# Large Deviations for Small Noise Stochastic Dynamical Systems

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based on joint works with
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Analysis of Stochastic Partial Differential Equations

# Outline.

- Background.
- Small noise asymptotics.
- A variational representation for infinite dimensional BM.
- Applications to large deviations.
- Systems driven by fractional Brownian motions.
- Poisson random measures.
- Moderate deviations.

# Large Deviation Principle.

Definition. Consider a sequence $\{X^\varepsilon\}_{\varepsilon>0}$ of $\mathcal{E}$ valued r.vs. $\mathcal{E}$ - Polish.

- A function $I$ from $\mathcal{E}$ to $[0,\infty]$ is called a rate function on $\mathcal{E}$ if for each $M<\infty$, $\{x \in \mathcal{E} : I(x) \le M\}$ is compact.
- $\{X^\epsilon\}$ is said to satisfy the large deviation principle on $\mathcal{E}$ (as $\varepsilon \to 0$) with rate function $I$ if:
  - For each closed subset F of $\mathcal{E}$

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^\epsilon \in F) \le -\inf_{x\in F} I(x).$$

  - For each open subset G of $\mathcal{E}$

$$\liminf_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^\epsilon \in G) \ge -\inf_{x\in G} I(x).$$

Formally, for small $\varepsilon$:

$$\mathbb{P}(X^\epsilon \in A) \approx \exp\left\{-\frac{\inf_{x\in A} I(x)}{\varepsilon}\right\}, \ A \in \mathcal{B}(\mathcal{E}).$$

# Stochastic Control Connection (Fleming 1978)

Consider a small noise $n$-dimensional SDE:

$$dX^{\varepsilon}(t) = b(X^{\varepsilon}(t))dt + \sqrt{\varepsilon}\sigma(X^{\varepsilon}(t))dW(t), \; X^{\varepsilon}(0) = x.$$

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- Let $G \subset \mathbb{R}^n$ be bounded open. Let $x \in G$ and $\tau^{\varepsilon} = \inf\{t : X^{\varepsilon}(t) \in \partial G\}$.
- Interested in $\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_x(X^{\varepsilon}(\tau^{\varepsilon}) \in N)$, where $N \subset \partial G$.

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- Formally, with $\Phi$ a nonnegative $C^2$ function, $\Phi(x) \approx M 1_{N^c}(x)$, $M$ a large scaler,

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_x(X^{\varepsilon}(\tau^{\varepsilon}) \in N) \approx \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_x \left\{ e^{-\Phi(X^{\varepsilon}(\tau^{\varepsilon}))/\varepsilon} \right\}.$$

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- Then $g^\varepsilon(x) = \mathbb{E}_x\left\{e^{-\Phi(X^\varepsilon(\tau^\varepsilon))/\varepsilon}\right\}$ solves

$$\begin{cases} \mathcal{L}^\varepsilon g^\varepsilon(x) = 0, & x \in G \\ g^\varepsilon(x) = e^{-\Phi(x)/\varepsilon}, & x \in \partial G \end{cases}$$

where $\mathcal{L}^\varepsilon g = \frac{\varepsilon}{2}\text{Tr}(\sigma D^2 g \sigma') + b \cdot \nabla g$.

# Stochastic Control Connection (ctd.)

- **log transform**: Let $J^\varepsilon = -\varepsilon \log g^\varepsilon$. Then $J^\varepsilon$ solves

$$\frac{\varepsilon}{2}\mathrm{Tr}(\sigma D^2 J^\varepsilon \sigma') + H(x, \nabla J^\varepsilon) = 0$$

where

$$H(x,p) = \min_{v \in \mathbb{R}^n}[L(x,v) + p \cdot v], \; x \in G, \; p \in \mathbb{R}^n$$

and $L(x,v) = \frac{1}{2}(b(x) - v)'[\sigma(x)\sigma'(x)]^{-1}(b(x) - v)$.

