STT 861 (Fall 2019): Homework 5 – SOLUTION

Question 1. Suppose (X, Y) are continuous random variables with joint pdf

$$f(x,y) = \begin{cases} k(x+y), & \text{if } 0 < x < 1, \quad 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find k.

(b) Find marginal pdf of X, i.e. $f_X(x)$.

(c) Find marginal pdf of Y, i.e. $f_Y(y)$.

(d) Find conditional pdf of X given $Y = y$, i.e. $f_{X|Y}(x|y)$.

(e) Find conditional pdf of Y given $X = x$, i.e. $f_{Y|X}(y|x)$.

(f) Compute $P(X + Y \leq 1)$.

(g) Find $P(Y < 0.5|X = 0.4)$.

(h) Compute $\mathrm{Cov}(X, Y)$.

Solution. (a) Since $f(x, y) \geq 0$ for all x, y, we must have $k \geq 0$. Moreover

$$1 = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f(x,y)dxdy = k \int_{y=0}^{1} \int_{x=0}^{1} (x+y)dxdy = k \int_{y=0}^{1} \left[\frac{x^2}{2} + xy\right]_{x=0}^{1} dy$$
$$= k \int_{y=0}^{1} \left[\frac{1}{2} + y\right] dy = k \left[\frac{y}{2} + \frac{y^2}{2}\right]_{y=0}^{1} dy = k \left[\frac{1}{2} + \frac{1}{2}\right] = k.$$

Thus $k = 1$ and hence the pdf is

$$f(x,y) = \begin{cases} x+y, & \text{if } 0 < x < 1, \quad 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Marginal pdf of X is

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)dy = \int_{y=0}^{1} (x+y)dy = \left[xy + \frac{y^2}{2}\right]_{y=0}^{1} = x + 0.5, \quad \text{if } \ 0 < x < 1,$$

and zero otherwise.

(c) Marginal pdf of Y is

$$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)dx = \int_{x=0}^{1} (x+y)dx = \left[\frac{x^2}{2} + xy\right]_{x=0}^{1} = y + 0.5, \quad \text{if } \ 0 < y < 1,$$

and zero otherwise.

(d) The conditional pdf of X given $Y = y$ is: if $0 < y < 1$, then

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x+y}{y+0.5}, \quad \text{if } 0 < x < 1,$$

and zero otherwise.

(e) The conditional pdf of Y given $X = x$ is: if $0 < x < 1$, then

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{x+0.5}, \quad \text{if } 0 < y < 1,$$

and zero otherwise

(f)

$$\begin{aligned}
P(X+Y \le 1) &= \iint_{\{(x,y):x+y\le 1\}} f(x,y)dxdy = \int_{y=0}^{1}\int_{x=0}^{1-y}(x+y)dxdy \\
&= \int_{y=0}^{1}\left[\frac{x^2}{2}+xy\right]_{x=0}^{1-y} dy = \int_{y=0}^{1}\left[\frac{(1-y)^2}{2}+y(1-y)\right] dy \\
&= \frac{1}{2}\int_{y=0}^{1}(1-y^2)dy = \frac{1}{2}\left[y-\frac{y^3}{3}\right]_{y=0}^{1} = \frac{1}{2}\left[1-\frac{1}{3}\right] = \frac{1}{3}.
\end{aligned}$$

(g) Notice that $f_{Y|X}(y|0.4) = \dfrac{0.4+y}{0.4+0.5} = \dfrac{y+0.4}{0.9}$, if $0 < y < 1$, and zero otherwise. Hence

$$\begin{aligned}
P(Y < 0.5|X = 0.4) &= \int_{-\infty}^{0.5} f_{Y|X}(y|0.4)dy = \frac{1}{0.9}\int_{0}^{0.5}(y+0.4)dy \\
&= \frac{1}{0.9}\left[\frac{y^2}{2}+0.4y\right]_0^{0.5} = \frac{0.5^2/2+0.4\times 0.5}{0.9} = 0.3611.
\end{aligned}$$

(h) We shall use $\text{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$. Firstly

$$\mathbb{E}(X) = \int_0^1 x(x+0.5)dx = \frac{x^3}{3} + 0.5\,\frac{x^2}{2}\bigg|_0^1 = \frac{1}{3}+\frac{0.5}{2} = \frac{7}{12} = 0.5833.$$

As y has the same marginal, similarly $\mathbb{E}(Y) = 0.5833$. Finally

$$\begin{aligned}
\mathbb{E}(XY) &= \int_{y=0}^{1}\int_{x=0}^{1} xy(x+y)dxdy = \int_{y=0}^{1}\left[\frac{x^3y}{3}+\frac{x^2y^2}{2}\right]_{x=0}^{1} dy = \int_{y=0}^{1}\left[\frac{y}{3}+\frac{y^2}{2}\right] dy \\
&= \left[\frac{y^2}{6}+\frac{y^3}{6}\right]_{y=0}^{1} = \frac{1}{3}.
\end{aligned}$$

So

$$\text{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y) = \frac{1}{3} - \frac{7}{12}\times\frac{7}{12} = -\frac{1}{144} = -0.00694.$$

□

Question 2. Suppose (X, Y) are continuous random variables with joint pdf

$$f(x,y) = \begin{cases} 2e^{-x-y}, & \text{if } 0 < y < x < \infty, \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the joint cdf $F(x, y)$ of (X, Y).

Hint: Consider three cases: (i) $0 < y < x$, (ii) $0 < x \le y$, (iii) otherwise.

