Hausdorff Measure of the Sample Paths of Gaussian Random Fields *

Yimin Xiao

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1 Introduction

Let $Y(t)$ $(t \in \mathbf{R}^N)$ be a real-valued, centered Gaussian random field with $Y(0) = 0$. We assume that $Y(t)$ $(t \in \mathbf{R}^N)$ has stationary increments and continuous covariance function $R(t,s) = EY(t)Y(s)$ given by

$$(1.1) \qquad R(t,s) = \int_{\mathbf{R}^N} (e^{i<t,\lambda>} - 1)(e^{-i<s,\lambda>} - 1)\Delta(d\lambda) \ ,$$

where $< x, y >$ is the ordinary scalar product in $\mathbf{R}^N$ and $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\mathbf{R}^N \backslash \{0\}$ satisfying

$$(1.2) \qquad \int_{\mathbf{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2} \ \Delta(d\lambda) < \infty \ .$$

Then there exists a centered complex-valued Gaussian random measure $W(d\lambda)$ such that

$$(1.3) \qquad Y(t) = \int_{\mathbf{R}^N} (e^{i<t,\lambda>} - 1) W(d\lambda)$$

and for any Borel sets $A, \ B \subseteq \mathbf{R}^N$

$$E\left(W(A)\overline{W(B)}\right) = \Delta(A \cap B) \quad \text{and} \quad W(-A) = \overline{W(A)} \ .$$

It follows from (1.3) that

$$(1.4) \qquad E[(Y(t+h) - Y(t))^2] = 2\int_{\mathbf{R}^N} (1 - \cos < h, \lambda >) \ \Delta(d\lambda) \ .$$

We assume that there exist constants $\delta_0 > 0, \ 0 < c_1 \le c_2 < \infty$ and a non-decreasing, continuous function $\sigma : [0, \delta_0) \to [0, \infty)$ which is regularly varying at the origin with index α $(0 < \alpha < 1)$ such that for any $t \in \mathbf{R}^N$ and $h \in \mathbf{R}^N$ with $|h| \le \delta_0$

$$(1.5) \qquad E[(Y(t+h) - Y(t))^2] \le c_1 \sigma^2(|h|) \ .$$

and for all $t \in \mathbf{R}^N$ and any $0 < r \le \min\{|t|, \ \delta_0\}$

$$(1.6) \qquad Var(Y(t)|Y(s) : r \le |s - t| \le \delta_0) \ge c_2 \sigma^2(r) \ .$$

If (1.5) and (1.6) hold, we shall say that $Y(t)$ $(t \in \mathbf{R}^N)$ is strongly locally σ-nondeterministic. We refer to Monrad and Pitt[14], Berman [4] [5] and Cuzick and Du Peez [6] for more information on (strongly) locally nondeterminism.

We associate with $Y(t)$ $(t \in \mathbf{R}^N)$ a Gaussian random field $X(t)$ $(t \in \mathbf{R}^N)$ in $\mathbf{R}^d$ by

$$X(t) = (X_1(t), \cdots, X_d(t)) \ , \tag{1.7}$$

where $X_1, \cdots, X_d$ are independent copies of Y. The most important example of such Gaussian random fields is the fractional Brownian motion of index α (see Example 4.1 below).

It is well known (see [1], Chapter 8) that with probability 1

$$\dim X([0,1]^N) = \min(d, \frac{N}{\alpha}) \ .$$

The objective of this paper is to consider the exact Hausdorff measure of the image set $X([0,1]^N)$. The main result is the following theorem, which generalizes a theorem of Talagrand [22].

Theorem 1.1 *If $N < \alpha d$, then with probability 1*

$$0 < \phi\text{-}m(X([0,1]^N)) < \infty \ , \tag{1.8}$$

where

$$\phi(s) = \psi(s)^N \log\log \frac{1}{s} \ ,$$

ψ is the inverse function of σ and where ϕ-$m(X([0,1]^N))$ is the ϕ-Hausdorff measure of $X([0,1]^N)$.

If $N > \alpha d$, then by a result of Pitt [17], $X([0,1]^N)$ a. s. has interior points and hence has positive d-dimensional Lebesgue measure. In the case of $N = \alpha d$, the problem of finding ϕ-$m(X([0,1]^N))$ is still open even in the fractional Brownian motion case.

The paper is organized as follows. In Section 2 we recall the definition and some basic facts of Hausdorff measure, Gaussian processes and regularly varying functions. In Section 3 we prove the upper bound and in Section 4, we prove the lower bound for ϕ-$m(X([0,1]^N))$. We also give some examples showing that the hypotheses in Theorem 1.1 are satisfied by a large class of Gaussian random fields including fractional Brownian motion.

Another important example of Gaussian random fields is the Brownian sheet or N-parameter Wiener process $W(t)$ $(t \in \mathbf{R}_+^N)$, see Orey and Pruitt [16]. Since $W(t)$ $(t \in \mathbf{R}_+^N)$ is not locally nondeterministic, Theorem 1.1 does not apply. The problem of finding the exact Hausdorff measure of $W([0,1]^N)$ was solved by Ehm [7].

We will use K to denote an unspecified positive constant which may be different in each appearance.

2 Preliminaries

Let Φ be the class of functions $\phi : (0, \delta) \to (0, 1)$ which are right continuous, monotone increasing with $\phi(0+) = 0$ and such that there exists a finite constant $K > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \leq K, \quad \text{for } 0 < s < \frac{1}{2}\delta.$$

For $\phi \in \Phi$, the ϕ-Hausdorff measure of $E \subseteq \mathbf{R}^N$ is defined by

$$\phi\text{-}m(E) = \lim_{\epsilon \to 0} \inf \left\{ \sum_i \phi(2r_i) : E \subseteq \cup_{i=1}^\infty B(x_i, r_i),\ r_i < \epsilon \right\} ,$$

where $B(x, r)$ denotes the open ball of radius r centered at x. It is known that ϕ-m is a metric outer measure and every Borel set in $\mathbf{R}^N$ is ϕ-m measurable. The Hausdorff dimension of E is defined by

$$\begin{aligned} \dim E &= \inf\{\alpha > 0 : \quad s^\alpha\text{-}m(E) = 0\} \\ &= \sup\{\alpha > 0 : \quad s^\alpha\text{-}m(E) = \infty\} . \end{aligned}$$

We refer to [F] for more properties of Hausdorff measure and Hausdorff dimension.

