corollary 2.3. Hence, we only concentrate on proving Theorems 1.1 and 1.5.

We divide the proofs of Theorems 1.1 and 1.5 in three parts:

(i) The easy half of Theorem 1.1, that is

$$\int_{\mathbb{R}^d} \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi < +\infty \implies \mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} > 0.$$

Of course, this statement is a special case of Part (iii) below. We give a simple and direct proof using Theorem 2.1.

(ii) The hard half of Theorem 1.5; i.e., for any fixed compact set $F \subset \mathbb{R}^d$,

$$\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N) \oplus F)\} > 0 \implies \exists\, \mu \in \mathcal{P}(F) \text{ such that } \mathcal{E}_\Psi(\mu) < \infty.$$

The hard half of Theorem 1.1 follows from this and (1.4) upon selecting $F = \{0\}$.

(iii) The easy half of Theorem 1.5, i.e.,

$$F \text{ carries a finite measure of finite energy} \implies \mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N) \oplus F)\} > 0.$$

For simplicity, we use the following suggestive notation: For all finite measures μ on $\mathbb{R}^N_+$, define

$$\|\mu\|_e^2 = (2\pi)^{-d}\,\mathbb{E}\left\{\|\widehat{\mathbb{O}_\mu}\|^2_{L^2(\mathbb{R}^d)}\right\}.$$

We may refer to $\|\mu\|_e$ as the *energy norm* of μ, although strictly speaking it is only a seminorm as the following shows.

Lemma 5.1 *For any two finite measures μ and ν on $\mathbb{R}^d$,*

$$\|\mu+\nu\|_e \leq \|\mu\|_e + \|\nu\|_e.$$

Proof Since $\mu \mapsto \mathbb{O}_\mu$ is linear, so is $\mu \mapsto \widehat{\mathbb{O}_\mu}$. The lemma follows from Minkowski's inequality. □

Throughout, we define the *1-killing measure*, $\kappa \in \mathcal{P}(\mathbb{R}^N_+)$, as

$$\kappa(ds) = e^{-\sum_{j=1}^N s_j}\,ds, \qquad \forall s \in \mathbb{R}^N_+. \tag{5.1}$$

Recalling (2.1), we note the *killed occupation measure* is defined by

$$\mathbb{O}_\kappa(f) = \int_{\mathbb{R}^N_+} f(X(s))e^{-\sum_{j=1}^N s_j}\,ds.$$

Note that $\mathbb{O}_\kappa$ is a random probability measure carried by $X(\mathbb{R}^N_+)$.

The relevance of the killing measure κ to the proofs of Theorems 1.1 and 1.5 is given by the following lemma.

Lemma 5.2 *Let κ be the killing measure defined above, then the energy norm of $\kappa \in \mathcal{P}(\mathbb{R}^N_+)$ is described by*

$$\|\kappa\|_e^2 = (2\pi)^{-d}\int_{\mathbb{R}^d}\prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right)d\xi.$$

Proof By Lemma 2.4,

$$\begin{aligned}
\|\kappa\|_e^2 &= (2\pi)^{-d}\int_{\mathbb{R}^d}\prod_{j=1}^N\left[\int_0^\infty\int_0^\infty e^{-|s-t|\Psi_j(\mathrm{sgn}(s-t)\xi)}e^{-s-t}\,ds\,dt\right]d\xi\\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\prod_{j=1}^N\left[\int_0^\infty\int_s^\infty(\cdots)\,dt\,ds + \int_0^\infty\int_t^\infty(\cdots)\,ds\,dt\right]d\xi\\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\prod_{j=1}^N\left[\frac{\frac12}{1+\Psi_j(\xi)} + \frac{\frac12}{1+\overline{\Psi_j(\xi)}}\right]d\xi\\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\prod_{j=1}^N\mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right)d\xi,
\end{aligned}$$

since $\{1+z\}^{-1} + \{1+\overline{z}\}^{-1} = 2\mathrm{Re}\,\{1+z\}^{-1}$ $(z \in \mathbb{C})$. □

Lemma 5.2 suffices for our

Proof of Theorem 1.1: Easy half The lower bound in Theorem 2.1 shows that

$$\mathbb{E}\big\{\lambda_d(X(E))\big\} \geq \|\kappa\|_e^{-2}. \tag{5.2}$$

Thus, Lemma 5.2 shows that the integrability of $\prod_{j=1}^N \mathrm{Re}\{1+\Psi_j\}^{-1}$ guarantees the positivity of $\mathbb{E}\{\lambda_d(X(E))\}$. This finishes Part (i) of the proof. □

We start working toward proving the hard half of Theorem 1.5. We begin with some prefatory results.

Recalling our definition of energy from (1.4), and the function Q_κ from (3.6), we have

Lemma 5.3 *For all $\xi \in \mathbb{R}^d$,*

$$Q_\kappa(\xi) = \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right).$$

Consequently, whenever $f:\mathbb{R}^d \to \mathbb{R}$ and its Fourier transform are both in $L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$,

$$\mathbb{E}_{\lambda_d}\left\{\left|\mathbb{O}_\kappa(f)\right|^2\right\} = \mathcal{E}_\Psi(f).$$

Proof In light of Lemma 3.5 and our definition of energy [(1.4)], it suffices to compute Q_κ as given. On the other hand,

$$\begin{aligned} Q_\kappa(\xi) &= \prod_{j=1}^N \int_0^\infty \int_0^\infty e^{-s-t-|s-t|\Psi_j(\mathrm{sgn}(s-t)\xi)}\, ds\, dt \\ &= \prod_{j=1}^N \left[\int_0^\infty \int_t^\infty (\cdots)\, ds\, dt + \int_0^\infty \int_s^\infty (\cdots)\, dt\, ds\right] \\ &= \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right). \end{aligned}$$

The few remaining details in the above are the same as those in the proof of Lemma 5.2. □

Recalling (4.1), we are interested in $\mathcal{M}_\kappa^\Pi f$, where we also recall that $\Pi = \{1,\ldots,N\}$. The operator U_+ below was defined in (3.3).

Lemma 5.4 *If $f:\mathbb{R}^d \to \mathbb{R}_+$ is in $L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, then*

$$\mathbb{E}_{\lambda_d}\left\{\sup_{s\in\mathbb{Q}_+^N}\left|\mathcal{M}_\kappa^\Pi f(s)\right|^2\right\} \leq 4^N \mathcal{E}_\Psi(f). \tag{5.3}$$

Furthermore, for any $r>0$, $f\in L^1(\mathbb{R}^d)$ whose Fourier transform is also in $L^1(\mathbb{R}^d)$, and for all $s\in(0,r)^N$, the following holds $\mathbb{P}_{\lambda_d}$-a.s.:

$$\mathcal{M}_\kappa^\Pi f(s) \geq e^{-Nr} U_+ f(X(s)). \tag{5.4}$$

Proof Equation (5.3) is a consequence of Lemmas 4.2 and 5.3 albeit in slightly different notation.

To prove (5.4), we proceed as in our proof of Lemma 4.2, but adapt the argument to the present setting. Since $f \geq 0$, the same reasoning as in the latter Lemma gives the following for all $s \in (0, r)^N$:

$$\begin{aligned}\mathcal{M}_\kappa^\Pi f(s) &\geq \mathbb{E}_{\lambda_d}\left\{ \int_{t \succcurlyeq_{(\Pi)} s} f(X(t))\, e^{-\sum_{j=1}^N t_j}\, dt \,\middle|\, \mathcal{F}^\Pi(s) \right\} \\ &= \int_{t \succcurlyeq_{(\Pi)} s} P_{t-s} f(X(s))\, e^{-\sum_{j=1}^N t_j}\, dt,\end{aligned}$$

$\mathbb{P}_{\lambda_d}$-a.s. Now, we move in a somewhat new direction by noticing that since $s \in (0, r)^N$,

$$\begin{aligned}\mathcal{M}_\kappa^\Pi f(s) &\geq e^{-Nr} \int_{t \succcurlyeq_{(\Pi)} s} P_{t-s} f(X(s))\, e^{-\sum_{j=1}^N (t_j - s_j)}\, dt \\ &\geq e^{-Nr} \int_{\mathbb{R}_+^N} P_t f(X(s)) e^{-\sum_{j=1}^N t_j}\, dt \\ &= e^{-Nr} U_+ f(X(s)),\end{aligned}$$

by (3.3). Thus, our lemma follows. □

Henceforth, we define the *capacity* of a compact set $F \subset \mathbb{R}^d$ by

$$\mathcal{C}_\Psi(F) = \left[\inf_{\mu \in \mathcal{P}(F)} \mathcal{E}_\Psi(\mu) \right]^{-1}, \tag{5.5}$$

where, we recall, $\mathcal{P}(F)$ denotes the collection of all probability measures on F.

Our proof of the hard half of Theorem 1.5 is based on the following.

Lemma 5.5 *Suppose X is an additive Lévy process in $\mathbb{R}^d$ that satisfies Condition (1.3), and that $\int_{\mathbb{R}^d} \prod_{j=1}^N |1 + \Psi_j(\xi)|^{-1}\, d\xi < +\infty$, where $\Psi = (\Psi_1, \ldots, \Psi_N)$ denotes the Lévy exponent of X. Then, for all compact sets $F \subset \mathbb{R}^d$, and for all $r > 0$,*

$$\mathbb{E}\big\{ \lambda_d(X([0, r]^N) \oplus F) \big\} \leq \vartheta^{-2} (4e^{2r})^N \cdot \mathcal{C}_\Psi(F),$$

where $\vartheta > 0$ is the constant in Condition (1.3).

Before proving it, we appeal to Lemma 5.5 to conclude Part (ii) of our proof, i.e., the hard half of Theorem 1.5. Clearly, the following will suffice.

