Lecture 3: Conditional Probability and Bayes' Theorem

MSU-STT-351-sum-19A

Contents

In this lecture, we will discuss:

Conditional Probability: Let A, B be two events. The conditional probability of A given B (that is, B has occurred), denoted by $P(A|B)$, is defined as

$$P(A|B) = P(A \text{ and } B)/P(B) = \frac{P(A \cap B)}{P(B)}.$$

The above realtion leads to:

$$P(A \text{ and } B) = P(A \cap B) = P(A|B)P(B);$$
$$P(A \text{ and } B) = P(A \cap B) = P(B|A)P(A).$$

If A and B are disjoint, then $P(A|B) = 0$.

Conditional Probability

Example 1 (Ex. 58): Show that for events A, B abd C with $P(C) > 0$,

$$P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C).$$

Proof: Note

$$\begin{aligned}
P(A \cup B|C) =& \frac{P\Big((A \cup B) \cap C\Big)}{P(C)} = \frac{P\Big((A \cap C) \cup (B \cap C)\Big)}{P(C)} \\
=& \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)} \\
=& P(A|C) + P(B|C) - P(A \cap B|C).
\end{aligned}$$

Note if A and B are disjoint, then

$$P(A \cup B|C) = P(A|C) + P(B|C).$$

Conditional Probability

It can be seen that

$$P(A^c|B) = 1 - P(A|B)$$
$$P(A|B) + P(A^c|B) = 1$$
$$P(B|B) = 1.$$

Also, the following facts holds in general:

$$P(A|B^c) \neq 1 - P(A|B);$$
$$P(A^c|B^c) \neq 1 - P(A|B);$$
$$P(A|B) + P(A^c|B^c) \neq 1.$$

Conditional Probability

Example 2: A box contains 4 red and 2 green balls. Draw successively two balls **without replacement** and observe the color.

Define the following events: $G_1 =$ green on the first draw, $G_2 =$ green on the second draw, $R_1 =$ red on the first draw, $R_2 =$ red on the second draw.

For this experiment, the sample space S is

$$S = \{G_1 G_2, G_1 R_2, R_1 G_2, R_1 R_2\}$$

First, we compute the probabilities of simple events. Note it involves conditional probabilities.

The probabilities of the simple events $G_1G_2, G_1R_2R_1G_2$ and R_1R_2 are:

$$\text{(i) } P(G_1G_2) = P(G_1)P(G_2|G_1) = \frac{2}{6} \times \frac{1}{5} = \frac{2}{30}$$

$$\text{(ii) } P(R_1R_2) = P(R_1)P(R_2|R_1) = \frac{4}{6} \times \frac{3}{5} = \frac{12}{30}$$

$$\text{(iii) } P(G_1R_2) = P(G_1)P(R_2|G_1) = \frac{2}{6} \times \frac{4}{5} = \frac{8}{30}$$

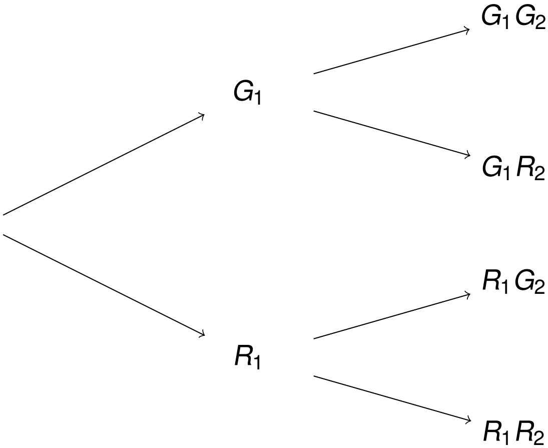
$$\text{(iv) } P(R_1G_2) = P(R_1)P(G_2|R_1) = \frac{4}{6} \times \frac{2}{5} = \frac{8}{30}$$

The probability that the second ball is green is

$$P(G_2) = P(R_1G_2) + P(G_1G_2) = \frac{8}{30} + \frac{2}{30} = \frac{10}{30}.$$

Conditional Probability

The tree diagram is



Conditional Probability

Suppose that green ball was observed in the second draw. What is the conditional probability that the first ball was also green?