The following lemma can be easily derived from the results in [18] (see [23]), which gives a way to get a lower bound for ϕ-$m(E)$. For any Borel measure μ on $\mathbf{R}^N$ and $\phi \in \Phi$, the upper ϕ-density of μ at $x \in \mathbf{R}^N$ is defined by

$$\overline{D}_\mu^\phi(x) = \limsup_{r \to 0} \frac{\mu(B(x, r))}{\phi(2r)} .$$

Lemma 2.1 *For a given $\phi \in \Phi$ there exists a positive constant K such that for any Borel measure μ on $\mathbf{R}^N$ and every Borel set $E \subseteq \mathbf{R}^N$, we have*

$$\phi\text{-}m(E) \geq K\mu(E) \inf_{x\in E}\{\overline{D}_\mu^\phi(x)\}^{-1} .$$

Now we summarize some basic facts about Gaussian processes. Let $Z(t)$ $(t \in S)$ be a Gaussian process. We provide S with the following metric

$$d(s,t) = \|Z(s) - Z(t)\|_2 ,$$

where $\|Z\|_2 = (E(Z^2))^{\frac{1}{2}}$. We denote by $N_d(S,\epsilon)$ the smallest number of open d-balls of radius ϵ needed to cover S and write $D = \sup\{d(s,t) : s,\ t \in S\}$.

The following lemma is well known. It is a consequence of the Gaussian isoperimetric inequality and Dudley's entropy bound([11], see also [22]).

Lemma 2.2 *There exists an absolute constant $K > 0$ such that for any $u > 0$, we have*

$$P\Big\{\sup_{s,\ t\in S}|Z(s) - Z(t)| \geq K(u + \int_0^D \sqrt{\log N_d(S,\epsilon)}d\epsilon)\Big\} \leq exp\Big(-\frac{u^2}{D^2}\Big) .$$

Lemma 2.3 *Consider a function Ψ such that $N_d(S,\epsilon) \leq \Psi(\epsilon)$ for all $\epsilon > 0$. Assume that for some constant $C > 0$ and all $\epsilon > 0$ we have*

$$\Psi(\epsilon)/C \leq \Psi(\frac{\epsilon}{2}) \leq C\Psi(\epsilon) .$$

Then

$$P\{\sup_{s,\ t\in S}|Z(s) - Z(t)| \leq u\} \geq exp\Big(-K\Psi(u)\Big) ,$$

where $K > 0$ is a constant depending only on C.

This is proved in [21]. It gives an estimate for the lower bound of the small ball probability of Gaussian processes. Similar problems have also been considered by Monrad and Rootzén [15] and by Shao [20].

We end this section with some lemmas about regularly varying functions. Let $\sigma(s)$ be a regularly varying function with index α $(0 < \alpha < 1)$. Then σ can be written as

$$\sigma(s) = s^\alpha L(s) ,$$

where $L(s) : [0, \delta_0) \to [0, \infty)$ is slowly varying at the origin in the sense of Karamata and hence can be represented by

$$L(s) = exp\Big(\eta(s) + \int_s^A \frac{\epsilon(t)}{t}\, dt\Big) \ , \tag{2.1}$$

where $\eta(s) : [0, \delta_0] \to \mathbf{R}$, $\epsilon(s) : (0, A] \to \mathbf{R}$ are bounded measurable functions and

$$\lim_{s\to 0} \eta(s) = c \ , \ |c| < \infty; \quad \lim_{s\to 0} \epsilon(s) = 0 \ .$$

In the following, Lemma 2.4 is an easy consequence of (2.1) and Lemma 2.5 can be deduced from Theorem 2.6 and 2.7 in Seneta [19] directly.

Lemma 2.4 *Let $L(s)$ be a slowly varying function at the origin and let $U = U(s) : [0, \infty) \to [0, \infty)$ satisfying*

$$\lim_{s\to 0} U(s) = \infty \quad \textit{and} \quad \lim_{s\to 0} sU(s) = 0 \ .$$

Then for any $\epsilon > 0$, as s small enough we have

$$U(s)^{-\epsilon} L(s) \le L(sU(s)) \le U(s)^{\epsilon} L(s)$$

and

$$U(s)^{-\epsilon} L(s) \le L(sU(s)^{-1}) \le U(s)^{\epsilon} L(s) \ .$$

Lemma 2.5 *Let σ be a regularly varying function at the origin with index $\alpha > 0$. Then there is a constant $K > 0$ such that for $r > 0$ small enough, we have*

$$\int_1^\infty \sigma(re^{-u^2})\ du \le K\sigma(r) \ , \tag{2.2}$$

$$\int_0^1 \sigma(rs)\ ds \le K\sigma(r) \ , \tag{2.3}$$

$$\int_0^1 \sigma(rs)s^{N-1}\ ds \le K\sigma(r) \ , \tag{2.4}$$

Let $\sigma : [0, \delta_0) \to [0, \infty)$ be non-decreasing and let ψ be the inverse function of σ, that is

$$\psi(s) = \inf\{t \ge 0 : \sigma(t) \ge s\} \ .$$

then $\psi(s) = s^{1/\alpha} f(s)$, where $f(s)$ is also a slowly varying function and

$$\sigma(\psi(s)) \sim s \quad \text{and} \quad \psi(\sigma(s)) \sim s \quad \text{as} \quad s \to 0 \ . \tag{2.5}$$

3 Upper bound for ϕ-$m(X([0,1]^N))$

Let $Y(t)$ $(t \in \mathbf{R}^N)$ be a real-valued, centered Gaussian random field with stationary increments and a continuous covariance function $R(t,s)$ given by (1.1). We assume that $Y(0)=0$ and (1.5) holds. Let $X(t)$ $(t \in \mathbf{R}^N)$ be the (N,d) Gaussian random field defined by (1.7).