Proposition 5.6 *Suppose X is an additive Lévy process in $\mathbb{R}^d$ that satisfies Condition (1.3). Let $\Psi = (\Psi_1, \ldots, \Psi_N)$ denote the Lévy exponent of X. Then, for all compact sets $F \subset \mathbb{R}^d$, and for all $r > 0$,*

$$\mathbb{E}\big\{ \lambda_d(X([0, r]^N) \oplus F) \big\} \leq \vartheta^{-2} (4e^{2r})^{N + \lfloor \frac{d}{2} \rfloor + 1} \cdot \mathcal{C}_\Psi(F).$$

Proof of Proposition 5.6 Let us bring in $M = \lfloor \frac{d}{2} \rfloor + 1$ continuous Brownian motions in $\mathbb{R}^d$, $B_1, \ldots, B_M$, all totally independent from one another, as well as X [under $\mathbb{P}$]. Having done so, for any $\delta > 0$, we can define an $(N + M)$-parameter process $\mathcal{X}_\delta$ in $\mathbb{R}^d$ by

$$\mathcal{X}_\delta = X_1 \oplus \cdots \oplus X_N \oplus \sqrt{2\delta} B_1 \oplus \cdots \oplus \sqrt{2\delta} B_M.$$

To be concrete, $\mathfrak{X}_\delta(t) = \sum_{j=1}^N X_j(t_j) + \sqrt{2\delta}\sum_{j=1}^M B_j(t_{j+N})$, for all $t \in \mathbb{R}_+^{N+M}$.

Define

$$\Phi_j(\xi) = \begin{cases} \Psi_j(\xi), & \text{if } j = 1, \ldots, N, \\ \delta\|\xi\|^2, & \text{if } j = N+1, \ldots, N+M, \end{cases}$$

where $\|x\|$ is the ℓ^2-norm of $x \in \mathbb{R}^d$. Then, $\mathfrak{X}_\delta$ is an $(N+M)$-parameter additive Lévy process in $\mathbb{R}^d$ whose Lévy exponent is $\Phi = (\Phi_1, \ldots, \Phi_{N+M})$. Furthermore,

$$\int_{\mathbb{R}^d} \prod_{j=1}^{N+M} |1 + \Phi_j(\xi)|^{-1}\, d\xi \le \int_{\mathbb{R}^d} \{1 + \delta\|\xi\|^2\}^{-M}\, d\xi < +\infty, \tag{5.6}$$

since $M > \frac{d}{2}$. Thus, we can apply Lemma 5.5 to the process $\mathfrak{X}_\delta$ and obtain

$$\mathbb{E}\left\{\lambda_d(\mathfrak{X}_\delta([0,r]^{N+M}) \oplus F)\right\} \le \vartheta^{-2}(4e^{2r})^{N+M} \cdot \mathcal{C}_\Phi(F). \tag{5.7}$$

Of course, by (5.6), the above capacity is strictly positive.

Now, consider a sequence of probability measures, $\mu_1, \mu_2, \ldots$, all on F, such that

$$\lim_{n\to\infty} \mathcal{E}_\Phi(\mu_n) = [\mathcal{C}_\Phi(F)]^{-1}.$$

Without loss of generality, we can assume that all these energies are finite, and by tightness, extract a subsequence n' and a probability measure μ_∞ on F such that $\mu_{n'}$ converges weakly to μ_∞. Thanks to (5.6) and to the continuity of the Φ_j's, we see that $\lim_{n'} \mathcal{E}_\Phi(\mu_{n'}) = \mathcal{E}_\Phi(\mu_\infty)$. Equivalently, we have found a probability measure μ_∞ on F whose Φ-energy is the reciprocal of the Φ-capacity of F. Changing the notation to allow for the dependence of μ_∞ on the parameter δ, we see that there exists a probability measure ν_δ on F such that it has finite Φ-energy, and

$$\mathbb{E}\{\lambda_d(\mathfrak{X}_\delta([0,r]^{N+M}) \oplus F)\} \le \frac{\vartheta^{-2}(4e^{2r})^{N+M}}{\mathcal{E}_\Phi(\nu_\delta)}.$$

This holds for all $\delta > 0$. Now, fix an arbitrarily small $\delta_0 > 0$, and for all $\delta \in (0, \delta_0)$ deduce the cruder bound:

$$\mathbb{E}\{\lambda_d(\mathfrak{X}_\delta([0,r]^{N+M}) \oplus F)\} \le \frac{\vartheta^{-2}(4e^{2r})^{N+M}}{\int_{\mathbb{R}^d} |\widehat{\nu_\delta}(\xi)|^2 \prod_{j=1}^N \operatorname{Re}\{1 + \Psi_j(\xi)\}^{-1} \cdot \{1 + \delta_0\|\xi\|^2\}^{-M}\, d\xi}.$$

By considering further subnets of δ, and by appealing to (5.6) once more, we can infer the existence of a probability measure ν_0 on F such that

$$\liminf_{\delta\to 0}\ \mathbb{E}\{\lambda_d(\mathfrak{X}_\delta([0,r]^{N+M}) \oplus F)\} \le \frac{\vartheta^{-2}(4e^{2r})^{N+M}}{\int_{\mathbb{R}^d} |\widehat{\nu_0}(\xi)|^2 \prod_{j=1}^N \operatorname{Re}\{1 + \Psi_j(\xi)\}^{-1} \cdot \{1 + \delta_0\|\xi\|^2\}^{-M}\, d\xi}.$$

On the other hand,

$$\mathfrak{X}_\delta([0,r]^{N+M}) = X([0,r]^N) \oplus \sqrt{2\delta} B([0,r]^M),$$

where $B = B_1 \oplus \cdots \oplus B_M$. By compactness, as $\delta \downarrow 0$, this random set converges downwards to $\overline{X([0,r]^N)}$, the closure of $X([0,r]^N)$. Consequently, by the monotone convergence theorem of Lebesgue,

$$\mathbb{E}\{\lambda_d(X([0,r]^N) \oplus F)\} \le \frac{\vartheta^{-2}(4e^{2r})^{N+M}}{\int_{\mathbb{R}^d} |\widehat{\nu_0}(\xi)|^2 \prod_{j=1}^N \operatorname{Re}\{1 + \Psi_j(\xi)\}^{-1} \cdot \{1 + \delta_0\|\xi\|^2\}^{-M}\, d\xi},$$

for all $\delta_0 > 0$. Let $\delta_0 \downarrow 0$, and apply Lebesgue monotone convergence one more time to finish. □

It remains to present our

Proof of Lemma 5.5 It follows from (1.4) that $\mathcal{C}_\Psi(-F) = \mathcal{C}_\Psi(F)$ for any compact set F, where $-F = \{-a;\, a \in F\}$. Hence, we can reduce our problem to showing that

$$\mathbb{E}\{\lambda_d(F \ominus X([0,r]^N))\} \leq \vartheta^{-2}(4e^{2r})^N \mathcal{C}_\Psi(F),$$

where $A \ominus B = \{a - b;\, a \in A,\, b \in B\}$.

Let F^ε denote the closed ε-enlargement of F. The integrability condition of the statement of the lemma, the continuity of the Ψ_j's, and Lebesgue dominated convergence theorem, together show that if $\varepsilon > 0$, then $\lim_{\varepsilon\to 0} \mathcal{C}_\Psi(F^\varepsilon) = \mathcal{C}_\Psi(F)$. [This involves a tightness argument that we have already utilized while proving Proposition 5.6.] Hence, it suffices to show that

$$\mathbb{E}\{\lambda_d(F \ominus X([0,r]^N))\} \leq \vartheta^{-2}(4e^{2r})^N \mathcal{C}_\Psi(F^\varepsilon). \tag{5.8}$$

The above holds trivially unless the left-hand side is strictly positive, which we will assume henceforth. We observe that, by Fubini's theorem,

$$\begin{aligned}
\mathbb{P}_{\lambda_d}\{X([0,r]^N) \cap F \neq \varnothing\} &= \int_{\mathbb{R}^d} \mathbb{P}\Big\{\big(x \oplus X([0,r]^N)\big) \cap F \neq \varnothing\Big\}\, dx \\
&= \int_{\mathbb{R}^d} \mathbb{P}\{x \in F \ominus X([0,r]^N)\}\, dx \\
&= \mathbb{E}\{\lambda_d(F \ominus X([0,r]^N))\}.
\end{aligned} \tag{5.9}$$

Thus, the above assumption is equivalent to assuming that

$$\mathbb{P}_{\lambda_d}\{X([0,r]^N) \cap F \neq \varnothing\} > 0. \tag{5.10}$$

Next, we add a cemetery point $\Delta \notin \mathbb{R}^N_+$ to $\mathbb{R}^N_+$, and consider any $\mathbb{Q}^N_+ \cup \{\Delta\}$-valued random variable T_ε such that

- $T_\varepsilon = \Delta$ if and only if $X([0,r]^N) \cap F^\varepsilon = \varnothing$; and
- on the event $\{X([0,r]^N) \cap F^\varepsilon \neq \varnothing\}$, $X(T_\varepsilon) \in F^\varepsilon$.

We remark that since F^ε has an open interior, and since X_i's are cadlag, we can always choose $T_\varepsilon \in \mathbb{Q}^N_+ \cup \{\Delta\}$ [as opposed to $\mathbb{R}^N_+ \cup \{\Delta\}$]. Consider

$$\begin{aligned}
\mu_{\varepsilon,k}(\bullet) &= \frac{\mathbb{P}_{\lambda_d}\{X(T_\varepsilon) \in \bullet\,,\, T_\varepsilon \neq \Delta\,,\, |X(0)| \leq k\}}{\mathbb{P}_{\lambda_d}\{T_\varepsilon \neq \Delta\,,\, |X(0)| \leq k\}}, \\
\varphi_\eta(x) &= (2\pi\eta^2)^{-\frac{d}{2}} \exp\Big(-\frac{\|x\|^2}{2\eta^2}\Big),
\end{aligned}$$

where $\eta, k > 0$ and $x \in \mathbb{R}^d$. Owing to (5.10), $\mu_{\varepsilon,k}$ is a probability measure on F^ε for all $\varepsilon > 0$ and $k > 0$ large. We can smooth $\mu_{\varepsilon,k}$ by convoluting it with φ_η:

$$f_{\varepsilon,k;\eta} = \mu_{\varepsilon,k} \star \varphi_\eta.$$

The function $f_{\varepsilon,k;\eta}$ has the following nice properties that are simple to check: (i) $f_{\varepsilon,k;\eta} \geq 0$ is bounded; and (ii) both $f_{\varepsilon,k;\eta}, \widehat{f_{\varepsilon,k;\eta}} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Thus, we can apply Lemma 5.4 [(5.4)] to obtain

$$\sup_{t\in[0,r]^N} \mathcal{M}^\Pi_\kappa f_{\varepsilon,k;\eta}(t) \geq e^{-Nr} U_+ f_{\varepsilon,k;\eta}\big(X(T_\varepsilon)\big) \cdot \mathbb{1}_{\{T_\varepsilon\neq\Delta\,,\,|X(0)|\leq k\}},$$

