STT 861 (Fall 2019): Homework 1 – SOLUTION

This homework will be collected at the start of the lecture of **Wednesday Sep 11, 2019**.

Question 1. Let A, B and C be three events and $P(\cdot)$ be a probability. Show that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Solution.

$$\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B \cup C) - P[A \cap (B \cup C)] \\
&= P(A) + [P(B) + P(C) - P(B \cap C)] - P[(A \cap B) \cup (A \cap C)] \\
&= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P\{(A \cap B) \cap (A \cap C)\}] \\
&= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).
\end{aligned}$$

$\square$

Question 2. A sequence of events $\{A_n : n = 1, 2, \ldots\}$ is said to increase to A if

$$\text{(i) } A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots, \qquad \text{and} \qquad \text{(ii) } A = \cup_{n=1}^{\infty} A_n.$$

We denote $A_n \uparrow A$.
Define $B_1 = A_1$, and $B_n = A_n \setminus A_{n-1}$, for $n = 2, 3, \ldots$. Show the following:

(a) Show that $B_1, B_2, \ldots$ are pairwise disjoint, i.e. for all $i \neq j$, $B_i \cap B_j = \emptyset$.

(b) $A_n = \cup_{j=1}^{n} B_j$, for any $n = 1, 2, 3, \ldots$.

(c) $A = \cup_{j=1}^{\infty} B_j$.

(d) $P(A) = \lim_{n\to\infty} P(A_n)$.

Solution. (a) Here for any $n = 1, 2, \ldots$, $B_n = A_n \setminus A_{n-1}$, where we define $A_0 = \emptyset$. We want to show $B_m \cap B_n = \emptyset$, for any $m \neq n$.

Without any loss of generality we take, $m < n$. As they are integers, $m \leq n - 1$, and so $A_m \subseteq A_{n-1}$ which implies $\emptyset \subseteq (A_m \cap A_{n-1}^c) \subseteq (A_{n-1} \cap A_{n-1}^c) = \emptyset$. Hence $A_m \cap A_{n-1}^c = \emptyset$. Thus

$$B_m \cap B_n = (A_m \setminus A_{m-1}) \cap (A_n \setminus A_{n-1}) = (A_m \cap A_{m-1}^c) \cap (A_n \cap A_{n-1}^c) = \emptyset.$$

Alternatively if possible $x \in B_m \cap B_n$, which means $X \in B_m$ and also $x \in B_n$. Now if $x \in B_n = A_n \setminus A_{n-1} \Rightarrow x \notin A_{n-1} \Rightarrow x \notin A_m$, because $A_m \subseteq A_{n-1}$. But $B_m = A_m \setminus A_{m-1} \subseteq A_m$, and so $x \notin B_m$. This contradicts the fact that $x \in B_m$.

(b) For any $j = 1, 2, \ldots, n$, $B_j = A_j \setminus A_{j-1} \subseteq A_j \subseteq A_n$. Hence $\cup_{j=1}^{n} B_j \subseteq A_n$. On the other hand, if $x \in A_n$, suppose $k = \min\{j : x \in A_j, \text{ but } x \notin A_{j-1},\ j =, \ldots n\}$. Then $x \in A_k \setminus B_{k-1} = B_k \subseteq \cup_{j=1}^{n} B_j$. Thus $A_n = \cup_{j=1}^{n} B_j$.

Alternatively, we can prove $A_n = \cup_{j=1}^{n} B_j$ by induction. Obviously true for $n = 1$, as $A_1 = B_1$. Now we assume it holds for n. Now

$$\begin{aligned}
\bigcup_{j=1}^{n+1} B_j &= \left(\bigcup_{j=1}^{n} B_j \right) \cup B_{n+1} \\
&= A_n \cup (A_{n+1} \setminus A_n) && \text{[by induction hypothesis]} \\
&= (A_n \cap A_{n+1}) \cup (A_{n+1} \cup A_n^c) && \text{[because } A_n \subseteq A_{n+1}\text{]} \\
&= A_{n+1} \cap (A_n \cup A_n^c) && \text{[by distributivity]} \\
&= A_{n+1} \cap S = A_{n+1}.
\end{aligned}$$

(c) For any n, $B_n = A_n \setminus A_{n-1} \subseteq A_n \subseteq \cup_{j=1}^{\infty} A_j = A$. Hence $\cup_{n=1}^{\infty} B_n \subseteq A$. On the other hand if $x \in A = \cup_{j=1}^{\infty} A_j$, hence for some $n \geq 1$, $x \in A_n = \cup_{j=1}^{n} B_j \subseteq \cup_{j=1}^{infty} B_j$. Hence $A \subseteq \cup_{j=1}^{\infty} B_j$. Thus $A = \cup_{j=1}^{\infty} B_j$.

(d) As B_n's are disjoint,

$$P(A_n) = P\left(\bigcup_{j=1}^{n} B_j \right) = \sum_{j=1}^{n} P(B_n)$$

$$\implies \quad \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \sum_{j=1}^{n} P(B_n) = \sum_{j=1}^{\infty} P(B_n) = P\left(\bigcup_{j=1}^{\infty} B_j \right) = P(A).$$

□

Question 3. Suppose $P(A^c) = 0.2$ and $P(B) = 0.3$. Can A and B be disjoint? Justify.

Solution. By Bonferroni's inequality

$$P(A \cap B) \geq 1 - P(A^c) - P(B^c) = 1 - 0.2 - (1 - 0.3) = 0.1 > 0.$$

So A and B cannot be disjoint, otherwise the probability would have been 0. □