We start with the following lemma.

Lemma 3.1 *Let $Y(t)$ $(t \in \mathbf{R}^N)$ be a Gaussian process with $Y(0)=0$ satisfying (1.5). Then*

(i) For any $r>0$ small enough and $u \geq K\sigma(r)$, we have

$$P\Big\{\sup_{|t|\leq r} |Y(t)| \geq u\Big\} \leq exp\Big(-\frac{u^2}{K\ \sigma^2(r)}\Big) \ . \tag{3.1}$$

(ii) Let

$$\omega_Y(h) = \sup_{t,\ t+s\in[0,1]^N,\ |s|\leq h} |Y(t+s)-Y(t)|$$

be the uniform modulus of continuity of $Y(t)$ on $[0,1]^N$. Then

$$\limsup_{h\to 0} \frac{\omega_Y(h)}{\sigma(h)\sqrt{2c_1\log\frac{1}{h}}} \leq 1 \ , \quad a.\ s. \tag{3.2}$$

Proof. Let $r<\delta_0$ and $S=\{t:\ |t|\leq r\}$. Since $d(s,t)\leq c_1\sigma(|t-s|)$, we have

$$N_d(S,\epsilon) \leq K\Big(\frac{r}{\psi(\epsilon)}\Big)^N$$

and

$$D = \sup\{d(s,t); s,\ t\in S\} \leq K\sigma(r) \ .$$

By simple calculations

$$\begin{aligned}
\int_0^D \sqrt{\log N_d(S,\epsilon)}\ d\epsilon &\leq K \int_0^{K\sigma(r)} \sqrt{\log(Kr)/\psi(\epsilon)}\ d\epsilon \\
&\leq K \int_0^{Kr} \sqrt{\log(Kr)/t}\ \ d\sigma(t) \\
&\leq K \left(\sigma(r) + \int_0^K \frac{1}{u\sqrt{\log K/u}}\ \sigma(ur)du\right) \\
&\leq K \left(\sigma(r) + \int_K^\infty \sigma(re^{-u^2})du\right) \\
&\leq K\sigma(r) \ ,
\end{aligned}$$

where the last inequality follows from (2.2). If $u \geq K\sigma(r)$, then by Lemma 2.2 we have

$$\begin{aligned} & P\Big\{\sup_{|t|\leq r}\ |Y(t)| \geq 2K\ u\Big\} \\ \leq\ & P\Big\{\sup_{|t|\leq r}|Y(t)| \geq K(u+\int_0^D \sqrt{\log N_d(S,\epsilon)}\ d\epsilon)\Big\} \\ \leq\ & exp\Big(-\frac{u^2}{K\sigma^2(r)}\Big)\ . \end{aligned}$$

This proves (3.1). The inequality (3.2) can be derived from Lemma 2.2 directly in a standard way (see also [13]).

In order to get the necessary independence, we will make use of the spectral representation (1.3). Given $0 < a < b < \infty$, we consider the process

$$Y(a,b,t) = \int_{a\leq|t|\leq b}(e^{i<t,\lambda>} - 1)W(d\lambda)\ .$$

Then for any $0 < a < b < a' < b' < \infty$, the processes $Y(a,b,t)$ and $Y(a',b',t)$ are independent. The next lemma expresses how well $Y(a,b,t)$ approximates $Y(t)$.

Lemma 3.2 *Let $Y(t)$ ($t \in \mathbf{R}^N$) be defined by (1.3). If (1.5) holds, then there exists a constant $B > 0$ such that for any $B < a < b$ we have*

$$(3.3) \qquad \|Y(a,b,t) - Y(t)\|_2 \leq K\left[|t|^2a^2\sigma^2(a^{-1}) + \sigma^2(b^{-1})\right]^{\frac{1}{2}}\ .$$

Proof. First we claim that for any $u > 0$ and any $h \in \mathbf{R}^N$ with $|h| = 1/u$ we have

$$(3.4) \qquad \int_{|\lambda|<u} < h,\lambda >^2 \Delta(d\lambda) \leq K\int_{\mathbf{R}^N}(1-\cos < h,\lambda >)\Delta(d\lambda)$$

$$(3.5) \quad \int_{|\lambda|\geq u} \Delta(d\lambda) \leq K(\frac{u}{2})^N\int_{[-1/u,1/u]^N} dv\int_{\mathbf{R}^N}(1-\cos < v,\lambda >)\Delta(d\lambda)\ .$$

For $N = 1$, (3.4) and (3.5) are the truncation inequalities in [12] p209. For $N > 1$ a similar proof yields (3.4) and (3.5).