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- $J^\varepsilon$ can be characterized as the value function of the stochastic control problem:

$$\begin{aligned} J^\varepsilon(x) &= \inf_{u\in\mathcal{A}} \mathbb{E}_x\left\{\int_0^{\tilde\tau^\varepsilon} L(\tilde X^\varepsilon(t), u(t))dt + \Phi(\tilde X^\varepsilon(\tilde\tau^\varepsilon))\right\} \\ d\tilde X^\varepsilon(t) &= u(t)dt + \sqrt{\varepsilon}\sigma(X^\varepsilon(t))dW(t),\ \tilde X^\varepsilon(0) = x \end{aligned}$$

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- One can argue $J^\varepsilon \to J$, where $J(x)$ is the value function of the deterministic control problem:

$$J(x) = \inf_{\phi,\theta}\int_0^\theta L(\phi(t),\dot{\phi}(t))dt + \Phi(\phi(\theta)),$$

  where inf is over all abs. cts. $\phi$ such that $\phi(0) = x$, and $\theta = \inf\{t : \phi(t) \in \partial G\}$.

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- Later works: Sheu (1985), Dupuis and Ellis(1997), Feng and Kurtz (2005).

# Freidlin-Wentzell Asymptotics.

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- ...also a basis for developing efficient importance sampling schemes.

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- ...also a basis for developing efficient importance sampling schemes.
- Here we revisit this problem – cover more general settings.
- infinite dimensional noise, Poisson random measures, fractional Brownian motions, moderate deviations problems.

# LDP and Laplace Principle.

- LDP is equivalent to Laplace principle if the state space is Polish (Varadhan(1966), Bryc(1990)):
  - A collection of $\mathcal{E}$ valued random variables $\{X^\varepsilon\}$ is said to satisfy Laplace principle with rate function $I$, if for all $h \in C_b(\mathcal{E})$

$$\lim_{\epsilon \to 0} -\epsilon \log \, \mathbb{E}\left\{\exp\left[-\frac{1}{\epsilon}h(X^\epsilon)\right]\right\} = \inf_{x \in \mathcal{E}}\{h(x) + I(x)\}.$$

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- Goal is to show the convergence of variational expressions:

$$\inf_{Q\in\mathcal{P}(E)}\left[\int h(x)dQ(x)+R(Q\|P^\varepsilon)\right] \xrightarrow{\varepsilon\to 0} \inf_{x\in\mathcal{E}}\{h(x)+I(x)\}.$$

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- Instead of PDE characterizations – argue that for a family of 'nice controls' the (state, control, cost) sequence converges to the right limits.

# Variational Representations for Exponential Functionals of BM.

- Suppose $X^\varepsilon = \mathcal{G}^\varepsilon(\beta)$, $\mathcal{G}^\varepsilon$ is a measurable map, $\beta$ an infinite dimensional BM.
- First Step: Find convenient variational formulas for

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- Thus we seek variational representations for

$$-\log \mathbb{E}(\exp\{-f(\beta)\}),$$

$f$ is bounded measurable.

# Brownian Sheet.

- Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^d$ and $\{B(t,x), (t,x) \in [0,T] \times \mathcal{O}\}$ be a Brownian sheet.
  I.e. it is a mean zero, continuous, Gaussian random field such that
  - $Cov(B(t,x), B(s,y)) = \text{Leb}(A_{t,x} \cap A_{s,y})$, where

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- $B$ is a $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ valued r.v., where $\mathbb{C} = C([0,T] \times \overline{\mathcal{O}} : \mathbb{R})$ and $\mathcal{B}(\mathbb{C})$ the Borel sigma-field.
- Denote by $\mu$ the induced Wiener measure.
- Henceforth B is the canonical process on $(\mathbb{C}, \mathcal{B}(\mathbb{C}), \mu)$.