$\mathbb{P}_{\lambda_d}$-a.s. [There are no problems with null sets, since on $\{T_\varepsilon \neq \Delta\}$, $T_\varepsilon \in \mathbb{Q}_+^N$.] We square this and take $\mathbb{E}_{\lambda_d}$-expectations to obtain the following as a consequence of Lemma 5.4 [(5.3)]:

$$\begin{aligned}
4^N \mathcal{E}_\Psi(f_{\varepsilon,k;\eta}) &\geq e^{-2Nr}\mathbb{E}_{\lambda_d}\left\{\left|U_+ f_{\varepsilon,k;\eta}(X(T_\varepsilon))\right|^2 \mathbb{1}_{\{T_\varepsilon\neq\Delta\,,\,|X(0)|\leq k\}}\right\} \\
&= e^{-2Nr}\int_{\mathbb{R}^d}\left|U_+ f_{\varepsilon,k;\eta}(x)\right|^2 \mu_{\varepsilon,k}(dx)\cdot \mathbb{P}_{\lambda_d}\{T_\varepsilon\neq\Delta\,,\,|X(0)|\leq k\} \\
&\geq e^{-2Nr}\left|\int_{\mathbb{R}^d} U_+ f_{\varepsilon,k;\eta}(x)\,\mu_{\varepsilon,k}(dx)\right|^2\cdot \mathbb{P}_{\lambda_d}\{T_\varepsilon\neq\Delta\,,\,|X(0)|\leq k\}, \qquad (5.11)
\end{aligned}$$

thanks to the Cauchy–Schwarz inequality. We can apply Lemma 3.1 to see that

$$\begin{aligned}
\int_{\mathbb{R}^d} U_+ f_{\varepsilon,k;\eta}(x)\,\mu_{\varepsilon,k}(dx) &= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\,\widehat{f_{\varepsilon,k;\eta}}(-\xi)\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\,d\xi\,\mu_{\varepsilon,k}(dx) \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d} |\widehat{\mu_{\varepsilon,k}}(\xi)|^2\,\widehat{\varphi_\eta}(-\xi)\,\mathrm{Re}\left(\prod_{j=1}^N\frac{1}{1+\Psi_j(\xi)}\right)d\xi \\
&\geq \vartheta(2\pi)^{-d}\int_{\mathbb{R}^d} |\widehat{\mu_{\varepsilon,k}}(\xi)|^2\, e^{-\frac{1}{2}\eta^2\|\xi\|^2}\prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right)d\xi \\
&\overset{(\eta\to 0)}{\longrightarrow} \vartheta\,\mathcal{E}_\Psi(\mu_{\varepsilon,k}).
\end{aligned}$$

In the above, the second equality follows from the fact that $|\widehat{\mu_{\varepsilon,k}}(\xi)|^2\,\widehat{\varphi_\eta}(-\xi)\geq 0$, and the inequality follows from Condition (1.3). On the other hand, by (5.9),

$$\begin{aligned}
\lim_{k\to\infty}\mathbb{P}_{\lambda_d}\{T_\varepsilon\neq\Delta\,,\,|X(0)|\leq k\} &= \mathbb{P}_{\lambda_d}\{T_\varepsilon\neq\Delta\} \\
&= \mathbb{E}\left\{\lambda_d\big(F^\varepsilon\ominus X([0,r]^N)\big)\right\} \\
&\geq \mathbb{E}\left\{\lambda_d\big(F\ominus X([0,r]^N)\big)\right\},
\end{aligned}$$

since $F\subset F^\varepsilon$. Finally, the integrability condition of our lemma allows us to take limits, and conclude that $\lim_{\eta\to 0}\mathcal{E}_\Psi(f_{\varepsilon,k;\eta}) = \mathcal{E}_\Psi(\mu_{\varepsilon,k})$. Thus, (5.11) implies (5.8) after letting $k\uparrow\infty$, from which our lemma follows. □

Next, we proceed to Part (iii) of our proof. The following proposition verifies the easy half of Theorem 1.5. Namely, if F carries a finite measure with finite energy, then the Lebesgue measure of $X(\mathbb{R}_+^N)\oplus F$ has a positive expectation.

Proposition 5.7 *Suppose X is an additive Lévy process in $\mathbb{R}^d$ with Lévy exponent $\Psi = (\Psi_1,\ldots,\Psi_N)$. If $F\subset\mathbb{R}^d$ is a compact set and $\mathcal{C}_\Psi(F)>0$, then for any $r>0$,*

$$\mathbb{E}\left\{\lambda_d(X([0,r]^N)\oplus F)\right\}>0.$$

Proof Since $\mathcal{C}_\Psi(-F)=\mathcal{C}_\Psi(F)$, as in the proof of Lemma 5.5 we only need to show that if $\mathcal{C}_\Psi(F)>0$, then for any $r>0$,

$$\mathbb{E}\left\{\lambda_d(X([0,r]^N)\ominus F)\right\}>0. \qquad (5.12)$$

We note that whenever $\mathbb{E}\{\lambda_d(X([0,r]^N))\}>0$, then $\mathbb{E}\{\lambda_d(X([0,r]^N)\ominus F)\}>0$ for all compact sets $F\subset\mathbb{R}^d$, and the proposition holds trivially. Moreover, if there exist some $n\leq N-1$, distinct $i_1,\ldots,i_n\in\{1,\ldots,N\}$, and a compact set $F\subset\mathbb{R}^d$ such that

$$\mathbb{E}\left\{\lambda_d(X_{i_1\ldots i_n}([0,r]^n)\ominus F)\right\}>0,$$

where $X_{i_1\cdots i_n} = X_{i_1} \oplus \cdots \oplus X_{i_n}$ is an n-parameter additive Lévy process, then inequality (5.12) also holds. Hence, without loss of generality, we can and will assume that

$$\mathbb{E}\big\{\lambda_d(X_{i_1\cdots i_n}([0,r]^n) \ominus F)\big\} = 0 \tag{5.13}$$

for all $n \leq N-1$ and distinct $i_1, \ldots, i_n \in \{1, \ldots, N\}$.

For any $\varepsilon > 0$, define

$$\varphi_\varepsilon(x) = (2\varepsilon)^{-d} \mathbb{1}_{\{|x| \leq \varepsilon\}}, \qquad x \in \mathbb{R}^d,$$

where $|x|$ is the ℓ^∞-norm of $x \in \mathbb{R}^d$. Then, whenever μ is a probability measure on F, $\mu_\varepsilon := \mu \star \varphi_\varepsilon$ is a probability measure on F^ε, where $\star$ denotes convolution, and F^ε is the closed ε-enlargement of F in the ℓ^∞-norm. To maintain some notational simplicity, we write μ_ε both for the measure and its density with respect to Lebesgue measure λ_d.

Since the $\mathbb{P}_{\lambda_d}$-distribution of $X(t)$ is λ_d for all $t \in \mathbb{R}^N_+$,

$$\mathbb{E}_{\lambda_d}\left\{\int_{[0,r]^N} e^{-\sum_{j=1}^N s_j} \mu_\varepsilon(X(s))\, ds\right\} = (1-e^{-r})^N. \tag{5.14}$$

On the other hand, by Lemma 5.3, and the definition of energy [(1.4)],

$$\mathbb{E}_{\lambda_d}\left\{\left[\int_{[0,r]^N} e^{-\sum_{j=1}^N s_j} \mu_\varepsilon(X(s))\, ds\right]^2\right\} \leq \mathcal{E}_\Psi(\mu_\varepsilon) \leq \mathcal{E}_\Psi(\mu), \tag{5.15}$$

since $|\widehat{\varphi_\varepsilon}(\xi)| \leq 1$. Recall the Paley–Zygmund inequality: For any measure ν on the underlying measure space, and for any nonnegative $g \in L^2(\nu) \cap L^1(\nu)$,

$$\nu\{g > 0\} \geq \frac{\|g\|^2_{L^1(\nu)}}{\|g\|^2_{L^2(\nu)}};$$

cf. Kahane (1985, page 8). We apply this with $\nu = \mathbb{P}_{\lambda_d}$ and

$$g(\omega) = \int_{[0,r]^N} e^{-\sum_{j=1}^N s_j} \mu_\varepsilon(X(s))\,(\omega)\, ds.$$

Thanks to (5.14) and (5.15), we have $\|g\|_{L^1(\nu)} = (1-e^{-r})^N$ and $\|g\|^2_{L^2(\nu)} \leq \mathcal{E}_\Psi(\mu)$. Therefore,

$$\mathbb{P}_{\lambda_d}\{X([0,r]^N) \cap F^\varepsilon \neq \varnothing\} \geq \mathbb{P}_{\lambda_d}\{g > 0\} \geq (1-e^{-r})^{2N} [\mathcal{E}_\Psi(\mu)]^{-1}.$$

It follows from (5.9) that

$$\mathbb{P}_{\lambda_d}\{X([0,r]^N) \cap F^\varepsilon \neq \varnothing\} = \mathbb{E}\{\lambda_d(X([0,r]^N) \ominus F^\varepsilon)\}.$$

Thus, we can let $\varepsilon \downarrow 0$ to obtain

$$\mathbb{E}\{\lambda_d(\overline{X([0,r]^N)} \ominus F)\} \geq (1-e^{-r})^{2N} [\mathcal{E}_\Psi(\mu)]^{-1}, \tag{5.16}$$

for all probability measures μ on F that have finite energy. Since each X_j has only a countable number of jumps, the assumption (5.13) implies that

$$\lambda_d\Big(\big(\overline{X([0,r]^N)} \setminus X([0,r]^N)\big) \ominus F\Big) = 0, \qquad \mathbb{P}\text{-a.s.}$$

Therefore, (5.16) becomes

$$\mathbb{E}\{\lambda_d(X([0,r]^N) \ominus F)\} \geq (1-e^{-r})^{2N} [\mathcal{E}_\Psi(\mu)]^{-1},$$

for all probability measures μ on F that have finite energy. Defining $1 \div 0 = \infty$ as we have, we can optimize over all probability measures μ on F to deduce that

$$\mathbb{E}\{\lambda_d(X([0,r]^N) \ominus F)\} \geq (1 - e^{-r})^{2N} \mathcal{C}_\Psi(F).$$

This proves (5.12), and our proposition follows. □

6 Convolution-Based Energies

This is a brief section on connections between the energy forms of the Introduction and the notion of mutual energy based on convolutions. Some of this material is classical, and can be found in standard references such as (Carleson 1983; Kahane 1985).