Now for any $a > \delta_0^{-1}$ and any $t \in \mathbf{R}^N\backslash\{0\}$, by (1.4) ,(1.5) and (3.4) we have

$$(3.6) \qquad \int_{|\lambda|<a}(1-\cos < t,\lambda >)\Delta(d\lambda) \leq \int_{|\lambda|<a} < t,\lambda >^2 \Delta(d\lambda)$$

$$= |t|^2 a^2 \int_{|\lambda|<a} \ < t/(a|t|), \lambda >^2 \ \Delta(d\lambda) \le K|t|^2 a^2 \sigma^2(a^{-1}) \ .$$

For $b > 0$ large enough, by (3.5), (1.4), (1.5) and (2.4) we have

$$\text{(3.7)} \qquad \int_{|\lambda|\ge b} \ \Delta(d\lambda) \le K(\frac{b}{2})^N \int_{[-1/b,1/b]^N} \ \sigma^2(|v|)dv$$

$$\le Kb^N \int_0^{\sqrt{N}b^{-1}} \ \sigma^2(\rho)\rho^{N-1}d\rho \ \le K\sigma^2(b^{-1}) \ .$$

Combining (3.6) and (3.7), we see that there exists a constant $B > 0$ such that $B < a < b$ implies

$$E\Big[(Y(a,b,t) - Y(t))^2\Big] = 2\int_{\{|\lambda|<a\}\cup\{|\lambda|>b\}} (1 - \cos < t, \lambda >) \ \Delta(d\lambda)$$

$$\le \ 2\int_{|\lambda|<a} (1 - \cos < t, \lambda >) \ \Delta(d\lambda) + 2\int_{|\lambda|>b} \ \Delta(d\lambda)$$

$$\le \ K\Big[|t|^2 a^2 \sigma^2(a^{-1}) + \sigma^2(b^{-1})\Big] \ .$$

This proves (3.3).

Lemma 3.3 *There exists a constant $B > 0$ such that for any $B < a < b$ and $0 < r < B^{-1}$ the following holds: let $A = r^2 a^2 \sigma^2(a^{-1}) + \sigma^2(b^{-1})$ such that $\psi(\sqrt{A}) \le \frac{1}{2}r$, then for any*

$$u \ge K\left(A \log \frac{Kr}{\psi(\sqrt{A})}\right)^{\frac{1}{2}}$$

we have

$$\text{(3.8)} \qquad P\Big\{\sup_{|t|\le r} |Y(t) - Y(a,b,t)| \ge u\Big\} \le exp\Big(-\frac{u^2}{KA}\Big) \ .$$

Proof. Let $S = \{t : \ |t| \le r\}$ and $Z(t) = Y(t) - Y(a,b,t)$. Then

$$d(s,t) = \|Z(t) - Z(s)\|_2 \le c_1\sigma(|t-s|) \ .$$

Hence

$$N_d(S,\epsilon) \le K\Big(\frac{r}{\psi(\epsilon)}\Big)^N \ .$$

By Lemma 3.2 we have $D \le K\sqrt{A}$. As in the proof of Lemma 3.1,

$$\begin{aligned}
&\int_0^D \sqrt{\log N_d(S,\epsilon)}\; d\epsilon \le K \int_0^{K\sqrt{A}} \sqrt{\log(Kr)/\psi(\epsilon)}\; d\epsilon \\
&\le K \int_0^{K\psi(\sqrt{A})/r} \sqrt{\log K/t}\; d\sigma(rt) \\
&\le K \left[\sqrt{\log K/t}\; \sigma(rt)|_0^{K\psi(\sqrt{A})/r} + \int_0^{K\psi(\sqrt{A})/r} \frac{1}{t\sqrt{\log K/t}}\; \sigma(rt)dt\right] \\
&\le K \sqrt{A \log Kr/\psi(\sqrt{A})} + K \int_{\sqrt{\log Kr/\psi(\sqrt{A})}}^{\infty} \sigma(Kre^{-u^2})du \\
&\le K\sqrt{A \log Kr/\psi(\sqrt{A})}\ ,
\end{aligned}$$

at least for $r > 0$ small enough, where the last step follows from (2.2). Hence (3.8) follows immediately from Lemma 2.2.

Let $X_1(a,b,t), \cdots, X_d(a,b,t)$ be independent copies of $Y(a,b,t)$ and let

$$X(a,b,t) = (X_1(a,b,t), \cdots, X_d(a,b,t)) \quad (t \in \mathbf{R}^N)\ .$$

Then we have the following corollary of Lemma 3.3.

Corollary 3.1 *Consider $B < a < b$ and $0 < r < B^{-1}$. Let*

$$A = r^2 a^2 \sigma^2(a^{-1}) + \sigma^2(b^{-1})$$

with $\psi(\sqrt{A}) \le \frac{1}{2} r$. Then for any

$$u \ge K\Big(A \log \frac{Kr}{\psi(\sqrt{A})}\Big)^{\frac{1}{2}}$$

we have

$$\text{(3.9)} \qquad P\Big\{\sup_{|t|\le r} |X(t) - X(a,b,t)| \ge u\Big\} \le exp\Big(-\frac{u^2}{KA}\Big)\ .$$

Lemma 3.4 *Given $0 < r < \delta_0$ and $\epsilon < \sigma(r)$. Then for any $0 < a < b$ we have*

$$\text{(3.10)} \qquad P\Big\{\sup_{|t|\le r} |X(a,b,t)| \le \epsilon\Big\} \ge exp\Big(-\frac{r^N}{K\psi(\epsilon)^N}\Big)\ .$$

Proof. It is sufficient to prove (3.10) for $Y(a,b,t)$. Let $S = \{t : |t| \le r\}$ and define a distance d on S by

$$d(s,t) = \|Y(a,b,t) - Y(a,b,s)\|_2 \ .$$

Then $d(s,t) \le c_1\sigma(|t-s|)$ and

$$N_d(S,\epsilon) \le K\Big(\frac{r}{\psi(\epsilon)}\Big)^N \ .$$

By Lemma 2.3 we have

$$P\Big\{\sup_{|t|\le r}|Y(a,b,t)| \le \epsilon\Big\} \ge exp\Big(-\frac{r^N}{K\psi(\epsilon)^N}\Big) \ .$$

This proves lemma 3.4.