# Representation for functionals of Brownian Sheet.

- Let $H \equiv L^2([0,T] \times \mathcal{O})$ and let

$$\mathcal{P}_2 \doteq \{u : u \text{ is } \mathcal{P} \otimes \mathcal{B}(\mathcal{O}) \text{ measurable and } u(\omega) \in H, \mu - a.s.\}.$$

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- For $\phi \in H$, define $\text{Int}(\phi) \in \mathbb{C}$ by

$$\text{Int}(\phi)(t,x) \doteq \int_{A_{t,x}} \phi(s,y) dsdy.$$

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- Theorem. [B., Dupuis (2000); B., Dupuis, Maroulas (2008).] Let $f : \mathbb{C} \to \mathbb{R}$ be a bounded measurable map. Let $B$ be a Brownian sheet. Then

$$-\log \mathbb{E}(\exp\{-f(B)\}) = \inf_{u \in \mathcal{P}_2} \left\{ \mathbb{E}\left( \frac{1}{2}||u||_H^2 + f(B + \mathrm{Int}(u)) \right) \right\}.$$

# Remarks.

$$-\log \mathbb{E}(\exp\{-f(B)\}) = \inf_{u\in\mathcal{P}_2} \left\{\mathbb{E}^{\mu}\left(\frac{1}{2}||u||_H^2 + f(B + \mathrm{Int}(u))\right)\right\}.$$

- From the Donsker-Varadhan formula:

  $$-\log \mathbb{E}(\exp\{-f(B)\}) = \inf_{Q\in\mathcal{P}(\mathbb{C})} \mathbb{E}^{Q}\left[f(B) + R(Q \parallel \mu)\right].$$

  From this one can deduce:

  $$-\log \mathbb{E}(\exp\{-f(B)\}) \le \inf_{\{\text{ nice } u\in\mathcal{P}_2\}} \left\{\mathbb{E}^{Q_u}\left(\frac{1}{2}||u||_H^2 + f(B^u + \mathrm{Int}(u))\right)\right\}.$$

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- Zhang (2009): Setting of an abstract Wiener space.

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- Zhang (2009): Setting of an abstract Wiener space.
- Üstünel(2009): Connections with Monge-Kantorovitch problem.
- Obtaining optimal controls:
  - Üstünel(2009) – in terms of Clark-Ocone formula.
  - Chen-Xiong (2010) – through solutions of BSDEs.

# A LDP for functionals of BS.

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- Then

$$-\epsilon \log \mathbb{E}\left\{\exp\left[-\frac{1}{\epsilon}h(X^{\epsilon})\right]\right\} = -\epsilon \log \mathbb{E}\left\{\exp\left[-\frac{1}{\epsilon}h(\mathcal{G}^{\varepsilon}(\sqrt{\epsilon}B))\right]\right\}.$$

# LDP for $X^\varepsilon \doteq \mathcal{G}^\varepsilon(\sqrt{\epsilon}B)$.

- Let
$$S^M \doteq \left\{\phi \in H : ||\phi||_H^2 \leq M\right\}.$$
- $S^M$ is compact with the weak topology.
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- Assumption. There exists a measurable map $\mathcal{G}^0 : \mathbb{C} \to \mathcal{E}$ such that: For every $M < \infty$:
  - Whenever $\{u_n\} \subset \mathcal{P}_2^M$ is such that $u_n \Rightarrow u$ (as $S^M$–valued random elements), and $\varepsilon_n \in [0, 1)$ is such that $\varepsilon_n \to 0$, we have
$$\mathcal{G}^{\epsilon_n}\left(\sqrt{\epsilon_n}B + \mathrm{Int}(u_n)\right) \Rightarrow \mathcal{G}^0\left(\mathrm{Int}(u)\right).$$

- Theorem.[B., Dupuis, Maroulas (2008).] Suppose Assumption holds. Then, the family $\{X^\epsilon\}$ satisfies LDP on $\mathcal{E}$, with rate function
$$I(f) \doteq \inf_{\{u \in H : f = \mathcal{G}^0(\mathrm{Int}(u))\}} \left\{\tfrac{1}{2}||u||_H^2\right\}.$$

# Sketch of Proof.

- Suffices to show that Laplace principle holds: For all $h \in C_b(\mathcal{E})$

$$\lim_{\epsilon \to 0} -\epsilon \log \mathbb{E}\left\{\exp\left[-\frac{1}{\epsilon} h(X^\epsilon)\right]\right\} = \inf_{x \in \mathcal{E}}\{h(x) + I(x)\}.$$