Any locally integrable function $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ defines a mutual energy on the space of all measures crossed itself. To be precise, the *K-mutual energy* of measures μ and ν is defined by

$$(\mu, \nu)_K = \tfrac{1}{2} \iint K(a-b)\, \mu(da)\, \nu(db) + \tfrac{1}{2} \iint K(b-a)\, \mu(da)\, \nu(db). \tag{6.1}$$

This is clearly a symmetric form, i.e., $(\mu,\nu)_K = (\nu,\mu)_K$. It also induces a capacity C_K on subsets of $\mathbb{R}^d$:

$$\mathsf{C}_K(F) = \left[\inf_{\mu \in \mathcal{P}(F)} (\mu,\mu)_K \right]^{-1},$$

where $\mathcal{P}(F)$ is the collection of all probability measures that are carried by F.

We say that K is the *1-potential density* of an additive Lévy process $X = \{X(t);\, t \in \mathbb{R}_+^N\}$ in $\mathbb{R}^d$ if for all $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\mathbb{E}\left\{ \int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N s_j} f(X(s))\, ds \right\} = \int_{\mathbb{R}^d} K(a) f(a)\, da.$$

It is easy to see that if the 1-potential density $K(a)$ exists, then $\int_{\mathbb{R}^d} K(a)\, da = 1$ and $K(a) > 0$ for almost every $a \in X(\mathbb{R}_+^N)$. A sufficient condition for the existence of the 1-potential density is that $X(s)$ has a density function $p_s(a)$ for all $s \in (0,\infty)^N$. In this case,

$$K(a) = \int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N s_j} p_s(a)\, ds. \tag{6.2}$$

See Hawkes (1979, Lemma 2.1) for a necessary and sufficient condition for the existence of a 1-potential density.

Lemma 6.1 *Let X be an additive Lévy process in $\mathbb{R}^d$ with Lévy exponent Ψ and 1-potential density K, and suppose Condition (1.3) holds. Then, for all finite measures μ on $\mathbb{R}^d$,*

$$2^{-N} (\mu,\mu)_K \leq \mathcal{E}_\Psi(\mu) \leq \vartheta^{-1} (\mu,\mu)_K.$$

Furthermore, if (1.7) holds, then $\mathcal{E}_\Psi(\mu) = (\mu,\mu)_K$.

Proof Define $K_\star(a) = \frac{1}{2}[K(a) + K(-a)]$ to be the symmetrization of K. Then,

$$\begin{aligned}\widehat{K_\star}(\xi) &= \frac{1}{2}\left[\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)} + \prod_{j=1}^N \frac{1}{1+\overline{\Psi_j(\xi)}}\right] \\ &= \mathrm{Re}\left(\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\right).\end{aligned} \tag{6.3}$$

We note that $\widehat{K_\star}$ is a real function and, under Condition (1.3), it is also nonnegative. Hence, by Fubini's theorem,

$$\begin{aligned}(\mu,\mu)_K &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} K_\star(a-b)\,\mu(da)\,\mu(db) \\ &= (2\pi)^{-d}\iint\int e^{-i\xi\cdot(a-b)}\widehat{K_\star}(\xi)\,d\xi\,\mu(da)\,\mu(db) \\ &= (2\pi)^{-d}\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2\,\widehat{K_\star}(\xi)\,d\xi.\end{aligned} \tag{6.4}$$

Together, this and (6.3) imply the second portion of the lemma, as well as the asserted upper bound on $\mathcal{E}_\Psi(\mu)$. It remains to verify the corresponding lower bound for $\mathcal{E}_\Psi(\mu)$.

Note that if $f : \mathbb{R}^d \to \mathbb{R}_+$ is measurable, $U_+ f \le Uf$, pointwise; cf. (3.2) and (3.3). In particular,

$$\int_{\mathbb{R}^d} f(x)U_+f(x)\,dx \le \int_{\mathbb{R}^d} f(x)Uf(x)\,dx. \tag{6.5}$$

Thanks to Lemma 3.1, for all nonnegative functions $f \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ with $\widehat{f} \in L^1(\mathbb{R}^d)$, (6.5) is *equivalent* to:

$$\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \mathrm{Re}\left(\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\right) d\xi \le 2^N \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi.$$

Now, given any finite measure μ on the Borel subsets of $\mathbb{R}^d$, we can replace, in the preceding display, f by $f_\varepsilon \star \mu$, where f_ε is the Gaussian mollifier of (4.6), and obtain

$$\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 e^{-\frac{\varepsilon}{2}\|\xi\|^2} \mathrm{Re}\left(\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\right) d\xi \le 2^N \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 e^{-\frac{\varepsilon}{2}\|\xi\|^2} \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) d\xi.$$

Thanks to (1.3), both integrands are nonnegative. Thus, we can let $\varepsilon \downarrow 0$, and appeal to Lebesgue monotone convergence theorem to deduce that

$$(2\pi)^{-d}\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \mathrm{Re}\left(\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\right) d\xi \le 2^N \mathcal{E}_\Psi(\mu).$$

Owing to (6.4), the left-hand side equals $(\mu,\mu)_K$, which completes our proof. □

Proposition 6.2 *Suppose X is an additive Lévy process on $\mathbb{R}^d$ with Lévy exponent Ψ that satisfies Condition (1.3) and X has a 1-potential density K. Then, for any compact set $F \subset \mathbb{R}^d$, the following are equivalent:*

(i) There exists a finite measure μ on F with $(\mu,\mu)_K < +\infty$.

(ii) $\lambda_d(\mathsf{H}_F) > 0$, where

$$\mathsf{H}_F = \Big\{ a \in \mathbb{R}^d : \mathbb{P}\big[X(\mathbb{R}^N_+) \cap (\{a\} \oplus F) \neq \varnothing\big] > 0 \Big\}.$$

(iii) $\mathsf{H}_F \neq \varnothing$.

If, in addition, K is almost everywhere positive, then the above is also equivalent to

(iv) $\mathsf{H}_F = \mathbb{R}^d$.

Remark 6.3 In the classical setting where $N = 1$, this theorem is well known. For instance, when F is a singleton, this proposition was considered first by Orey (1967), and later on by Kesten (1968) and Bretagnolle (1971). The same remark applies to the equivalence of (*iv*) and (*ii*). When $F \subset \mathbb{R}^d$ is a general closed set, this result can be found in Bertoin (1996, Chapter II). Furthermore, the equivalence of (*ii*) and (*iii*) appears in Hawkes (1979, Theorem 2.1). □

Proof We observe that by (5.9), and by Fubini's theorem,

$$\mathbb{E}\big\{\lambda_d(X(\mathbb{R}^N_+) \ominus F)\big\} = \int_{\mathbb{R}^d} \mathbb{P}\{X(\mathbb{R}^N_+) \cap [\{a\} \oplus F] \neq \varnothing\}\, da. \tag{6.6}$$

Hence, (*i*) $\iff$ (*ii*) follows from Theorem 1.5 and Lemma 6.1. It is also clear that (*ii*) $\Rightarrow$ (*iii*). To prove (*iii*) $\Rightarrow$ (*ii*), note that for all $s \in \mathbb{R}^N_+$ and all $a \in \mathbb{R}^d$,

$$\mathbb{P}\big\{X\big((s,\infty)\big) \cap [\{a\} \oplus F] \neq \varnothing\big\} = \int_{\mathbb{R}^d} \mathbb{P}\big\{X(\mathbb{R}^N_+) \cap [\{a-b\} \oplus F] \neq \varnothing\big\}\, \mathbb{P}\{X(s) \in db\},$$

where $(s,\infty) = \{t \in \mathbb{R}^N_+;\, t \succcurlyeq s\}$. We multiply the above by $\exp(-\sum_j s_j)$ and integrate $[ds]$ to obtain

$$\begin{aligned} &\int_{\mathbb{R}^N_+} \mathbb{P}\big\{X\big((s,\infty)\big) \cap [\{a\} \oplus F] \neq \varnothing\big\} e^{-\sum_{j=1}^N s_j}\, ds \\ &\qquad = \int_{\mathbb{R}^d} \mathbb{P}\big\{X(\mathbb{R}^N_+) \cap [\{a-b\} \oplus F] \neq \varnothing\big\} K(b)\, db. \end{aligned} \tag{6.7}$$

If (*iii*) holds, then for some $a \in \mathbb{R}^d$, the left-hand side of (6.7) is positive. Therefore,

$$\mathbb{P}\big\{X(\mathbb{R}^N_+) \cap [\{a-b\} \oplus F] \neq \varnothing\big\} > 0,$$

for b in a set of positive Lebesgue measure, which implies $\lambda_d(\mathsf{H}_F) > 0$. This prove the first half of the proposition. Since it is clear that (*iv*) $\Rightarrow$ (*i*), it remains to prove (*i*) $\Rightarrow$ (*iv*).

If (*iv*) did not hold, there would exist an $a \in \mathbb{R}^d$ such that the left-hand side of (6.7) would equal 0. This would then imply that

$$\int_{\mathbb{R}^d} \mathbb{P}\big\{X(\mathbb{R}^N_+) \cap [\{a-b\} \oplus F] \neq \varnothing\big\} K(b)\, db = 0.$$

Since $K > 0$ almost everywhere, we would have

$$\mathbb{P}\big\{X(\mathbb{R}^N_+) \cap [\{a-b\} \oplus F] \neq \varnothing\big\} = 0 \quad \lambda_d\text{-almost every } b \in \mathbb{R}^d.$$

Equation (6.6) would then imply that $\mathbb{E}\{\lambda_d(X(\mathbb{R}^N_+) \ominus F)\} = 0$. Using Theorem 1.5 and Lemma 6.1 again, we would derive a contradiction to (*i*). We have shown that (*i*) $\Rightarrow$ (*iv*), which completes our proof. □

Remark 6.4 The almost everywhere positivity of the function K is indispensable, as can be seen by considering a nonnegative stable subordinator X, and by letting $F = [-2, -1]$. In this case (iii) clearly does not hold. $\square$

In the following, we prove a zero-infinity law for $\lambda_d(X(\mathbb{R}_+^N))$.