Proposition 3.1 *There exists a constant $\delta_1 > 0$ such that for any $0 < r_0 \le \delta_1$, we have*

$$(3.11) \qquad P\Big\{\exists r \in [r_0^2,\ r_0] \ \ \textit{such that} \ \sup_{|t|\le r} |X(t)| \le K\sigma(r(\log\log\frac{1}{r})^{-\frac{1}{N}})\Big\}$$

$$\ge 1 - exp\Big(-(\log\frac{1}{r_0})^{\frac{1}{2}}\Big) \ .$$

Proof. We follow the line of Talagrand [22]. Let $U = U(r_0) \ge 1$, where $U(r)$ satisfying

$$(3.12) \qquad U(r) \to \infty \ \text{ as } \ r \to 0$$

and for any $\epsilon > 0$

$$(3.13) \qquad r^\epsilon U(r) \to 0 \ \text{ as } \ r \to 0 \ ,$$

will be chosen later. For $k \ge 0$, let $r_k = r_0U^{-2k}$. Let k_0 be the largest integer such that

$$k_0 \le \frac{\log\frac{1}{r_0}}{2\log U} \ ,$$

then for any $0 \le k \le k_0$ we have $r_0^2 \le r_k \le r_0$. In order to prove (3.11), it suffices to show that

$$(3.14) \qquad P\Big\{\exists k \le k_0 \ \text{ such that } \ \sup_{|t|\le r_k} |X(t)| \le K\sigma(r_k(\log\log\frac{1}{r_k})^{-\frac{1}{N}})\Big\}$$

$$\geq 1 - exp\Big(-(\log \frac{1}{r_0})^{\frac{1}{2}}\Big) \ .$$

Let $a_k = r_0^{-1} U^{2k-1}$ and we define for $k = 0,\ 1, \cdots$

$$X_k(t) = X(a_k, a_{k+1}, t) \ ,$$

then $X_0,\ X_1, \cdots$ are independent. By Lemma 3.4 we can take a constant K_1 such that for $r_0 > 0$ small enough

$$\text{(3.15)} \qquad P\Big\{\sup_{|t|\leq r_k} |X_k(t)| \leq K_1 \sigma(r_k(\log\log \frac{1}{r_k})^{-\frac{1}{N}})\Big\}$$
$$\geq exp(-\frac{1}{4}\log\log\frac{1}{r_k})$$
$$= \frac{1}{(\log\frac{1}{r_k})^{\frac{1}{4}}} \ .$$

Thus, by independence we have

$$\text{(3.16)} \qquad P\Big\{\exists k \leq k_0,\ \sup_{|t|\leq r_k} |X_k(t)| \leq K_1 \sigma(r_k(\log\log \frac{1}{r_k})^{-1/N})\Big\}$$
$$\geq 1 - \Big(1 - \frac{1}{(2\log 1/r_0)^{1/4}}\Big)^{k_0}$$
$$\geq 1 - exp\Big(-\frac{k_0}{(2\log 1/r_0)^{1/4}}\Big) \ .$$

Let

$$\begin{aligned} A_k &= r_k^2 a_k^2 \sigma^2(a_k^{-1}) + \sigma^2(a_{k+1}^{-1}) \\ &= U^{-2+2\alpha} r_k^{2\alpha} L^2(r_k U) + U^{-2\alpha} r_k^{2\alpha} L^2(r_k/U) \ . \end{aligned}$$

Let $\beta = 2\min\{1-\alpha, \alpha\}$ and fix an $\epsilon < \frac{1}{2}\beta$. Then by Lemma 2.4, we see that as r_0 small enough

$$U^{-\beta-\epsilon}\sigma^2(r_k) \leq A_k \leq U^{-\beta+\epsilon}\sigma^2(r_k) \ .$$

Notice that for r_0 small enough we have

$$\begin{aligned} \psi(\sqrt{A_k}) &\geq \psi(U^{-(\beta+\epsilon)/2}\sigma(r_k)) \\ &= (U^{-\beta/2}\sigma(r_k))^{1/\alpha} f(U^{-\beta/2}\sigma(r_k)) \\ &= U^{-\beta/(2\alpha)} r_k L(r_k)^{1/\alpha} f(U^{-\beta/2}\sigma(r_k)) \\ &\geq K U^{-(\beta+\epsilon)/(2\alpha)} r_k \ , \end{aligned}$$

the last inequality follows from (2.5). It follows from Corollary 3.1 that for

$$u \geq K\sigma(r_k)U^{-\frac{\beta-\epsilon}{2}}(\log U)^{1/2} ,$$

we have

$$P\Big\{\sup_{|t|\leq r_k} |X(t) - X_k(t)| \geq u\Big\} \leq \exp\Big(-\frac{u^2 U^{\beta-\epsilon}}{K\sigma^2(r_k)}\Big) . \tag{3.17}$$

Hence, if we take

$$U = (\log 1/r_0)^{\frac{1}{\beta-\epsilon}} ,$$

then as r_0 small enough

$$\sigma(r_k)U^{-\frac{\beta-\epsilon}{2}}(\log U)^{1/2} \leq \sigma(r_k(\log\log\frac{1}{r_0})^{-\frac{1}{N}}) .$$

Hence by taking

$$u = \frac{K_1}{2}\sigma(r_k(\log\log\frac{1}{r_0})^{-\frac{1}{N}})$$

in (3.17), we obtain

(3.18)

$$P\Big\{\sup_{|t|\leq r_k} |X(t) - X_k(t)| \geq \frac{K_1}{2}\sigma(r_k(\log\log\frac{1}{r_0})^{-\frac{1}{N}})\Big\} \leq \exp\Big(-\frac{u^2 U^{\beta-\epsilon}}{K\sigma^2(r_k)}\Big) .$$

Combining (3.16) and (3.18) we have

$$P\Big\{\exists k \leq k_0 \text{ such that } \sup_{|t|\leq r_k} |X(t)| \leq 2K_1\sigma(r_k(\log\log\frac{1}{r_k})^{-1/N})\Big\} \tag{3.19}$$

$$\geq 1 - exp\Big(-\frac{k_0}{2(\log 1/r_0)^{1/4}}\Big) - k_0\exp\Big(-\frac{U^{\beta-\epsilon}}{K(\log\log 1/r_0)^{(2\alpha)/N+\epsilon}}\Big) .$$

We recall that

$$\frac{\log\frac{1}{r_0}}{4\log U} \leq k_0 \leq \log\frac{1}{r_0} .$$

and hence for r_0 small enough, (3.11) follows from (3.19).