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- Recall $X^\varepsilon \doteq \mathcal{G}^\varepsilon(\sqrt{\epsilon}B)$. Applying repn. with $f = \frac{1}{\epsilon} h o \mathcal{G}^\varepsilon(\sqrt{\varepsilon}\cdot)$ we have

$$-\epsilon \log \mathbb{E}\left\{\exp\left[-\frac{1}{\epsilon} h(X^\epsilon)\right]\right\} = \inf_u \mathbb{E}\left(\frac{1}{2}||u||_H^2 + h(X^{\epsilon,u})\right),$$

where $X^{\epsilon,u} = \mathcal{G}^\varepsilon(\sqrt{\epsilon}B + \text{Int}(u))$.

## Proof: $\inf_u \mathbb{E}\left(\frac{1}{2}||u||_H^2 + h(X^{\epsilon,u})\right) \to \inf_{x\in\mathcal{E}}\{h(x)+I(x)\}.$

Proof of Upper Bound. Recall: $X^{\epsilon,u} = \mathcal{G}^{\varepsilon}(\sqrt{\epsilon}B + \mathrm{Int}(u))$
Fix $\delta \in (0,1)$ and choose for each $\varepsilon$, $u^{\epsilon}$ such that

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&\quad\geq \inf_{\{(x,u)\in\mathcal{E}\times H : x=\mathcal{G}^0(\mathrm{Int}(u))\}} \left\{\frac{1}{2}||u||_H^2 + h(x)\right\}
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\end{aligned}$$

# Applications.

- Hilbert Space Valued Diffusions. Unique solvability studied in Leha and Ritter (1984). Small noise LDP in B.-Dupuis(2000).
- Stochastic reaction diffusion equations. Prior works on LDP: Freidlin(1988), Zabczyk(1988), Sowers(1992), Kallianpur and Xiong(1995). These papers assume diffusion coefficient is bounded, "cone condition" on domain... conditions needed for tail probability estimates on certain stochastic convolutions in Holder norms – Garsia's Theorem.

  Conditions relaxed in B.-Dupuis-Maroulas(2008).
- Stochastic flows of diffeomorphisms. B.-Dupuis-Maroulas(2009). Prior works include Millet, Nualart and Sanz-Sole(1992), Ben Arous and Castell(1995)–these concern finite dimensional flows.

  – asymptotic relation, in terms of the rate function, between (small noise) Bayesian solution of an image matching problem with the solution of a deterministic variational problem.

# Advantages of the Approach.

- No approximations or discretizations.
- Exponential prob. estimates are completely bypassed.
- Uniqueness results for HJ equations not needed.
- Proofs of LDP reduce to demonstrating basic qualitative properties of certain perturbations of the original system (eg. existence, uniqueness, stability under $L^2$-bounded perturbations).

# Other Applications.

- Fluid dynamics models.
  - 2D Navier-Stokes equation with multiplicative noise (Sritharan and Sundar (2006)), stochastic tamed 3D Navier-Stokes equations (Rockner, Zhang, Zhang (2010)), Boussinesq equations under random influences (Duan and Millet (2009)), inviscid shell models (Bessaih and Millet (2009)), 2D Navier Stokes equations with a free boundary condition (Bessaih and Millet (2010)), stochastic shell model of turbulence (Manna, Sritharan and Sundar (2010)), stochastic 2D hydrodynamical type systems (Chueshov and Millet (2010)), stochastic derivative Ginzburg-Landau equation with multiplicative noise (Yang and Hou (2008)).
- Less Regular Coefficients.
  - Homeomorphism flows of non-Lipschitz multi-dimensional SDEs (Ren and Zhang (2005)), degenerate SDEs with Sobolev coefficients (Zhang (2010)), multivalued stochastic differential equations (Ren, Xu and Zhang (2010)), stochastic variational inequalities (Bo and Jiang (2011)).
- Other examples.
  - Stochastic partial differential equations under fast dynamical boundary conditions (Wang and Duan (2009)), SPDEs with reflection (Xu and Zhang (2009)), 3D stochastic wave equation (Ortiz-Lopez and Sanz-Sole (2010)), stochastic Volterra equations in Banach spaces (Zhang (2010)).