(b) Find conditional pdf of X given $Y = y$, i.e. $f_{X|Y}(x|y)$.

(c) Compute $\mathbb{E}(X|Y = 5)$.

(d) Compute $\mathrm{Var}(X|Y = 5)$.

Solution. (a) $F(x,y) = \int_{s=-\infty}^{x} \int_{t=-\infty}^{y} f(s,t)dtds.$

If either $x \le 0$ or $y \le 0$, $f(s,t) = 0$, for $s \le x$ and $t \le y$ and hence $F(x, y) = 0$.

If $x > 0$, and $y > 0$ then $F(x,y) = \int_{s=-\infty}^{x} \int_{t=-\infty}^{y} f(s,t)dtds = \int_{s=0}^{x} \int_{t=0}^{y} f(s,t)dtds.$

Now $f(s,t) = 2e^{-s-t}$, if $0 < t < s$. But in the integral $t < y$, and hence if $0 < s < x$, then $0 < t < \min\{s, y\}$.

If $0 < y < x$, then for $0 < s < y$, $0 < t < s$, and for $y \le s < x$, $0 < t < y$, and so

$$\begin{aligned}
F(x,y) &= \int_{s=0}^{x} \int_{t=0}^{y} f(s,t)dtds = \int_{s=0}^{x} \int_{t=0}^{\min\{s,y\}} 2e^{-s-t}dtds \\
&= 2\left[\int_{s=0}^{y} \int_{t=0}^{s} e^{-s-t}dtds + \int_{s=y}^{x} \int_{t=0}^{y} e^{-s-t}dtds\right] \\
&= 2\left[\int_{s=0}^{y} e^{-s}[-e^{-t}]_{t=0}^{s}\, ds + \int_{s=y}^{x} e^{-s}[-e^{-t}]_{t=0}^{y}\, ds\right] \\
&= 2\left[\int_{s=0}^{y} (e^{-s} - e^{-2s})\, ds + (1 - e^{-y}) \int_{s=y}^{x} e^{-s}\, ds\right] \\
&= 2\left(\left[-e^{-s} + \frac{1}{2}e^{-2s}\right]_{s=0}^{y} + (1 - e^{-y})\left[-e^{-s}\right]_{s=y}^{x}\right) \\
&= 2\left(1 - e^{-y} + \frac{1}{2}(e^{-2y} - 1) + (1 - e^{-y})(e^{-y} - e^{-x})\right) \\
&= 1 - e^{-y} - e^{-x} - e^{-x-y} \\
&= (1 - e^{-x})(1 - e^{-y}).
\end{aligned}$$

If $0 < x \le y$, then $0 < s < x \le y$, and therefore $0 < t < s$. Thus

$$\begin{aligned} F(x,y) &= \int_{s=0}^{x}\int_{t=0}^{y} f(s,t)dtds = \int_{s=0}^{x}\int_{t=0}^{\min\{s,y\}} 2e^{-s-t}dtds \\ &= 2\int_{s=0}^{x} e^{-s}\int_{t=0}^{s} e^{-t}dtds = 2\int_{s=0}^{x} e^{-s}(1-e^{-s})ds = 2\int_{s=0}^{x}(e^{-s}-e^{-2s})ds \\ &= 2\left[(1-e^{-x}) + \frac{1}{2}(1-e^{-2x})\right] = 1 - 2e^{-x} + e^{-2x} = (1-e^{-x})^2. \end{aligned}$$

Hence

$$F(x,y) = \begin{cases} 0, & \text{if } \ x \le 0, \text{ or } \ y \le 0, \\ (1-e^{-x})(1-e^{-y}), & \text{if } \ 0 < y < x, \\ (1-e^{-x})^2, & \text{if } \ 0 < x \le y. \end{cases}$$

(b) The marginal pdf of Y is

$$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)dx = 2\int_{x=y}^{\infty} e^{-x-y}dx = 2e^{-y}\int_{x=y}^{\infty} e^{-x}dx = 2e^{-y}[-e^{-x}]_{x=y}^{\infty} = 2e^{-2y},$$

if $y > 0$, and zero otherwise.

So the conditional pdf of X given $Y = y$ is: if $0 < y < x < \infty$,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2e^{-x-y}}{2e^{-2y}} = e^{-(x-y)}.$$

Notice that X given $Y = y$ is a location scale *Exponential*$(1, y)$.

(c) If $y = 5$, then $f_{X|Y}(x|5) = e^{-(x-5)}$, for $x > 5$.

$$\mathbb{E}(X|Y=5) = \int_{5}^{\infty} xe^{-(x-5)}dx = \int_{0}^{\infty}(x+5)e^{-x}dx = \Gamma(2) + 5\Gamma(1) = 6.$$

Alternatively, since $X|Y = 5 \sim$ *Exponential*$(1, 5)$. Therefore $\mathbb{E}(X|Y = 5) = 1 + 5 = 6$.

(d) Now

$$\begin{aligned} &\mathbb{E}(X^2|Y=5) = \int_{5}^{\infty} x^2e^{-(x-5)}dx = \int_{0}^{\infty}(x+5)^2e^{-x}dx = \Gamma(3) + 10\Gamma(2) + 25\Gamma(1) = 37. \\ &\text{Var}(X|Y=5) = \mathbb{E}(X^2|Y=5) - \{\mathbb{E}(X|Y=5)\}^2 = 37 - 6^2 = 1. \end{aligned}$$

Alternatively, since $X|Y = 5 \sim$ *Exponential*$(1, 5)$. Hence $\text{Var}(X|Y = 5) = 1$.

□