It is given by

$$P(G_1|G_2) = \frac{P(G_1G_2)}{P(G_2)} = \frac{\frac{2}{30}}{\frac{10}{30}} = 0.2.$$

Exercise 1. Find also

(a): the probability that exactly one ball selected is green.

(b): the probability that at least one ball selected is green.

(**Do it yourself!**).

Example 3: Given is a contingency table of 100 students cross-classified by their school goal and gender.

		Goals			
		Grades	Popular	Sports	Total
Gender	Boy	24	10	13	47
	Girl	27	19	7	53
	Total	51	29	20	100

A student is selected at random. Let $G = \{\text{a girl is selected}\}$ and $S = \{\text{wants to excel at sports}\}$.

(i) Find $P(G)$ and P(S).

Note

$$P(G) = (53/100) = 0.53; \quad P(S) = (20/100) = 0.2.$$

(ii) Find the probability that "a girl is selected **and** she wants to excel at sports."

It is given by

$$P(G \cap S) = (7/100) = 0.07$$

(iii) Find the probability that "a student wants to excel at sports, given that a girl is selected."

It is given by

$$P(S|G) = \frac{P(GS)}{P(G)} = \frac{(7/100)}{(53/100)} = \frac{7}{53}.$$

Multiplication Rule:

From

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

we get

$$\begin{aligned} P(A \cap B) &= P(A|B)P(B) \\ &= P(B|A)P(A). \end{aligned}$$

Exercise 2. In a lot of 10 elements, there are three defectives. Two elements are selected at random and without replacement, one after another.
(i) What is the probability that the first is good, and the second is defective?
(ii) What is the probability that the second is defective?
(iii) What is the probability that the second is defective given the first one is good?
(iv) What is the probability that the first one is good given the second is defective?

The Law of Total Probability:

Let $A_1, \ldots, A_k$ be mutually exclusive ($A_i \cap A_j = \phi$ for $i \neq j$) and exhaustive ($\cup_{i=1}^{k} A_i = S$) events. Then for any other event B,

$$\begin{aligned} P(B) &= P(B|A_1)P(A_1) + \ldots + P(B|A_k)P(A_k) \\ &= \sum_{i=1}^{k} P(B|A_i)P(A_i) \end{aligned}$$

Note the event B is a subset of sample space S.

Proof:

Since A_i's are mutually exclusive and exhaustive, if B occurs it must be in conjunction with exactly one of the A_i's. That is,

$$B = (B \cap A_1) \cup ... \cup (B \cap A_k),$$

where the events $(B \cap A_i)$ are mutually exclusive. Thus,

$$P(B) = \sum_{i=1}^{k} P(B \cap A_i) = \sum_{i=1}^{k} P(B|A_i)P(A_i),$$

using the multiplication rule.

We next discuss the **Bayes formula** which is very useful to compute certain conditional probabilities. Suppose A and B are any two events. Given that $P(A), P(B|A), P(B|A^c)$, how to find $P(A|B)$?

Solution: Note first that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}.$$

But $P(B)$ is not given. However,

$$\begin{aligned} P(B) =& P(B \cap A) + P(B \cap A^c) \\ =& P(B|A)P(A) + P(B|A^c)P(A^c) \end{aligned}$$

Hence, the Bayes formula for two events is

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}.$$

In the above result, $P(A)$ is called prior probability and $P(A|B)$ is called the posterior probability of A, given that B has occurred. The above result can be extended to k events.