Now we are in a position to prove the upper bound for ϕ-$m(X([0,1]^N))$.

Theorem 3.1 *Let* $\phi(s) = \psi(s)^N \log\log\frac{1}{s}$. *Then with probability 1*

$$\phi\text{-}m(X([0,1]^N)) < \infty .$$

Proof. For $k \geq 1$, consider the set

$$R_k = \Big\{ t \in [0,1]^N : \exists r \in [2^{-2k}, 2^{-k}] \text{ such that } \sup_{|s-t|\leq r} |X(s) - X(t)| \leq K\sigma(r(\log\log\frac{1}{r})^{-1/N}) \Big\} .$$

By Proposition 3.1 we have

$$P\{t \in R_k\} \geq 1 - \exp(-\sqrt{k/2}) .$$

Denote the Lebesgue measure in $\mathbf{R}^N$ by L_N. It follows from Fubini's theorem that $P(\Omega_0) = 1$, where

$$\Omega_0 = \{\omega : L_N(R_k) \geq 1 - \exp(-\sqrt{k/4}) \text{ infinitely often}\}.$$

To see this, let $\Omega_k = \{\omega : L_N(R_k) \geq 1 - \exp(-\sqrt{k/4})\ \}$. Then

$$\Omega_0 = \limsup_{k\to\infty} \Omega_k.$$

We also define $A_k = \{(t,\omega) :\ t \in R_k(\omega)\}$ and $Y_k(\omega) = L_N(\{t : (t,\omega) \in A_k\})$. Then

$$E(Y_k) = P \otimes L_N(A_k) \geq 1 - \exp(-\sqrt{k/2}).$$

For simplicity, write $a_k = 1 - \exp(-\sqrt{k/4}\ )$. It follows from Fubini's theorem that

$$\begin{aligned} P\{\Omega_k\} &= P\{Y_k \geq a_k\} \\ &= 1 - P\{Y_k < a_k\} \\ &= 1 - P\{1 - Y_k > 1 - a_k\} \\ &\geq 1 - \frac{E(1 - Y_k)}{1 - a_k} \\ &= 1 - \frac{1}{1 - a_k} + \frac{E(Y_k)}{1 - a_k} \\ &\geq 1 - \frac{1}{1 - a_k} + \frac{1 - e^{-\sqrt{k/2}}}{1 - a_k} \\ &= 1 - \frac{e^{-\sqrt{k/2}}}{1 - a_k} \\ &\geq 1 - \exp\Big(- \sqrt{k}(\frac{1}{\sqrt{2}} - \frac{1}{2})\Big). \end{aligned}$$

Therefore we have

$$P(\limsup_{k\to\infty}\Omega_k) \geq \lim_{k to \infty} P\{\Omega_k\} = 1.$$

On the other hand, by Lemma 3.1 ii), there exists an event Ω_1 such that $P(\Omega_1) = 1$ and for all $\omega \in \Omega_1$, there exists $n_1 = n_1(\omega)$ large enough such that for all $n \geq n_1$ and any dyadic cube C of order n in $\mathbf{R}^N$, we have

$$\sup_{s,t\in C} |X(t) - X(s)| \leq K\sigma(2^{-n})\sqrt{n}\ . \tag{3.20}$$

Now fix an $\omega \in \Omega_0 \cap \Omega_1$, we show that ϕ-$m(X([0,1]^N)) < \infty$. Consider $k \geq 1$ such that

$$L_N(R_k) \geq 1 - exp(-\sqrt{k/4})\ .$$

For any $x \in R_k$ we can find n with $k \leq n \leq 2k + k_0$ (where k_0 depends on N only) such that

$$\sup_{s,t\in C_n(x)} |X(t) - X(s)| \leq K\sigma(2^{-n}(\log\log 2^n)^{-1/N})\ , \tag{3.21}$$

where $C_n(x)$ is the unique dyadic cube of order n containing x. Thus we have

$$R_k \subseteq V = \cup_{n=k}^{2k+k_0} V_n$$

and each V_n is a union of dyadic cubes C_n of order n for which (3.21) holds. Clearly $X(C_n)$ can be covered by a ball of radius

$$\rho_n = K\sigma(2^{-n}(\log\log 2^n)^{-1/N})\ .$$

Since $\phi(2\rho_n) \leq K2^{-nN} = KL_N(C_n)$, we have

$$\sum_n \sum_{C\in V_n} \phi(2\rho_n) \leq \sum_n \sum_{C\in V_n} KL_N(C_n) \tag{3.22}$$

$$= KL_N(V) < \infty\ .$$

On the other hand, $[0,1]^N \backslash V$ is contained in a union of dyadic cubes of order $q = 2k + k_0$, none of which meets R_k. There can be at most

$$2^{Nq} L_N([0,1]^N \backslash V) \leq K 2^{Nq} exp(-\sqrt{k}/4)$$

such cubes. For each of these cubes, $X(C)$ is contained in a ball of radius $\rho = K\sigma(2^{-q})\sqrt{q}$. Thus for any $\epsilon > 0$

$$(3.23) \qquad \sum \phi(2\rho) \leq K 2^{Nq} exp(-\sqrt{k}/4) 2^{-Nq} q^{N/(2\alpha)+\epsilon} \leq 1$$

for k large enough. Since k can be arbitrarily large, Theorem 3.1 follows from (3.22) and (3.23).