Proposition 6.5 *Suppose X is an additive Lévy process on $\mathbb{R}^d$ with Lévy exponent Ψ that satisfies Condition (1.3), and has an a.e. positive 1-potential density K. Then,*

$$\lambda_d(X(\mathbb{R}_+^N)) \in \{0, +\infty\}, \qquad \mathbb{P}\text{-}a.s.$$

Proof Assuming that $\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} < \infty$, we first show that the value of this expectation is, in fact, zero. Bearing this goal in mind, we note that for any $n > 0$,

$$\begin{aligned}\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} &\geq \mathbb{E}\{\lambda_d(X([0,n]^N))\} + \mathbb{E}\{\lambda_d(X((n,\infty)^N))\} \\ &\qquad - \mathbb{E}\{\lambda_d[X([0,n]^N) \cap X((n,\infty)^N)]\} \\ &= \mathbb{E}\{\lambda_d(X([0,n]^N))\} + \mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} \\ &\qquad - \mathbb{E}\{\lambda_d[X([0,n]^N) \cap X'(\mathbb{R}_+^N)]\},\end{aligned}$$

where X' is an independent copy of X. Consequently, we see that if $\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} < \infty$,

$$\mathbb{E}\{\lambda_d(X([0,n]^N))\} \leq \mathbb{E}\{\lambda_d[X([0,n]^N) \cap X'(\mathbb{R}_+^N)]\}.$$

Let $n \uparrow \infty$ to see that as long as $\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} < \infty$,

$$\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} \leq \mathbb{E}\{\lambda_d[X(\mathbb{R}_+^N) \cap X'(\mathbb{R}_+^N)]\}.$$

Define $\varphi(a) = \mathbb{P}\{a \in X(\mathbb{R}_+^N)\}$, and note that the above is equivalent to

$$\int_{\mathbb{R}^d} \varphi(a)\, da \leq \int_{\mathbb{R}^d} \varphi^2(a)\, da.$$

Since $0 \leq \varphi(a)(1 - \varphi(a)) \leq 1$ for all $a \in \mathbb{R}^d$, we have

$$\varphi(a) \in \{0, 1\}, \quad \lambda_d\text{-almost every} \ \ a \in \mathbb{R}^d.$$

Consequently, if $\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\}$ is finite,

$$\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} = \lambda_d(\varphi^{-1}\{1\}). \tag{6.8}$$

It follows from Proposition 6.2 that either $\varphi(a) = 0$ for all $a \in \mathbb{R}^d$ or > 0 for all $a \in \mathbb{R}^d$. This means that $\lambda_d(\varphi^{-1}\{1\}) = 0$; for, otherwise, $\varphi^{-1}\{1\} = \mathbb{R}^d$, which has infinite λ_d-measure, and this would contradict (6.8). In order words, we have demonstrated that

$$\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} < \infty \implies \lambda_d(\varphi^{-1}\{1\}) = 0 \implies \mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} = 0.$$

We now "remove the expectation" from this statement and finish our proof. Suppose $\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} > 0$ (which means that $\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} = \infty$), and note that for any $\nu > 0$,

$$\lambda_d(X(\mathbb{R}_+^N)) \geq \sup_{n \geq 0} \lambda_d(X([n, n+\nu]^N)) = \sup_{n \geq 0} \Gamma_n^\nu.$$

Since $\Gamma_0^\nu, \Gamma_{1+\nu}^\nu, \Gamma_{1+2\nu}^\nu, \ldots$ are i.i.d., by the Borel-Cantelli lemma, for any $\nu > 0$,

$$\lambda_d(X(\mathbb{R}_+^N)) \geq \mathbb{E}\{\Gamma_0^\nu\} = \mathbb{E}\{\lambda_d(X([0,\nu]^N))\}, \qquad \mathbb{P}\text{-a.s.}$$

We can let $\nu \uparrow \infty$ along a sequence of rational numbers to deduce from $\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N))\} = \infty$ that $\lambda_d(X(\mathbb{R}_+^N)) = \infty$, a.s. $\square$

7 Bessel–Riesz Capacities

The α-dimensional Bessel-Riesz (sometimes only Riesz-) energies and capacities on $\mathbb{R}^d$ are those that correspond to $K = \mathcal{R}_{(\alpha)}$, where

$$\mathcal{R}_{(\alpha)}(a) = \begin{cases} 1, & \text{if } \alpha < 0, \\ \ln(1/\|a\|), & \text{if } \alpha = 0, \\ \|a\|^{-\alpha}, & \text{if } \alpha > 0, \end{cases} \qquad \forall a \in \mathbb{R}^d. \tag{7.1}$$

In this case we will write $(\mu, \mu)_{(\alpha)}$, in place of the more cumbersome $(\mu, \mu)_{\mathcal{R}_{(\alpha)}}$, and write $\mathsf{C}_{(\alpha)}$ for the corresponding capacity.

There are deep connections between α-dimensional Bessel-Riesz capacities and ordinary one-parameter Lévy processes when $0 < \alpha \le 2$. In this section, we show that, by considering additive Lévy processes, one can have a probabilistic interpretation of sets of positive α-dimensional capacity for any $\alpha > 0$.

Example 7.1 Suppose $B = B_1 \oplus \cdots \oplus B_N$ is additive Brownian motion in $\mathbb{R}^d$. That is, B_i's are independent d-dimensional Brownian motions. Then, by (6.2), it is easy to see that the 1-potential density of B is

$$K(a) = \frac{1}{(2\pi)^{\frac{d}{2}}(N-1)!} \int_0^\infty e^{-t-\frac{\|a\|^2}{2t}} t^{-\frac{d}{2}+N-1}\, dt, \qquad \forall a \in \mathbb{R}^d.$$

This calculation only requires the elementary fact that

$$\lambda_k\big(\{q \in \mathbb{R}_+^k : q_1 + \cdots + q_k \le x\}\big) = \frac{1}{k!} x^k, \qquad \forall x \ge 0,\, k = 1, 2, \ldots,$$

which, itself, follows form symmetry considerations. Furthermore, it is a simple matter to check that

$$\lim_{a \to 0} \frac{K(a)}{\mathcal{R}_{(d-2N)}(a)} = \frac{\Lambda\left(\frac{d}{2} - N\right)}{(2\pi)^{\frac{d}{2}}(N-1)!},$$

where $\mathcal{R}_{(\alpha)}$ is defined in the Riesz kernel of (7.1), and for all $x \in \mathbb{R}$,

$$\Lambda(x) = 2^{x_+}\Gamma(|x|),$$

where $x_+ = \max(x, 0)$. Consequently, we can deduce that for any compact set $F \subset \mathbb{R}^d$, there are two constants, A_1 and A_2 such that for all $x \in F \ominus F$,

$$A_1 \mathcal{R}_{(d-2N)}(x) \le K(x) \le A_2 \mathcal{R}_{(d-2N)}(x).$$

This, Lemma 6.1, and Theorem 1.5, together, combine to show that $B(\mathbb{R}_+^N) \oplus F$ can have positive Lebesgue measure if and only if $\mathsf{C}_{(d-2N)}(F) > 0$. □

With a little more work, and motivated by this example, we can find additive Lévy processes that correspond to *any* Bessel–Riesz capacity of interest. Recall that $X_1 \oplus \cdots \oplus X_N$ is *additive stable* of index $\alpha \in (0, 2]$, if $X_1, \ldots, X_N$ are independent isotropic stable processes with index α each.

Theorem 7.2 *Suppose $X = X_1 \oplus \cdots \oplus X_N$ is an additive stable process of index $\alpha \in (0, 2]$, and in $\mathbb{R}^d$. Then, for any compact set $F \subset \mathbb{R}^d$,*

$$\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^N) \oplus F)\} > 0 \iff \mathsf{C}_{(d-\alpha N)}(F) > 0.$$

Remark 7.3 Upon varying $d, N \in \mathbb{N}$, and $\alpha \in (0,2]$, we see that this theorem associates an additive Lévy process to any Bessel–Riesz capacity, including those with dimension > 2. $\square$

Theorem 7.2 follows from Theorem 1.5 and the arguments of Example 7.1, once we establish

Proposition 7.4 *Let X denote an N-parameter additive stable process in $\mathbb{R}^d$. Then, X has a 1-potential density, K, whose asymptotics at the origin are described by the following:*

$$\lim_{a\to 0} \frac{K(a)}{\mathcal{R}_{(d-N\alpha)}(a)} = C(\alpha, d, N), \tag{7.2}$$

where $\mathcal{R}_{(\bullet)}$ is the Riesz kernel of (7.1), and $C(\alpha,d,N)$ is a positive and finite constant depending on α, d and N only. Moreover, there exists a positive constant $\tilde{C} = \tilde{C}(\alpha,d,N)$ such that

$$K(a) \le \tilde{C}\,\mathcal{R}_{(d-N\alpha)}(a), \qquad \forall a \in \mathbb{R}^d. \tag{7.3}$$

Proof In light of Example 7.1, we can assume, without loss generality, that $0 < \alpha < 2$.

We denote the density function of $X_1(1)$ by $p_1(x)$. Here, p_1 is scaled as

$$e^{-\|\xi\|^\alpha} = \int_{\mathbb{R}^d} e^{ix\cdot\xi} p_1(x)dx.$$

It is possible to show that $p_1(x)$ is a continuous and strictly positive function on $\mathbb{R}^d$ that is isotropic, i.e., it depends on x only through $\|x\|$.

Direct calculations reveal that the 1-potential density of X is

$$K(a) = \frac{1}{(N-1)!}\int_0^\infty t^{N-1-d/\alpha} e^{-t}\, p_1(at^{-1/\alpha})\, dt, \quad \forall a \in \mathbb{R}^d. \tag{7.4}$$

On the other hand, by using Bochner's subordination, we can write

$$p_1(x) = \|x\|^{-d}\int_0^\infty \nu\left(\frac{s}{\|x\|^2}\right) g^{(\alpha/2)}(s)\, ds, \tag{7.5}$$

where the function ν is defined as

$$\nu(s) = (4\pi s)^{-d/2}\exp\left(-\frac{1}{4s}\right),$$

$g^{(\alpha/2)}(s)$ is the density function of the random variable $\tau(1)$, and where $\tau = \{\tau(t), t \ge 0\}$ is a stable subordinator of index $\frac{1}{2}\alpha$; cf. for example, Bendikov (1994). It should be recognized that that $a \mapsto K(a)$ is isotropic and strictly decreasing in $\|a\|$.