# Variational Representation for PRM.

- Let $\mathbb{X}$ be a complete separable, locally compact metric space.
- $\mathbb{M}$ be the space of all locally finite measures on $\mathbb{X}_T = [0, T] \times \mathbb{X}$.
- The space is endowed with the weakest topology such that $\mathbb{M} \ni \lambda \mapsto \int f d\lambda$ is continuous for all $f \in C_c(\mathbb{X}_T)$.

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- Fix a locally finite measure $\nu$ on $\mathbb{X}$ and for $\theta > 0$, let $N^\theta$ be a PRM on $\mathbb{X}_T$ with intensity $\theta dt \times \nu(dx)$.

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- $N^\theta$ can be represented as

$$N^\theta((0, t] \times U)) = \int_{(0,t] \times U \times (0,\infty)} 1_{[0,\theta]}(r) \bar{N}(dsdxdr),$$

where $\bar{N}$ is a PRM on $\mathbb{X}_T \times \mathbb{R}_+$ with intensity $dt \times \nu(dx) \times dr$.

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- Let $\bar{\mathcal{A}} = \{\varphi : [0, T] \times \mathbb{X} \times \Omega \to [0, \infty),\ \text{predictable, measurable}\}$. For $\varphi \in \bar{\mathcal{A}}$, $N^\varphi$ defined similarly:

$$N^\varphi((0,t] \times U)) = \int_{(0,t]\times U\times(0,\infty)} 1_{[0,\varphi(s,x,\omega)]}(r)\bar{N}(dsdxdr),$$

# Variational Representation for PRM (ctd.).

- Let $\ell : [0, \infty) \to (0, \infty)$

$$\ell(r) = r \log r - r + 1, \;\; r \in [0, \infty).$$

For $\varphi \in \bar{\mathcal{A}}$, define

$$L_T(\varphi)(\omega) = \int_{\mathbb{X}_T} \ell(\varphi(t, x, \omega))\, \nu_T(dt\, dx), \;\; \omega \in \bar{\mathbb{M}}.$$

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- Theorem [B., Dupuis and Maroulas(2010).] Let $f$ be a bounded measurable map from $\mathbb{M} \to \mathbb{R}$. Then, for $\theta > 0$

$$-\log \mathbb{E}(e^{-f(N^\theta)}) = \inf_{\varphi \in \bar{\mathcal{A}}} \mathbb{E}\left[\theta L_T(\varphi) + f(N^{\theta\varphi})\right].$$

- A different repn obtained in Zhang(2009) - not suitable for large deviation applications.

# Application to Large Deviations.

- $\beta \equiv (\beta_i)_{i=1}^{\infty}$ is an i.i.d. family of standard Brownian motions.
- $N^{\varepsilon^{-1}}$ is a PRM with intensity measure $\varepsilon^{-1}\nu_T$.
- Let $\mathbb{V} = C([0,\infty) : \mathbb{R}^{\infty}) \times \mathbb{M}$. Let $\mathcal{G}^{\varepsilon} : \mathbb{V} \to \mathbb{U}$, where $\mathbb{U}$ is a Polish space, be a sequence of measurable maps.
- Interested in large deviation principle for

$$Z^{\varepsilon} = \mathcal{G}^{\varepsilon}(\sqrt{\varepsilon}\beta, \varepsilon N^{\varepsilon^{-1}}).$$

- For $M \in \mathbb{N}$, let

$$\tilde{S}^M = \left\{ \phi \in L^2([0,T] : \ell_2) : \tilde{L}_T(f) \equiv \int_0^T ||\phi||_2^2 \leq M \right\}.$$

$$\bar{S}^M = \{\psi : [0,T] \times \mathbb{X} \to (0,\infty) : L_T(\psi) \leq M\}.$$

- Identify a function $\psi \in \bar{S}^M$ with the measure $\nu_T^{\psi} \in \mathbb{M}$, through

$$\nu_T^{\psi}(A) = \int_A \psi(s,x)\, \nu_T(dsdx).$$

- With 'weak' topology $S^M = \bar{S}^M \times \tilde{S}^M$ is a compact metric space. Let $\mathcal{U}^M$ be the space of $S^M$ valued controls that are 'non-anticipative'.

