# STT 861 (Fall 2019): Homework 3 – SOLUTION

## Solution of Exercise 1.52

For $x < x_0$, $g(x) = 0$. Since for all $x \in \mathbb{R}$, $f(x) \geq 0$ and also $0 \leq F(x_0) < 1$, and hence $g(x) = \dfrac{f(x)}{1 - F(x_0)} \geq 0$, for all $x \geq x_0$. Thus $g(x) \geq 0$, for all $x \in \mathbb{R}$.

Moreover

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{x_0} g(x)dx + \int_{x_0}^{\infty} g(x)dx = \int_{-\infty}^{x_0} 0\ dx + \int_{x_0}^{\infty} \frac{f(x)}{1 - F(x_0)}dx$$
$$= 0 + \frac{\int_{x_0}^{\infty} f(x)dx}{1 - F(x_0)} = \frac{1 - F(x_0)}{1 - F(x_0)} = 1.$$

Hence $g(x)$ is a pdf.

## Solution of Exercise 1.53

(a) Since $Y$ cannot take value less than 1, thus for $y < 1$, $F_Y(y) = P(Y \leq y) = 0$. So for $y < 1$, thus $\lim_{y \to -\infty} F_Y(y) = 0$.

Also $\lim_{y \to \infty} F_Y(y) = \lim_{y \to \infty}(1 - \frac{1}{y^2}) = 1$.

If $y_1 < y_2 \leq 1$, then $F_Y(y_1) = F_Y(y_2) = 0$.

If $y_1 < 1 \leq y_2 \leq 1$, then $y_2^2 \geq 1 \Rightarrow \frac{1}{y_2^2} \leq 1 \Rightarrow 1 - \frac{1}{y_2^2} \geq 0$. Hence $F_Y(y_1) = 0 \geq 1 - \frac{1}{y_2^2} = F_Y(y_2)$.

If $1 \leq y_1 < y_2$, then $y_1^2 < y_2^2 \Rightarrow \frac{1}{y_1^2} > \frac{1}{y_2^2} \Rightarrow 1 - \frac{1}{y_1^2} < \frac{1}{y_2^2} \Rightarrow F_Y(y_1) < F_Y(y_2)$. Hence $F_Y(y)$ is a non-decreasing function.

[*Alternatively*, for $y \geq 1$, $\frac{d}{dy}F_Y(y) = \frac{2}{y^3} > 0$ is an increasing function.]

For $y < 1$, $F_Y(y) = 0$ is a continuous function.

For $y > 1$, $F_Y(y) = 1 - \frac{1}{y^2}$ is also a continuous function.

Also $\lim_{y \downarrow 1} F_Y(y) = \lim_{y \downarrow 1}(1 - \frac{1}{y^2}) = 0 = F_Y(1)$. Hence $F_Y(y)$ is a right continuous function. [Actually $F_Y(y)$ is also left continuous and hence a continuous function]. Thus $F_Y$ satisfies all the properties of a cdf.

(b) $f_Y(y) = \dfrac{d}{dy}F_Y(y) = \begin{cases} 2y^{-3}, & \text{if } y \geq 1, \\ 0, & \text{if } y < 1. \end{cases}$.

(c) $F_Z(z) = P(Z \leq z) = P[10(Y-1) \leq z] = P(Y \leq 0.1z+1) = F_Z(0.1z+1)$. Now $0.z+1 \geq 1$ if and only if $z \geq 0$. Thus

$$F_Z(z) = \begin{cases} 1 - (0.1z+1)^{-2}, & \text{if } z \geq 0, \\ 0, & \text{if } z < 0. \end{cases}$$

## Solution of Exercise 1.54

(a) Since on $(0, \frac{\pi}{2})$, $\sin x > 0$, hence $c$ must be non-negative. Moreover, as $f$ is a pdf

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\pi/2} c \sin x dx = c\left[-\cos x\right]_0^{\pi/2} = c(0+1) = c.$$

Hence $c = 1$.

(b) As $e^{-|x|} > 0$, hence $c$ must be non-negative. Furthermore, since $f$ is a pdf

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} ce^{-|x|}dx &= c\left[\int_{-\infty}^{0} e^{x}dx + \int_{0}^{\infty} e^{-x}dx\right] \\ &= c\left[e^{x}\Big|_{-\infty}^{0} + -e^{-x}\Big|_{0}^{\infty}\right] = c(1+1) = 2c. \end{aligned}$$

Hence $c = \frac{1}{2}$.