4 Lower bound for ϕ-$m(X([0,1]^N))$

Let $Y(t)$ $(t \in \mathbf{R}^N)$ be a real-valued, centered Gaussian random field with stationary increments and a continuous covariance function $R(t,s)$ given by (1.1). We assume that $Y(0) = 0$ and (1.6) holds. Let $X(t)$ $(t \in \mathbf{R}^N)$ be the (N,d) Gaussian random field defined by (1.7). In this section, we prove that if $N < \alpha d$,, then

$$\phi\text{-}m(X([0,1]^N)) > 0 \quad a.\ s.$$

For simplicity we assume $\delta_0 = 1$ and let $I = [0,1]^N \cap B(0,1)$ (otherwise we consider a smaller cube). For any $0 < r < 1$ and $y \in \mathbf{R}^d$. let

$$T_y(r) = \int_I 1_{B(y,r)}(X(t))dt$$

be the sojourn time of $X(t)$ $(t \in I)$ in the open ball $B(y,r)$. If $y = 0$, we write $T(r)$ for $T_0(r)$.

Proposition 4.1 *There exist* $\delta_2 > 0$ *and* $b > 0$ *such that for any* $0 < r < \delta_2$

$$(4.1) \qquad E\Big(exp(b\psi(r)^{-N}T(r))\Big) \leq K < \infty\ .$$

Proof. We first prove that there exists a constant $0 < K < \infty$ such that for any $n \geq 1$

$$(4.2) \qquad E(T(r))^n \leq K^n n! \psi(r)^{Nn}\ .$$

For $n = 1$, by (2.4) and (2.5) we have

$$(4.3) \qquad ET(r) = \int_I P\{X(t) \in B(0,r)\}dt$$

$$
\begin{aligned}
&\leq \int_I \min\{1, K(\frac{r}{\sigma(|t|)})^d\}\ dt \\
&\leq K\int_0^1 \ \min\{1, \frac{Kr^d}{\sigma(\rho)^d}\}\ \rho^{N-1} d\rho \\
&\leq K\int_0^{K\psi(r)} \ \rho^{N-1} d\rho + K\int_{K\psi(r)}^1 \ \frac{r^d\rho^{N-1}}{\sigma(\rho)^d}\ d\rho \\
&\leq K\psi(r)^N + Kr^d\psi(r)^{N-\alpha d}\int_1^\infty \ \frac{1}{t^{1+\alpha d-N}L(\psi(r)t)^d}\ dt \\
&\leq K\psi(r)^N + Kr^d\psi(r)^{N-\alpha d}/L(\psi(r))^d \\
&\leq K\psi(r)^N\ .
\end{aligned}
$$

For $n \geq 2$

$$
E(T(r)^n) = \int_{I^n} P\{|X(t_1)| < r, \cdots, |X(t_n)| < r\} dt_1 \cdots dt_n\ . \tag{4.4}
$$

Consider $t_1, \cdots, t_n \in I$ satisfying

$$
t_j \neq 0 \quad \text{for} \ \ j = 1, \cdots, n, \quad t_j \neq t_k \quad \text{for} \quad j \neq k\ .
$$

Let $\eta = \min\{|t_n|,\ |t_n - t_i|,\ i = 1,\ \cdots, n-1\}$. Then by (1.6) we have

$$
Var(X(t_n)|X(t_1), \cdots, X(t_{n-1})) \geq c_2\sigma^2(\eta)\ . \tag{4.5}
$$

Since conditional distributions in Gaussian processes are still Gaussian, it follows from (4.5) that

$$
\begin{aligned}
&P\{|X(t_n)| < r | X(t_1) = x_1, \cdots, X(t_{n-1}) = x_{n-1}\} \\
&\qquad \leq K\int_{|u|<r} \frac{1}{\sigma(\eta)^d} exp\Big(-\frac{|u|^2}{K\sigma^2(\eta)}\Big) du\ .
\end{aligned} \tag{4.6}
$$

Similar to (4.3), we have

$$
\int_I dt_n \int_{|u|<r} \frac{1}{\sigma(\eta)^d} exp\Big(-\frac{|u|^2}{K\sigma^2(\eta)}\Big) du \tag{4.7}
$$

$$
\begin{aligned}
&\leq \int_I \min\{1,\ K(\frac{r}{\sigma(\eta)})^d\} dt_n \\
&\leq K\int_I \sum_{i=0}^{n-1} \min\{1,\ K(\frac{r}{\sigma(|t_n - t_i|)})^d\}\ dt_n \qquad (t_0 = 0) \\
&\leq Kn\int_0^1 \ \min\{1,\ \frac{Kr^d}{\sigma(\rho)^d}\} \rho^{N-1}\ d\rho \\
&\leq Kn\psi(r)^N\ .
\end{aligned}
$$

By (4.4), (4.6) and (4.7), we obtain

$$\begin{aligned} E(T(r))^n &\leq K\int_{I^{n-1}} P\{|X_1(t_1)|<r,\cdots,|X(t_{n-1})|<r\}dt_1\cdots dt_{n-1} \\ &\qquad \cdot\int_I dt_n\int_{|u|<r}\frac{1}{\sigma(\eta)^d}exp\Big(-\frac{|u|^2}{K\sigma^2(\eta)}\Big)du \\ &\leq Kn\psi(r)^N E(T(r))^{n-1}\ . \end{aligned}$$

Hence, the inequality (4.2) follows from (4.3) and induction. Let $0 < b < 1/K$, then by (4.2) we have

$$Eexp(b\psi(r)^{-N}T(r)) = \sum_{n=0}^{\infty}(Kb)^n < \infty\ .$$

This proves (4.1).

Proposition 4.2 *With probability 1*

$$\limsup_{r\to 0}\frac{T(r)}{\phi(r)} \leq \frac{1}{b}\ , \tag{4.8}$$

where $\phi(r) = \psi(r)^N\log\log 1/r$.