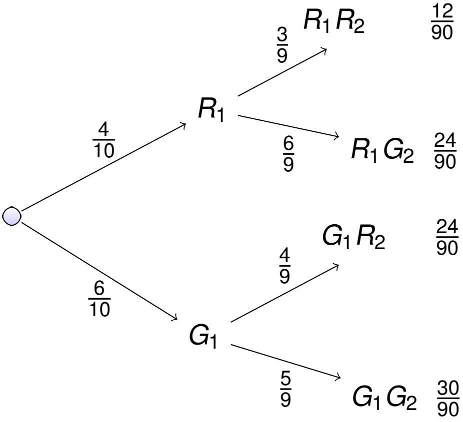
Bayes' Theorem

Let $A_1, \ldots, A_k$ be a collection of k mutually exclusive and exhaustive events with prior probabilities $P(A_i),\ i = 1, \ldots, k$. Let B be any event with $P(B) > 0$. Then the posterior probability of A_i given that B has occurred is

$$P(A_i|B) = \frac{P(A_i \bigcap B)}{P(B)}$$
$$= \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)},$$

for $i = 1, \ldots, k$.

Example 3. Suppose a box contains 4 red and 6 green balls. Two balls are drawn at random (and **without** replacement), one after another. Then the tree diagram for the outcomes, with associated probabilities, is



Bayes' Theorem

In the tree diagram, R_i (G_i) denotes the event of having a red (green) ball at the i-th draw, $i = 1, 2$. Note the second set of branches denotes the set S of all possible outcomes of the experiment. That is, given by

$$S = \{R_1 R_2, R_1 G_2, G_1 R_2, G_1 G_2\}.$$

Note, for example,

$$\begin{aligned} P(G_1 R_2) &= P(G_1)P(R_2|G_1) \\ &= \frac{6}{10} \times \frac{4}{9} \\ &= \frac{4}{15}. \end{aligned}$$

Bayes' Theorem

Similarly, we find

$$P(R_1R_2) = \frac{12}{90}; \;\; P(R_1G_2) = \frac{24}{90}; \;\; P(G_1G_2) = \frac{30}{90}.$$

Note: The sum of the probabilities of all the branches (at the end) add up to 1.

Also, $P(R_2)$ can be found as follows:

$$\begin{aligned} P(R_2) &= P(R_1R_2) + P(G_1R_2) \\ &= \frac{12+24}{90} = \frac{36}{90} = \frac{2}{5}. \end{aligned}$$

Bayes' Theorem

Reverse conditioning: How to find $P(R_1|R_2)$? This is called reverse in conditioning.

Note this probability is not shown in the tree, but can be computed as follows.

Note as seen earlier,

$$P(R_2) = \frac{2}{5}; \;\; P(R_1R_2) = \frac{12}{90} = \frac{2}{15}.$$

Hence,

$$P(R_1|R_2) = \frac{P(R_1R_2)}{P(R_2)} = \frac{2 \times 5}{15 \times 2} = \frac{1}{3}.$$

Independent Events

We now give a formal definition of independence of two events.

Definition 1 (Independence)

Two events A and B are independent if $P(B|A) = P(B)$ or
$P(AB) = P(A)P(B)$.
Two events A and B are dependent if $P(B|A) \neq P(B)$ or
$P(AB) \neq P(A)P(B)$.

Example 5

Let $P(A) = 0.4$ and $P(B) = 0.5$. If A and B are independent, find $P(AB)$.

Solution: Note $P(AB) = P(A)P(B) = 0.4 \times 0.5 = 0.2$.

Independent Events

Example 6
Given $P(A) = 0.6,\ P(B) = 0.5,\ P(A \cup B) = 0.8$, are A and B independent?

Solution: Note

$$\begin{aligned} P(AB) &= P(A) + P(B) - P(A \cup B) \\ &= (0.6 + 0.5 - 0.8) \\ &= 0.30 \\ &= P(A)P(B). \end{aligned}$$

Hence, A and B are independent.

Example 7 (Benefits): A survey shows 56% of all American workers have a workplace retirement plan, 68% have health insurance, and 49% have both benefits. We select a worker at random.

(a) What is the probability he has neither health insurance nor a retirement plan?