It follows from (7.4) and (7.5), combined with Fubini's theorem, that

$$\begin{aligned} K(a) &= \frac{\|a\|^{-d}}{(N-1)!}\int_0^\infty\int_0^\infty t^{N-1}\, e^{-t}\, \nu\left(\frac{st^{2/\alpha}}{\|a\|^2}\right) g^{(\alpha/2)}(s)\, ds\, dt \\ &= \frac{\alpha/2}{(4\pi)^{d/2}(N-1)!}\|a\|^{\alpha N-d}\int_0^\infty s^{-\alpha N/2} g^{(\alpha/2)}(s)\, ds \times \\ &\qquad \times \int_0^\infty u^{\frac{1}{2}(\alpha N-d)-1}\exp\left(-\frac{\|a\|^\alpha}{s^{\alpha/2}}u^{\alpha/2}\right)\exp\left(-\frac{1}{4u}\right) du. \end{aligned} \tag{7.6}$$

When $d-\alpha N>0$, (7.6) and the monotone convergence theorem imply

$$\lim_{a\to 0}\frac{K(a)}{\mathcal{R}_{(d-N\alpha)}(a)}=\frac{\alpha/2}{(4\pi)^{d/2}(N-1)!}\int_0^\infty s^{-\frac{\alpha N}{2}}g^{(\alpha/2)}(s)\,ds\int_0^\infty u^{\frac{1}{2}(\alpha N-d)-1}\exp\big(-\frac{1}{4u}\big)\,du.$$

Hence, when $d-\alpha N>0$, we can identify the constant $C(\alpha,d,N)$ in (7.2) as

$$C(\alpha,d,N)=\frac{\alpha 2^{d-\alpha N-1}}{(4\pi)^{d/2}(N-1)!}\Gamma\big(\frac{1}{2}(d-\alpha N)\big)\int_0^\infty s^{-\frac{\alpha N}{2}}g^{(\alpha/2)}(s)\,ds$$

In the above, note that

$$\int_0^\infty s^{-\frac{\alpha N}{2}}g^{(\alpha/2)}(s)ds=\mathbb{E}[\tau(1)^{-\frac{\alpha N}{2}}]=\int_0^\infty \mathbb{P}\left\{\tau(1)\le x^{-\frac{2}{\alpha N}}\right\}dx<+\infty, \tag{7.7}$$

by a well known estimates for $\mathbb{P}\{\tau(1)\le\varepsilon\}$ as $\varepsilon\to 0$; see, e.g., Hawkes (1971), or Bertoin (1996, p. 88).

If $d-\alpha N=0$, we split the last integral in (7.6) as

$$\int_0^{\|a\|^{-1}}(\cdots)\,du+\int_{\|a\|^{-1}}^\infty(\cdots)\,du=I_1(a)+I_2(a). \tag{7.8}$$

We note that

$$\exp\left(-\frac{\|a\|^{\alpha/2}}{s^{\alpha/2}}\right)\int_0^{\|a\|^{-1}}u^{-1}\exp\left(-\frac{1}{4u}\right)\,du\le I_1(a)\le\int_0^{\|a\|^{-1}}u^{-1}\exp\left(-\frac{1}{4u}\right)\,du,$$

and, thanks to L'Hopital's rule,

$$\lim_{a\to 0}\frac{\int_0^{\|a\|^{-1}}u^{-1}\exp\big(-\frac{1}{4u}\big)\,du}{\log\|a\|^{-1}}=1.$$

Thus,

$$\lim_{a\to 0}\frac{I_1(a)}{\log\|a\|^{-1}}=1. \tag{7.9}$$

On the other hand,

$$\exp\left(-\frac{\|a\|}{4}\right)\int_{\|a\|^{-1}}^\infty u^{-1}\exp\left(-\frac{\|a\|^\alpha}{s^{\alpha/2}}u^{\alpha/2}\right)\,du\le I_2(a)\le\int_{\|a\|^{-1}}^\infty u^{-1}\exp\left(-\frac{\|a\|^\alpha}{s^{\alpha/2}}u^{\alpha/2}\right)\,du,$$

and by using L'Hopital's rule again, we obtain

$$\begin{aligned}\lim_{a\to 0}\frac{I_2(a)}{\log\|a\|^{-1}}&=\lim_{a\to 0}\frac{1}{\log\|a\|^{-1}}\int_{\|a\|^{-1}}^\infty u^{-1}\exp\big(-\frac{\|a\|^\alpha}{s^{\alpha/2}}u^{\alpha/2}\big)\,du\\&=\lim_{a\to 0}\left[-\exp\left(-\frac{\|a\|^{\alpha/2}}{s^{\alpha/2}}\right)+\frac{\alpha\|a\|^\alpha}{s^{\alpha/2}}\int_{\|a\|^{-1}}^\infty u^{-1+\alpha/2}\exp\left(-\frac{\|a\|^\alpha}{s^{\alpha/2}}u^{\alpha/2}\right)\,du\right]\\&=1.\end{aligned} \tag{7.10}$$

It follows from (7.6), (7.8), (7.9) and (7.10), combined with Lebesgue dominated convergence theorem (the above upper bounds for $I_1(a)$ and $I_2(a)$ are used here), that when $d-\alpha N=0$,

$$\lim_{a\to 0}\frac{K(a)}{\mathcal{R}_{(0)}(a)}=\frac{\alpha}{(4\pi)^{d/2}(N-1)!}\int_0^\infty s^{-\frac{\alpha N}{2}}g^{\alpha/2}(s)\,ds.$$

In case $d - \alpha N < 0$, (7.6) tells us that we only need to show that the following limit exists:

$$\lim_{a\to 0} \|a\|^{\alpha N-d} \int_0^\infty u^{\frac{1}{2}(\alpha N-d)-1} \exp\left(-\frac{\|a\|^\alpha}{s^{\alpha/2}} u^{\alpha/2}\right) \exp\left(-\frac{1}{4u}\right) du.$$

After changing variables and appealing to the monotone convergence theorem, we can see that the above limit equals

$$\int_0^\infty v^{\frac{1}{2}(\alpha N-d)-1} \exp\left(-\frac{v^{\alpha/2}}{s^{\alpha/2}}\right) dv.$$

Hence, in this case, we apply (7.6), and change variables once more, to show that (7.2) holds with

$$C(\alpha,d,N) = \frac{\Gamma\left(N-\frac{d}{\alpha}\right)}{(4\pi)^{d/2}(N-1)!} \int_0^\infty s^{-d/2} g^{(\alpha/2)}(s)\, ds,$$

where, as in (7.7), the last integral is finite.

Finally, the inequality (7.3) follows readily by adapting the aforementioned arguments. For example, when $d - \alpha N > 0$, it follows from (7.6) that

$$\frac{K(a)}{\mathcal{R}_{(d-N\alpha)}(a)} \le \frac{\alpha/2}{(4\pi)^{d/2}(N-1)!} \int_0^\infty s^{-\alpha N/2} g^{\alpha/2}(s) ds \cdot \int_0^\infty u^{\frac{1}{2}(\alpha N-d)-1} \exp\left(-\frac{1}{4u}\right) du.$$

We omit the other two cases, and declare the proof of Proposition 7.4 complete. □

8 Proof of Theorem 1.6

We will use Frostman's theorem of potential theory; cf. Kahane (1985; Chapter 10) or Carleson (1983). Recall that the latter states that for any Borel set $G \subset \mathbb{R}^d$, the capacitary and the Hausdorff dimensions of G agree. That is,

$$\dim(G) = \sup\left\{\gamma > 0 : \mathsf{C}_{(\gamma)}(G) > 0\right\}, \tag{8.1}$$

where $\mathsf{C}_{(\gamma)}$ is the Bessel–Riesz capacity of §7, and $\sup \varnothing = 0$.

Now, we introduce an M-parameter additive stable process Y in $\mathbb{R}^d$ whose index is $\alpha \in (0, 2]$. The process Y is totally independent of X, and we will determine the constants M and α shortly. Note that $X \oplus Y$ is an $(N+M)$-parameter additive Lévy process in $\mathbb{R}^d$ whose Lévy exponent $\Phi = (\Phi_1, \ldots, \Phi_{N+M})$ is given by

$$\Phi_j(\xi) = \begin{cases} \Psi_j(\xi), & \text{if } j = 1, \ldots, N, \\ \frac{1}{2}\|\xi\|^\alpha, & \text{if } j = N+1, \ldots, N+M. \end{cases}$$

Clearly,

$$\int_{\mathbb{R}^d} \prod_{j=1}^{N+M} \mathrm{Re}\left(\frac{1}{1+\Phi_j(\xi)}\right) d\xi < +\infty \iff \mathcal{I}^{(M\alpha)} < +\infty,$$

where for all $\gamma \in \mathbb{R}$,

$$\mathcal{I}^{(\gamma)} = \int_{\xi\in\mathbb{R}^d:\ \|\xi\|>1} \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) \|\xi\|^{-\gamma}\, d\xi.$$

Thus, we can apply Theorem 1.1 to the process $X \oplus Y$, and see that

$$\mathbb{E}\{\lambda_d(X(\mathbb{R}^N_+) \oplus Y(\mathbb{R}^M_+))\} > 0 \iff \mathcal{J}^{(M\alpha)} < +\infty.$$

On the other hand, we can also apply Theorem 7.2, conditionally on $F = X(\mathbb{R}^N_+)$, to deduce that

$$\mathbb{E}\{\lambda_d(X(\mathbb{R}^N_+) \oplus Y(\mathbb{R}^M_+))\} > 0 \iff \mathbb{E}\{\mathsf{C}_{(d-M\alpha)}(X(\mathbb{R}^N_+))\} > 0.$$

We combine the latter two displays to obtain:

$$\mathbb{E}\left\{\mathsf{C}_{(d-M\alpha)}(X(\mathbb{R}^N_+))\right\} > 0 \iff \mathcal{J}^{(M\alpha)} < +\infty.$$

Consequently, when $\mathcal{J}^{(M\alpha)} = +\infty$, $\mathsf{C}_{(d-M\alpha)}(X(\mathbb{R}^N_+)) = 0$, $\mathbb{P}$-almost surely. From Frostman's theorem [(8.1)], we deduce that $\mathcal{J}^{(M\alpha)} = +\infty$ implies that $\mathbb{P}$-a.s., $\dim(X(\mathbb{R}^N_+)) \le d - M\alpha$. On the other hand, we can choose $M \in \{1, 2, \ldots\}$, and rational $\alpha_1, \alpha_2, \ldots \in (0, 2]$ such that $M\alpha_j \uparrow \eta$; this shows that $\dim(X(\mathbb{R}^N_+)) \le d - \eta$, $\mathbb{P}$-almost surely. In particular, if $\eta = d$, then $\dim(X(\mathbb{R}^N_+)) = 0$, $\mathbb{P}$-almost surely, and this constitutes half of our theorem. For the other half, we use the same argument, but quantitatively.