# Application to Large Deviations.

$$Z^{\varepsilon} = \mathcal{G}^{\varepsilon}(\sqrt{\varepsilon}\beta, \varepsilon N^{\varepsilon^{-1}}).$$

- Main Condition. There exists a measurable map $\mathcal{G}^0 : \mathbb{V} \to \mathbb{U}$ such that: For every $M < \infty$:
  - Whenever $\{u_n = (\psi_n, \varphi_n)\} \subset \mathcal{U}^M$ is such that $u_n \Rightarrow u$ (as $S^M$–valued random elements), and $\varepsilon_n \in [0,1)$ is such that $\varepsilon_n \to 0$, we have

$$\mathcal{G}^{\epsilon_n}\left(\sqrt{\epsilon_n}\beta + \int_0^{\cdot} \psi_n(s)ds,\ \epsilon_n N^{\epsilon_n^{-1}\varphi_n}\right) \Rightarrow \mathcal{G}^0\left(\int_0^{\cdot} \psi(s)ds, \nu_T^{\varphi}\right).$$

- Let $\mathbb{S} = \cup_{M \in \mathbb{N}} S^M$. For $\phi \in \mathbb{U}$, define

$$\mathbb{S}_{\phi} = \left\{(f,g) \in \mathbb{S} : \phi = \mathcal{G}^0(\int_0^{\cdot} f(s)ds, g)\right\}.$$

  Let $I$ be the rate function defined as

$$I(\phi) = \inf_{q=(f,g)\in\mathbb{S}_{\phi}} \left\{\frac{1}{2}||f||_H^2 + L_T(g)\right\}.$$

- Theorem [B., Dupuis and Maroulas(2010)] Under the condition above $\{Z^{\epsilon}\}_{\epsilon>0}$ satisfies a LDP with rate function $I$.

# Applications.

- Advection-Diffusion Equation with Poissonian Sources (B., Chen, Dupuis(2013))
- Large Deviations for Stochastic Averaging Problems for jump-diffusions (B., Chen Dupuis(201?)).

# Fractional Brownian Motion.

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- This kernel describes a Hilbert space $\mathcal{H}$ as the collection of all $h : [0,1] \to \mathbb{R}^d$ such that

$$h(t) = (K_H\dot{h})(t) = \int_0^1 K_H(t,s)\dot{h}(s)ds, \; t \in [0,1],$$

  for some $\dot{h} \in L^2([0,1] : \mathbb{R}^d)$. Inner product on $\mathcal{H}$:

$$\langle h, g\rangle_{\mathcal{H}} = \langle K_H\dot{h}, K_H\dot{g}\rangle_{\mathcal{H}} = \langle \dot{h}, \dot{g}\rangle_{L^2}.$$

# Fractional Brownian Motion.

- Let $\{B_t^H, t \in [0,1]\}$ be a $d$-dimensional fBM with Hurst parameter $H \in (0,1)$ on $(\Omega, \mathcal{F}, P)$.
- $B^H$ has a representation:

$$B_t^H = \int_0^1 K_H(t,s) dB_s,$$

where $K_H : [0,1] \times [0,1] \to \mathbb{R}$ and $B$ is a standard $d$-dimensional BM.
- This kernel describes a Hilbert space $\mathcal{H}$ as the collection of all $h : [0,1] \to \mathbb{R}^d$ such that

$$h(t) = (K_H \dot{h})(t) = \int_0^1 K_H(t,s) \dot{h}(s) ds, \; t \in [0,1],$$

for some $\dot{h} \in L^2([0,1] : \mathbb{R}^d)$. Inner product on $\mathcal{H}$:

$$\langle h, g \rangle_{\mathcal{H}} = \langle K_H \dot{h}, K_H \dot{g} \rangle_{\mathcal{H}} = \langle \dot{h}, \dot{g} \rangle_{L^2}.$$

- Let $\mathcal{F}_t^H = \overline{\sigma\{B_s^H : s \leq t\}}$ and let $\mathcal{A}$ be the family of all $\mathcal{F}_t^H$ adapted $\mathcal{H}$ valued random variables.