(b) What is the probability he has health insurance, if he has retirement plan?

(c) Are having health insurance and a retirement plan independent events? Are these two benefits mutually exclusive?

Solution:

Let $R = \{\text{retirement plan}\}$; $H = \{\text{health insurance}\}$. Given

$$P(R) = 0.56; \;\; P(H) = 0.68; \;\; P(RH) = 0.49.$$

(a) By Demorgan's rule

$$\begin{aligned} P(R^c \cap H^c) &= P(R \cup H)^c \\ &= 1 - P(R \cup H). \end{aligned}$$

Now,

$$\begin{aligned} P(R \cup H) &= P(R) + P(H) - P(R \cap H) \\ &= 0.56 + 0.68 - 0.49 = 0.75. \end{aligned}$$

Therefore,

$$P(R^c \cap H^c) = 1 - (0.75) = 0.25.$$

Bayes' Theorem

(b) Also,

$$
\begin{aligned}
P(H|R) &= \frac{P(H \cap R)}{P(R)} \\
&= \frac{0.49}{0.56} = \frac{7}{8}.
\end{aligned}
$$

(c) Since $P(H|R) = \frac{7}{8} \neq P(H) = 0.68$, the events H and R are not independent.

(d) Are H and R mutually exclusive?

Since $P(HR) = 0.49 > 0$, the events H and R are not mutually exclusive.

Example 8 (Ex 71): An oil exploration company currently has two active projects, one in Asia and the other in Europe. Let A be the event that the Asian project is successful and B be the event that the European project is successful. Suppose that A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.7$.
(a) If the Asian project is not successful, what is the probability that the European project is also not successful? Explain your reasoning.
(b) What is the probability that at least one of the two projects will be successful?
(c) Given that at least one of the two projects is successful, what is the probability that only the Asian project is successful?

Solution:

(a) Since the events A and B are independent, then A^c and B^c are also independent (Prove this fact).

Hence,

$$P(B^c|A^c) = P(B^c) = 1 - 0.7 = 0.30.$$

(b) $P(A \cup B) = P(A) + P(B) - P(A)P(B) = 0.4 + 0.7 - (0.4 \times 0.7) = 0.82.$

(c) $P(AB^c|A \cup B) = \dfrac{P(AB^c \cap (A \cup B))}{P(A \cup B)} = \dfrac{P(AB^c)}{P(A \cup B)} = \dfrac{0.12}{0.82} = 0.146.$

Exercise 4. Items coming off a production line are categorized as good (G), slightly blemished (B), and defective (D), and the percentages are 80%, 15% for good and for slightly blemished, respectively. Suppose that two items will be selected randomly for inspection and the selections are independent.

(a) Find the probability that at least one of the items is slightly blemished.

(b) Find the probability that neither of the items is good.

(c) Suppose that one of the selected items is good, what is the (conditional) probability that both of them are good?

(d) Now suppose that 3 items are selected. What is the probability that at least one is not good?

Parallel and Series Systems

Parallel and Series Systems: Suppose a system consists of three components, say, C_1, C_2 and C_3, which function independently and correctly with probabilities $p_i = P(C_i = 1)$, for $1 \leq i \leq 3$ (probability that a component works is often referred to as the reliability of the component).

Series systems: A series system n components functions if all the components function. Hence,the probability that the series system of n components functions is $p_1 p_2 \dots p_n$.

A series system of n **independent** components fails if at least one of the components fails. Let $F_i = \{C_i = 0\}$ denote the failure of the i-th component so that $P(F_i) = q_i = 1 - p_i$, $i = 1, 2, \dots, n$. Then the probability that the **series** system fails is

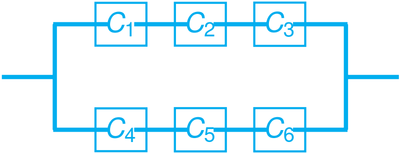
$$\begin{aligned} P(F_1 \cup F_2 \cup \dots F_n) &= 1 - P(F_1 \cup F_2 \cup \dots \cup F_n)^c \\ &= 1 - P(F_1^c F_2^c \dots F_n^c) \\ &= 1 - p_1 p_2 \dots p_n, \end{aligned}$$

since the components are independent. This is expected.