For the converse half, we only need to consider the case when $\eta < d$. With this in mind, choose $\alpha \in (0, 2]$ and $M \in \{1, 2, \ldots\}$ such that $\eta < M\alpha < d$. Thus, we can deduce from the preceding paragraph that

$$\mathcal{J}^{(M\alpha)} < +\infty.$$

We now recall the killed occupation measure $\mathbb{O}_\kappa$ from (2.1) and (5.1). This is a Borel probability measure carried by $X(\mathbb{R}^N_+)$, and we claim that as long as $0 < d - \alpha M < d - \eta$,

$$(\mathbb{O}_\kappa, \mathbb{O}_\kappa)_{(d-M\alpha)} < +\infty, \qquad \mathbb{P}\text{-a.s.,} \tag{8.2}$$

where $(\mathbb{O}_\kappa, \mathbb{O}_\kappa)_{(\gamma)}$ is the γ-dimensional Bessel–Riesz energy of $\mathbb{O}_\kappa$ as defined in §7. Together with Frostman's theorem [(8.1)], this shows that with probability one, $\dim(X(\mathbb{R}^N_+)) \ge d - M\alpha$. This is the key part of our proof, since we can approximate η from above arbitrarily well by numbers of the form $M\alpha$ ($M \in \{1, 2, \ldots\}$, $\alpha \in (0, 2] \cap \mathbb{Q}$). In this way, we deduce that with probability one, $\dim(X(\mathbb{R}^N_+)) \ge d - \eta$, as asserted.

At this point, we only need to establish (8.2). For this purpose, recall the process Y, as above, and consider its Lévy exponent $\Lambda = (\Lambda_1, \ldots, \Lambda_M)$, where $\Lambda_i(\xi) = \frac{1}{2}\|\xi\|^\alpha$, $i = 1, \ldots, M$. The process Y has a 1-potential density K whose asymptotics are described by Proposition 7.4. After applying Lemma 6.1 to Y, we deduce that $(\mathbb{O}_\kappa, \mathbb{O}_\kappa)_K = \mathcal{E}_\Lambda(\mathbb{O}_\kappa)$, $\mathbb{P}$-almost surely. In particular, Lemma 5.2 gives

$$\begin{aligned}
\mathbb{E}\{(\mathbb{O}_\kappa, \mathbb{O}_\kappa)_K\} &= (2\pi)^{-d} \int_{\mathbb{R}^d} |\widehat{\mathbb{O}_\kappa}(\xi)|^2 \cdot \{1 + \tfrac{1}{2}\|\xi\|^\alpha\}^{-M} \, d\xi \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1 + \Psi_j(\xi)}\right) \cdot \{1 + \tfrac{1}{2}\|\xi\|^\alpha\}^{-M} \, d\xi,
\end{aligned}$$

which is finite, since $\mathcal{J}^{(M\alpha)} < +\infty$. Therefore, we have found a random measure $\mathbb{O}_\kappa$ on the random set $X(\mathbb{R}^N_+)$ such that with probability one, $(\mathbb{O}_\kappa, \mathbb{O}_\kappa)_K < +\infty$. Thanks to Proposition 7.4, there is a positive and finite constant C', depending on α, d and N only, such that

$$\mathcal{R}_{(d-N\alpha)}(a) \le C' K(a), \qquad \forall a \in \mathbb{R}^d \text{ with } \|a\| \le 1.$$

Hence,

$$(\mathbb{O}_\kappa, \mathbb{O}_\kappa)_{(d-M\alpha)} \le 1 + C' \times (\mathbb{O}_\kappa, \mathbb{O}_\kappa)_K < +\infty.$$

This verifies (8.2), whence Theorem 1.6 follows. □

9 Concluding Remarks

A number of interesting questions remain unresolved some of which are listed below.

Question 9.1 Do Theorems 1.1 and 1.5 hold for all additive Lévy processes? One only needs to worry about the necessity since in both theorems the sufficiency has already been shown to hold generally.

A possible approach for proving Theorem 1.1 without Condition (1.3) is as follows: In light of Lemma 5.2 and the upper bound in Theorem 2.1, it suffices to show that

$$\exists \mu \in \mathcal{P}(E) : \|\mu\|_e < +\infty \implies \|\kappa\|_e < +\infty. \tag{9.1}$$

When $\| \bullet \|_e$ is an energy norm based on a positive definite convolution kernel, one can prove such a result by appealing to simple Fourier analytical arguments. In the present general setting, however, we do not know how to proceed. In the Appendix below, we derive an analytical estimate that may be of independent interest, and which barely falls short of settling this open problem by way of verifying the preceding display. □

Question 9.2 A simpler, but still interesting open problem is to find a necessary and sufficient condition for $X(\mathbb{R}^N_+) \oplus F$ to have positive Lebesgue measure with positive probability when X_i's are independent [but, otherwise, arbitrary] subordinators. Equivalently, we ask for a necessary and sufficient condition for the existence of local times of N-parameter additive subordinators without a condition such as (1.3). □

Question 9.3 In light of Theorem 1.6, it would be interesting to determine an exact Hausdorff measure function that gauges the size of $X([0,1]^N)$. For stable sheets and two-parameter additive subordinators, related results can be found in Ehm (1981) and Hu (1994). □

10 Appendix

In this Appendix we present a possible alternative approach for proving Theorem 1.1 without Condition (1.3) that involves an estimate that may be of independent analytical interest. As we mentioned in Section 9, the key is to prove (9.1). In the present general setting, we are only able to verify a partial derivation; cf. Proposition 10.3 and Remark 10.4 below.

Given any measure μ on $\mathbb{R}^N_+$, and given $s \in \mathbb{R}^N$, write

$$\mu_{(s)}(\bullet) = \mu(\bullet + s)$$

for the *s-shift* of μ. We note that $\mu_{(s)}$ need not be a measure on $\mathbb{R}^N_+$ although μ is assumed to be.

Lemma 10.1 (Shift invariance of energy norm) *The map $\mu \mapsto \|\mu\|_e$ is shift invariant in the sense that whenever μ and $\mu_{(s)}$ are both finite measures on $\mathbb{R}^N_+$,*

$$\|\mu\|_e = \|\mu_{(s)}\|_e.$$

Proof This follows immdiately from our computation of $\|\mu\|_e$ in Lemma 2.4. □

Next, we prove that convolutions reduce the norm. To be more precise,

Lemma 10.2 (Norm reduction of convolutions) *Suppose φ is a probability density function on $\mathbb{R}^N$ such that $\varphi \star \mu$ and μ are both in $\mathcal{P}(\mathbb{R}^N_+)$. Then,*

$$\|\varphi \star \mu\|_e \le \|\mu\|_e.$$

Proof We write $\varphi \star \mu$ for both the measure, and its density with respect to Lebesgue measure λ_N. In this way we can write the corresponding occupation measure as

$$\mathbb{O}_{\varphi\star\mu}(f) = \int_{\mathbb{R}^N_+} f(X(s))\varphi \star \mu(s)\, ds.$$

Using Fubini's theorem, twice in succesion, we obtain

$$\begin{aligned}
\widehat{\mathbb{O}_{\varphi\star\mu}}(\xi) &= \int_{\mathbb{R}^N_+} e^{i\xi\cdot X(s)}\, ds \int_{\mathbb{R}^N_+} \varphi(s-t)\,\mu(dt) \\
&= \int_{\mathbb{R}^N_+} \mu(dt) \int_{\mathbb{R}^N_+\ominus\{t\}} e^{i\xi\cdot X(v+t)}\,\varphi(v)dv \\
&= \int_{\mathbb{R}^N} \varphi(v)dv \int_{\mathbb{R}^N_+\ominus\{v\}} e^{i\xi\cdot X(v+t)}\,\mu(dt) \\
&= \int_{\mathbb{R}^N} \widehat{\mathbb{O}_{\mu_{(v)}}}(\xi)\varphi(v)\,dv.
\end{aligned}$$

To this, we apply the Cauchy–Schwarz inequality twice as follows:

$$\begin{aligned}
\|\varphi\star\mu\|_e^2 &= (2\pi)^{-d}\int_{\mathbb{R}^d}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \mathbb{E}\left\{\widehat{\mathbb{O}_{\mu_{(u)}}}(\xi)\overline{\widehat{\mathbb{O}_{\mu_{(v)}}}(\xi)}\right\}\varphi(u)\varphi(v)\,du\,dv\,d\xi \\
&\le (2\pi)^{-d}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^N}\sqrt{\mathbb{E}\Big\{\left|\widehat{\mathbb{O}_{\mu_{(v)}}}(\xi)\right|^2\Big\}}\,\varphi(v)\,dv\right|^2 d\xi \\
&\le (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^N}\mathbb{E}\left\{|\widehat{\mathbb{O}_{\mu_{(v)}}}(\xi)|^2\right\}\varphi(v)\,dv\,d\xi \\
&= \|\mu\|_e^2,
\end{aligned}$$

thanks to Lemma 10.1. □

We conclude this appendix by showing that if there are any probability measures of finite norm on a given compact set F, Lebesgue measure on F is also of finite norm.

Proposition 10.3 *Suppose $F \subset \mathbb{R}^N_+$ is compact. Then,*

$$\|\lambda_N\rfloor_F\|_e \le 2^N \sup_{t\in F}|t|^N \inf_{\mu\in\mathcal{P}(F)}\|\mu\|_e,$$

where $\lambda_N\rfloor_F$ denotes the restriction of λ_N to F.