# Representation for fBM:

- Let $f$ be a real bounded measurable function on $C([0,1]:\mathbb{R}^d)$. Then

$$-\log E\left(e^{-f(B^H)}\right) = \inf_{v\in\mathcal{A}} E\left(f(B^H+v)+\frac{1}{2}||v||^2_{\mathcal{H}}\right).$$

# SDE driven by fBM ($H > 1/2$).

- Consider the SDE

$$X_t^\varepsilon = x_0 + \int_0^t b(s, X_s^\varepsilon) ds + \sqrt{\varepsilon} \int_0^t \sigma(s, X_s^\varepsilon) dB_s^H, \; t \in [0,1].$$

  - For some $L > 0$

$$|b(t,x) - b(t,y)| \leq L|x-y|, \; |b(t,x)| \leq L(1+|x|) \, \forall x, \, y \in \mathbb{R}^d, \, \forall t \in [0,1].$$

  - $\sigma(t,x) : [0,1] \times \mathbb{R}^m \to \mathbb{R}^{m \times d}$ is differentiable in $x$, and for some $M > 0$, $1 - H < \lambda \leq 1$, $\frac{1}{H} - 1 < \gamma \leq 1$ and $\forall \; N > 0$ there exists $M_N > 0$ s.t.

$$|\sigma(t,x) - \sigma(t,y)| \leq M|x-y|, \, \forall x \in \mathbb{R}^m, \, \forall t \in [0,1],$$

$$|\partial_{x_i}\sigma(t,x) - \partial_{y_i}\sigma(t,y)| \leq M_N|x-y|^\gamma, \, \forall |x|, \, |y| \leq N, \, \forall t \in [0,1],$$

$$|\sigma(t,x) - \sigma(s,x)| + |\partial_{x_i}\sigma(t,x) - \partial_{x_i}\sigma(s,x)| \leq M|t-s|^\lambda, \, \forall x \in \mathbb{R}^m, \, \forall t, \, s \in [0,1],$$

    for each $i = 1, \ldots, m$.
  - There exist $0 \leq \rho \leq 2 - \frac{1}{H}$ and $K > 0$ such that

$$|\sigma(t,x)| \leq K(1 + |x|^\rho), \, \forall x \in \mathbb{R}^m, \, \forall t \in [0,1].$$

- Existence and uniqueness of solutions shown in Nualart and Rascanu(2002).

# SDE driven by fBM (ctd.)

- Theorem (B., Pipiras and Song(201?).) Under the above conditions, $\{X^\varepsilon\}_{\varepsilon>0}$ satisfies a LDP in $C^\alpha([0,1];\mathbb{R}^m)$ for any $\alpha \in (1-H, \min\{\frac{1}{2}, \lambda, \frac{\gamma}{1+\gamma}\})$, with the rate function

$$I(f) = \inf_v \left\{ \frac{1}{2} \|v\|_{\mathcal{H}}^2 \right\}$$

where the infimum is taken over $\{v \in \mathcal{H} :\ f_t = x_0 + \int_0^t b(s, f_s) ds + \int_0^t \sigma(s, f_s) dv_s\}$.

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Let $\{a_n\}$ be a sequence such that $a_n \uparrow \infty$ and $n^{-1/2}a_n \to 0$. Then

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- Definition. A collection of random variables $\{X^\varepsilon\}$ of $\mathcal{E}$ valued random variables satisfies a LDP on $\mathcal{E}$ with speed $b(\varepsilon)^{-1}$ and rate function $I$ of for all $h \in C_b(\mathcal{E})$.

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- The example says
  - $\frac{S_n}{n}$ satisfies a LDP with speed $n$ and rate function $I$.
  - $\frac{S_n}{n^{1/2}a_n}$ satisfies a LDP with speed $a_n^2$ and rate function $I^0$.