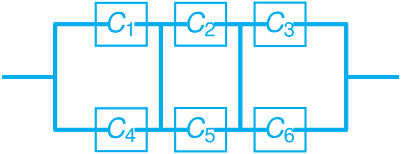
Parallel Systems:

A parallel system PS_3 of three components functions if **at least** one of the components functions. Let $p_i = P(C_i = 1)$ that component C_i works. Then $P(PS_3 = 1) = 1 - P(\text{none of the components functions}) = 1 - (1-p_1)(1-p_2)(1-p_3)$.

Some Complex Systems

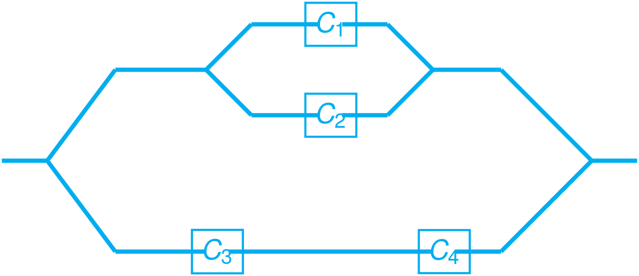


(a) Series parallel system



(b) Parallel series sytem

Example 9 (Exercise 80) Consider the system of components connected as in the figure given below. Components C_1 and C_2 are connected in parallel, so that subsystem S_1 works if either C_1 or C_2 works; since C_3 and C_4 are connected in series, that subsystem S_2 works if both C_3 and C_4 work. If components work independently of one another, with $P(\text{component works}) = 0.9$, calculate $P(\text{system works})$.



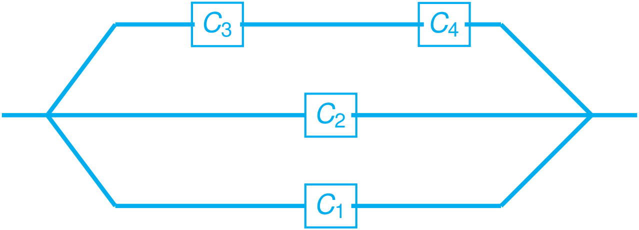
Solution:

Let S_1= parallel subsystem of components C_1 and C_2; $S_2 =$ series subsystems of C_3 and C_4. Then, $S = S_1 \cup S_2$ (since S_1 and S_2 are in parallel). Also, given $P(C_i = 1) = 0.9$, for $1 \leq i \leq 4$.

Therefore,

$$\begin{aligned} P(\text{system works}) = P(S) &= P(S_1 \cup S_2) \\ &= P(S_1) + P(S_2) - P(S_1)P(S_2) \\ &= (0.9 + 0.9 - 0.81) + (0.9 \times 0.9) \\ &\quad - (0.9 + 0.9 - 0.81)(0.9)(0.9) \\ &= 0.99 + 0.81 - 0.8019 = 0.9981 \end{aligned}$$

Exercise 5 Consider a system of components connected as shown below. If all components work independently, and the probability that a given component works correctly is 0.9 for each, what is the probability that the entire system works correctly?



Exercise

Exercise 6. On each of its two wings a plane has 2 engines. We assume that the engines operate independently and $P(\text{engine fails}) = p = 0.2$. A plane will not crash if at least one engine operates on each wing.

(a) What is the probability that it will not crash?

(b) How many engines should be installed on each wing to have the probability of not crashing at least 0.99?

(c) The plane has not crashed. What is the chance that all four engines are in a good shape?

Homework

Sect 2.4: 45, 51, 55, 60, 67, 69

Sect 2.5: 71, 73, 81, 89

Supplementary: 93, 98, 99, 104, 114.