Proof First, we suppose that F is a compact subset of $(0,\infty)^N$. At the end of our proof, we show how this condition can be removed.

Since $F\subset(0,\infty)^N$, its closed ε-enlargement (written as F^ε) is a subset of $\mathbb{R}^N_+$ for all $\varepsilon>0$ small enough.

For all $\varepsilon>0$, define the function φ_ε, on $\mathbb{R}^N$, by

$$\varphi_\varepsilon(r) = (2\varepsilon)^{-N}\mathbb{1}_{\mathcal{B}(0,\varepsilon)}(r), \qquad \forall r\in\mathbb{R}^N,$$

where

$$\mathcal{B}(s,r) = \big\{t\in\mathbb{R}^N : |t-s|\le r\big\}, \qquad \forall s\in\mathbb{R}^N,\ r>0,$$

and $|t| = \max_{1\le i\le N} |t_i|$ is the ℓ^∞-norm on $\mathbb{R}^N$. We note the following logical sequence:

$$\begin{aligned}\mu \in \mathcal{P}(F) &\implies \varphi_\varepsilon \star \mu \in \mathcal{P}(F^\varepsilon)\\ &\implies \forall \varepsilon > 0 \text{ small},\ \varphi_\varepsilon \star \mu \in \mathcal{P}(\mathbb{R}^N_+)\\ &\implies \forall \varepsilon > 0 \text{ small},\ \|\varphi_\varepsilon \star \mu\|_e \le \|\mu\|_e,\end{aligned}$$

the last inequality following from the norm reduction of convolutions; cf. Lemma 10.2. For each $\eta > 0$, let f_η denote the density of $\eta^{\frac{1}{2}}$ times a d-dimensional vector of independent standard Gaussians; cf. (4.6). We note that for all $\varepsilon > 0$ sufficiently small,

$$\mathbb{O}_{\varphi_\varepsilon \star \mu}(f_\eta) = (2\varepsilon)^{-N} \int f_\eta(X(s))\mu\big(\mathcal{B}(s,\varepsilon)\big)\, ds.$$

[The only reason for our insisting on the smallness of is to ensure that $\varphi_\varepsilon \star \mu \in \mathcal{P}(\mathbb{R}^N_+)$. Of course, here, "*small*" means "*small enough to ensure that $F^\varepsilon \subset \mathbb{R}^N_+$.*"] We will use this formula, and a covering argument, to obtain a simple bound.

For any compact set $K \subset \mathbb{R}^N_+$, and for all $\varepsilon > 0$, let $\mathcal{N}_K(\varepsilon)$ denote the minimum number of ℓ^∞-balls of radius ε needed to cover K. $\mathcal{N}_K$ is sometimes called the *metric entropy* of K. Plainly, if $\varepsilon > 0$ is fixed but sufficiently small, we can find $s^1, \ldots, s^{\mathcal{N}_F(\varepsilon)} \in \mathbb{R}^N_+$ such that $\mathcal{B}(s^j, \varepsilon) \subset \mathbb{R}^N_+$ and $\cup_{j=1}^{\mathcal{N}_F(\varepsilon)} \mathcal{B}(s^j,\varepsilon) \supseteq F$. Consequently, $\sum_{j=1}^{\mathcal{N}_F(\varepsilon)} \mu(\mathcal{B}(s^j,\varepsilon)) \ge \mu(F) = 1$. We can deduce the existence of a point $s^\star \in \mathbb{R}^N_+$ such that $\mu(\mathcal{B}(s^\star,\varepsilon)) \ge [\mathcal{N}_F(\varepsilon)]^{-1}$. [Warning: $s^\star$ may depend on ε.] On the other hand, for all $s \in \mathcal{B}(s^\star, \varepsilon)$, $\mathcal{B}(s, 2\varepsilon) \supseteq \mathcal{B}(s^\star, \varepsilon)$, thanks to the triangle inequality. In other words, we have shown that for

$$\inf_{s \in \mathcal{B}(s^\star,\varepsilon)} \mu\big(\mathcal{B}(s, 2\varepsilon)\big) \ge \frac{1}{\mathcal{N}_F(\varepsilon)}.$$

The display preceding the above, then, shows that

$$\begin{aligned}\mathbb{O}_{\varphi_{2\varepsilon} \star \mu}(f_\eta) &\ge \frac{1}{(4\varepsilon)^N \mathcal{N}_F(\varepsilon)} \int_{\mathcal{B}(s^\star,\varepsilon)} f_\eta(X(s))\, ds\\ &= \frac{1}{(4\varepsilon)^N \mathcal{N}_F(\varepsilon)}\ \mathbb{O}_{\lambda_N \rfloor_{\mathcal{B}(s^\star,\varepsilon)}}(f_\eta).\end{aligned}$$

In other words,

$$\mathbb{E}_{\lambda_d}\big\{\big|\mathbb{O}_{\varphi_{2\varepsilon}\star\mu}(f_\eta)\big|^2\big\} \ge \frac{1}{(4\varepsilon)^{2N}[\mathcal{N}_F(\varepsilon)]^2}\ \mathbb{E}_{\lambda_d}\big\{\big|\mathbb{O}_{\lambda_N \rfloor_{\mathcal{B}(s^\star,\varepsilon)}}(f_\eta)\big|^2\big\}.$$

We apply Lemma 3.5 to see that for any finite measure ν, on $\mathbb{R}^N_+$,

$$\mathbb{E}_{\lambda_d}\big\{\big|\mathbb{O}_\nu(f_\eta)\big|^2\big\} \uparrow \|\nu\|_e^2, \quad \text{as } \eta \downarrow 0. \tag{10.1}$$

Thus, we have shown that for all $\varepsilon > 0$ sufficiently small,

$$\|\varphi_{2\varepsilon} \star \mu\|_e^2 \ge \frac{1}{(4\varepsilon)^{2N}[\mathcal{N}_F(\varepsilon)]^2}\ \|\lambda_N \rfloor_{\mathcal{B}(s^\star,\varepsilon)}\|_e^2.$$

Since convolutions decrease the energy norm, the first term is bounded above by $\|\mu\|_e^2$; cf. Lemma 10.1. By shift invariance, we then conclude that for any ℓ^∞-ball B of radius ε,

$$\|\mu\|_e^2 \ge \frac{1}{(4\varepsilon)^{2N}[\mathcal{N}_F(\varepsilon)]^2}\ \|\lambda_N \rfloor_B\|_e^2,$$

as long as $\varepsilon > 0$ is sufficiently small. But $\| \bullet \|_e$ is a seminorm [Lemma 5.1]. Thus,

$$\begin{aligned}\|\lambda_N\rfloor_F\|_e &\leq \sum_{j=1}^{\mathcal{N}_F(\varepsilon)} \|\lambda_N\rfloor_{\mathcal{B}(s^j,\varepsilon)}\|_e \\ &\leq 4^N\varepsilon^N|\mathcal{N}_F(\varepsilon)|^2\cdot\|\mu\|_e, \qquad (10.2)\end{aligned}$$

for any $\varepsilon > 0$ that makes $F^\varepsilon \subset \mathbb{R}^N_+$. We now claim that this holds for all $\varepsilon > 0$, and remove the assumption that $F \subset (0,\infty)^N$ in one sweep.

For any real number $c > 0$, consider the set $\langle c\rangle + F = \{\langle c\rangle + s;\ s \in F\}$, where $\langle c\rangle$ is the N-vector all of whose coordinates are c. By shift invariance, $\lambda_N\rfloor_F$ and $\lambda_N\rfloor_{\langle c\rangle+F}$ have the same energy norm [Lemma 10.1]. Furthermore, $\mathcal{N}_{\langle c\rangle+F} = \mathcal{N}_F$. Thus, if we only know that $F \subset \mathbb{R}^N_+$, by considering $\langle c\rangle + F$ in place of F in (10.2), we arrive at the following:

$$\|\lambda_N\rfloor_F\|_e \leq 4^N\varepsilon^N|\mathcal{N}_F(\varepsilon)|^2\cdot\|\mu\|_e,$$

whenever $(\langle c\rangle + F)^\varepsilon \subset \mathbb{R}^N_+$, which holds when $c > \varepsilon$. Since the above inequality is independent of c, we can deduce that (10.2) holds for all compact sets $F \subset \mathbb{R}^N_+$, and for all $\varepsilon > 0$. Let ε equal the ℓ^∞-radius of F and note that, for this choice of ε, $\mathcal{N}_F(\varepsilon) = 1$, while $\varepsilon^N \leq 2^{-N}\sup_{t\in F}|t|^N$. □

Remark 10.4 Proposition 10.3 comes very close to showing that Theorem 1.1 holds without restrictions such as Condition (1.3). Indeed, suppose that $\mathbb{E}\{\lambda_d(X(\mathbb{R}^N_+))\} > 0$. Then, by Theorem 2.1, there exists a nonrandom $t \in \mathbb{R}^N_+$, and a $\mu \in \mathcal{P}([0,t])$ such that $\|\mu\|_e < +\infty$. Proposition 10.3, then, shows that $\|\lambda_N\rfloor_{[0,t]}\|_e < +\infty$. We now appeal to Theorem 1.3 to deduce the existence of a local time process, $\mathbb{R}^N_+ \ni t \mapsto \ell_t(\bullet) = L_{\lambda_N\rfloor[0,t]}(\bullet)$, such that for all $t \in \mathbb{R}^N_+$, and all bounded measurable $f : \mathbb{R}^d \to \mathbb{R}$, the following holds $\mathbb{P}$-a.s.:

$$\int_{[0,t]} f(X(s))\,ds = \int_{\mathbb{R}^d} f(a)\ell_t(a)\,da.$$

Moreover, $\ell_t \in L^2(\mathbb{R}^d)$, $\mathbb{P}$-almost surely. Theorem 1.1 can be shown to follow, if we could show that this fact would imply that $\int_{\mathbb{R}^N_+}\exp(-\sum_{j=1}^N t_j)\|\ell_t\|_{L^2(\mathbb{R}^d)}\,dt$ has a finite expectation. When $N = 1$, this follows from the strong Markov property. (Of course, when $N = 1$, Condition (1.3) holds tautologically.) However, when $N > 1$, we do not know if such a fact holds. □

Acknowledgment The authors would like to thank the referee for his/her careful reading of the paper and for several thoughtful comments that have led to several improvements of the manuscript.

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