Proof. For any $\epsilon > 0$, it follows from (4.1) that

$$P\{T(r) \geq (1/b+\epsilon)\psi(r)^N\log\log 1/r\} \leq \frac{K}{(\log 1/r)^{1+b\epsilon}}\ . \tag{4.9}$$

Take $r_n = exp(-n/\log n)$, then by (4.9) we have

$$P\{T(r_n) \geq (1/b+\epsilon)\psi(r_n)^N\log\log 1/r_n\} \leq \frac{K}{(n/\log n)^{1+b\epsilon}}\ .$$

Hence by Borel-Cantelli lemma we have

$$\limsup_{n\to\infty}\frac{T(r_n)}{\phi(r_n)} \leq \frac{1}{b}+\epsilon\ . \tag{4.10}$$

It is easy to verify that

$$\lim_{n\to\infty}\frac{\phi(r_n)}{\phi(r_{n+1})} = 1\ . \tag{4.11}$$

Hence by (4.10) and (4.11) we have

$$\limsup_{r\to 0}\frac{T(r)}{\phi(r)} \leq \frac{1}{b}+\epsilon\ .$$

Since $\epsilon > 0$ is arbitrary, we obtain (4.8).

Since $X(t)$ $(t\in\mathbf{R}^N)$ has stationary increments, we derive the following

Corollary 4.1 *Fix $t_0 \in I$, then with probability 1*

$$\limsup_{r\to 0} \frac{T_{X(t_0)}(r)}{\phi(r)} \le \frac{1}{b} \ .$$

Theorem 4.1 *If $N < \alpha d$, then with probability 1*

$$\phi\text{-}m(X([0,1]^N)) > 0 \ , \tag{4.12}$$

where $\phi(r) = \psi(r)^N \log\log 1/r$.

Proof. We define a random Borel measure μ on $X(I)$ as follows. For any Borel set $B \subseteq \mathbf{R}^d$, let

$$\mu(B) = L_N\{t \in I,\ X(t) \in B\} \ .$$

Then $\mu(\mathbf{R}^d) = \mu(X(I)) = L_N(I)$. By Corollary 4.1, for each fixed $t_0 \in I$, with probability 1

$$\limsup_{r\to 0} \ \frac{\mu(B(X(t_0),r))}{\phi(r)} \tag{4.13}$$

$$\le \limsup_{r\to 0} \ \frac{T_{X(t_0)}(r)}{\phi(r)} \le \frac{1}{b} \ .$$

Let $E(\omega) = \{X(t_0) :\ \ t_0 \in I$ and (4.13) holds $\}$. Then $E(\omega) \subseteq X(I)$. A Fubini argument shows $\mu(E(\omega)) = 1,\ a.\ s.$. Hence by Lemma 2.1, we have

$$\phi\text{-}m(E(\omega)) \ge Kb \ .$$

This proves (4.12).

Proof of Theorem 1.1. It follows from Theorems 3.1 and 4.1 immediately.

Example 4.1. Let $Y(t)$ $(t \in \mathbf{R}^N)$ be a real-valued fractional Brownian motion of index α $(0 < \alpha < 1)$ (see [10], Chapter 18). Its covariance function has the representation

$$\begin{aligned} R(s,t) &= \frac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha}) \\ &= c(\alpha)\int_{\mathbf{R}^N} (e^{i<t,\lambda>} - 1)(e^{-i<s,\lambda>} - 1)\frac{d\lambda}{|\lambda|^{N+2\alpha}} \ , \end{aligned}$$

where $c(\alpha)$ is a normalizing constant. Then (1.5) is verified and by a result of Pitt [17], (1.6) is also verified. In this case, Theorem 1.1 is proved by Goldman [9] for $\alpha = 1/2$ and by Talagrand [22] for $0 < \alpha < 1$.

Example 4.2. Let $Z(t)$ $(t \in \mathbf{R}^N)$ be a real-valued mean zero stationary random field with covariance function

$$R(s,t) = exp(-c|s-t|^{2\alpha}) \qquad \text{with } c > 0 \ \text{ and } \ 0 < \alpha < 1 \ .$$

Then $Y(t) = Z(t) - Z(0)$ verifies the conditions (1.5) and (1.6). We can apply Theorem 1.1 to obtain the Hausdorff measure of $X([0,1]^N)$, where

$$X(t) = (X_1(t), \cdots, X_d(t))$$

and $X_1, \cdots, X_d$ are independent copies of Z. Other examples with absolutely continuous spectral measure can be found in Berman [2] p289, and Berman [4].

Example 4.3. Now we give an example with discrete spectral measure. Let X_n $(n \geq 0)$ and Y_n $(n \geq 0)$ be independent standard normal random variables and a_n $(n \geq 0)$ real numbers such that $\sum_n a_n^2 < \infty$. Then for each t, the random series

$$Z(t) = \sum_{n=0}^{\infty} a_n (X_n \cos nt + Y_n \sin nt) \tag{4.14}$$

converges with probability 1 (see [10]), and $Z(t)$ $(t \in \mathbf{R})$ represents a stationary Gaussian process with mean 0 and covariance function

$$R(s,t) = \sum_{n=0}^{\infty} a_n^2 \cos n(t-s) \ .$$

By a result of Berman [4], there are many choices of a_n $(n \geq 0)$ such that the process $Y(t) = Z(t) - Z(0)$ satisfies the hypotheses of Theorem 1.1 with

$$\sigma^2(s) = 2 \sum_{n=0}^{\infty} a_n^2 (1 - \cos ns) \ .$$

Let $X(t)$ $(t \in \mathbf{R})$ be the Gaussian process in $\mathbf{R}^d$ associated with $Z(t)$ or $Y(t)$ $(t \in \mathbf{R})$ by (1.7). If $1 < \alpha d$, then

$$0 < \phi\text{-}m(X([0,1])) < \infty \ ,$$

where $\phi(s) = \psi(s) \log\log \frac{1}{s}$ and ψ is the inverse function of σ. A special case of (4.14) is Example 3.5 in Monrad and Rootzén [15].

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Yimin Xiao
Department of Mathematics
Ohio State University

Columbus, Ohio 43210 - 1174
U. S. A.