# Moderate Deviations for SDEs.

- Consider the SDE:

$$X^{\varepsilon}(t) = x_0 + \int_0^t b(X^{\varepsilon}(s))ds + \sqrt{\varepsilon}\int_0^t \sigma(X^{\varepsilon}(s))dW(s) + \varepsilon \int_{\mathbb{X}\times[0,t]} G(X^{\varepsilon}(s-), y)N^{\varepsilon^{-1}}(dsdy).$$

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- Also, under additional conditions on $L_G, M_G$, $X^\varepsilon$ satisfies a LDP.
- Let $a(\varepsilon) \to 0$ and $b(\varepsilon) = \varepsilon/a^2(\varepsilon) \to 0$, Consider $Y^\varepsilon = \frac{1}{a(\varepsilon)}(X^\varepsilon - X^0)$.

# Moderate Deviations for SDEs.

- Theorem. (B., Dupuis and Ganguly (201?).) In addition to the above conditions on $b, \sigma$ and $G$, suppose that
  - $L_G$ and $M_G$ are in
    $$\left\{ h : \mathbb{X} \to \mathbb{R} : \exists\ \delta_1 > 0, \text{ s.t. } \forall \Gamma \text{ with } \nu(\Gamma) < \infty \int_\Gamma \exp(\delta_1 h^2(y))\nu(dy) < \infty \right\}.$$
  - The maps $x \mapsto b(x)$ and, for every $y \in \mathbb{X}$, $x \mapsto G(x, y)$ are differentiable. For some $L_{Db} \in (0, \infty)$ and $L_{DG} \in L_2(\nu)$
    $$|Db(x) - Db(x')| \le L_{Db}|x - x'|,\ x, x' \in \mathbb{R}^d,$$
    $$|D_x G(x, y) - D_x G(x', y)| \le L_{DG}(y)|x - x'|,\ x, x' \in \mathbb{R}^d,\ y \in \mathbb{X}.$$
  - For every $\rho > 0$,
    $$\sup_{|x| \le \rho} \int_{\mathbb{X}} |D_x G(x, y)|\nu(dy) < \infty.$$

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  - For every $\rho > 0$,
  $$\sup_{|x| \le \rho} \int_{\mathbb{X}} |D_x G(x, y)|\nu(dy) < \infty.$$
- Then $\{Y^\varepsilon\}$ satisfies a LDP in $D([0, T] : \mathbb{R}^d)$ with speed $b^{-1}(\varepsilon)$ and the rate function given by
  $$I(\eta) = \inf_{\psi, u} \{\frac{1}{2}||\psi||_2^2 + \frac{1}{2}|u|^2\}$$
  where the infimum is taken over all $\psi \in L^2(\nu_T)$, $u \in L^2(\mathbb{R}^d)$ such that
  $$\eta(t) = \int_0^t [Db(X^0(s))](\eta(s))\ ds + \int_{\mathbb{X} \times [0,t]} [D_x G(X^0(s), y)](\eta(s))\ \nu(dy)ds$$
  $$+ \int_{\mathbb{X} \times [0,t]} \psi(y, s) G(X^0(s), y)\nu(dy)ds + \int_{[0,t]} \sigma(X^0(s))u(s)ds.$$

# Moderate Deviations for SDEs.

In other words the rate function is the same as that for LDP (with speed $\varepsilon$) for a Gaussian process $U^\varepsilon$ that solves

$$U^\varepsilon(t) = \int_0^t A(s)U^\varepsilon(s)ds + \sqrt{\varepsilon}\int_0^t B_1(s)dW_1(s) + \sqrt{\varepsilon}\int_0^t B_2(s)dW_2(s)$$

- $W_1$, $W_2$ are independent $d$-dimensional Brownian motions.
- $A(s) = DB(X^0(s)) + \int_{\mathbb{X}} D_x G(x,y)\nu(dy)$.
- $B_1(s) = ||G(X^0(s),\cdot)||_2 I_{d\times d}$.
- $B_2(s) = \sigma(X^0(s))$.