Chaotic properties for a family of SPDEs.

(More on Lecture #9 ...)

Daniel Conus

(Lehigh University)

NSF/CBMS conference on Analysis of SPDEs, Michigan State University

August 22, 2013.

This is joint work with:

- Mathew Joseph (Sheffield)
- Davar Khoshnevisan (U of Utah)
- Shang-Yuan Shiu (NCU Taiwan)

The non-linear stochastic heat equation.

We study the nonlinear stochastic heat equation, namely

$$\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2} \Delta u_t(x) + \sigma(u_t(x)) \dot{W}(t,x), \qquad \text{(SHE)}$$

where

- $t > 0$, $x \in \mathbb{R}$;
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ;
- $\dot{W}$ is a noise that is white in time and (possibly) correlated in space, i.e.

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)f(x-y),$$

  where f is a positive definite function (possibly δ_0);
- the initial function $u_0 : \mathbb{R} \to \mathbb{R}_+$ is bounded.

We study the nonlinear stochastic heat equation, namely

$$\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2} \Delta u_t(x) + \sigma(u_t(x)) \dot{W}(t,x), \qquad \text{(SHE)}$$

where

- $t > 0, \ x \in \mathbb{R}$;
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ;
- $\dot{W}$ is a noise that is white in time and (possibly) correlated in space, i.e.

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s) f(x-y),$$

  where f is a positive definite function (possibly δ_0);
- the initial function $u_0 : \mathbb{R} \to \mathbb{R}_+$ is bounded.

We study the nonlinear stochastic heat equation, namely

$$\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2} \Delta u_t(x) + \sigma(u_t(x)) \dot{W}(t,x), \qquad \text{(SHE)}$$

where

- $t > 0, \ x \in \mathbb{R}$;
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ;
- $\dot{W}$ is a noise that is white in time and (possibly) correlated in space, i.e.

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s) f(x-y),$$

  where f is a positive definite function (possibly δ_0);
- the initial function $u_0 : \mathbb{R} \to \mathbb{R}_+$ is bounded.

We study the nonlinear stochastic heat equation, namely

$$\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2} \Delta u_t(x) + \sigma(u_t(x))\dot{W}(t,x), \qquad \text{(SHE)}$$

where

- $t > 0,\ x \in \mathbb{R}$;
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ;
- $\dot{W}$ is a noise that is white in time and (possibly) correlated in space, i.e.

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)f(x-y),$$

  where f is a positive definite function (possibly δ_0);
- the initial function $u_0 : \mathbb{R} \to \mathbb{R}_+$ is bounded.

We study the nonlinear stochastic heat equation, namely

$$\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2} \Delta u_t(x) + \sigma(u_t(x)) \dot{W}(t,x), \qquad \text{(SHE)}$$

where

- $t > 0$, $x \in \mathbb{R}$;
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ;
- $\dot{W}$ is a noise that is white in time and (possibly) correlated in space, i.e.

  $$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s) f(x-y),$$

  where f is a positive definite function (possibly δ_0);
- the initial function $u_0 : \mathbb{R} \to \mathbb{R}_+$ is bounded.

(SHE) arises in different settings. For instance,

- With $\sigma(x) = \lambda x$, (SHE) is the continuous version of the Parabolic Anderson Model. It models branching processes in a random environment, when the spatial motion is a Brownian motion.

  Ref:
  - Carmona & Molchanov (1994).

- (SHE) is connected to the so-called KPZ equation, modelling growing interfaces: $\log u_t(x)$ "solves" the KPZ equation.

  Refs:
  - Kardar, Parisi & Zhang (1986);
  - Hairer (2012).

Motivation.

(SHE) arises in different settings. For instance,

- With $\sigma(x) = \lambda x$, (SHE) is the continuous version of the Parabolic Anderson Model. It models branching processes in a random environment, when the spatial motion is a Brownian motion.

  Ref:
  - Carmona & Molchanov (1994).

- (SHE) is connected to the so-called KPZ equation, modelling growing interfaces: $\log u_t(x)$ "solves" the KPZ equation.

  Refs:
  - Kardar, Parisi & Zhang (1986);
  - Hairer (2012).

Motivation.

(SHE) arises in different settings. For instance,

- With $\sigma(x) = \lambda x$, (SHE) is the continuous version of the Parabolic Anderson Model. It models branching processes in a random environment, when the spatial motion is a Brownian motion.

  Ref:
  - Carmona & Molchanov (1994).

- (SHE) is connected to the so-called KPZ equation, modelling growing interfaces: $\log u_t(x)$ "solves" the KPZ equation.

  Refs:
  - Kardar, Parisi & Zhang (1986);
  - Hairer (2012).

Solution to the non-linear stochastic heat equation.

We consider the *mild solution* to (SHE), i.e. a stochastic process $(u_t(x); t > 0, x \in \mathbb{R})$ satisfying:

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y))W(ds, dy),$$

where p_t is the heat kernel and the stochastic integral is defined in the sense of Walsh (1986).

Theorem (Dalang (1999))

The non-linear stochastic heat equation (SHE) *has a unique random-field solution such that, for all* $T > 0$,

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}} \mathbb{E}[u_t(x)^2] < \infty.$$

We consider the *mild solution* to (SHE), i.e. a stochastic process $(u_t(x); t > 0, x \in \mathbb{R})$ satisfying:

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y))W(ds, dy),$$

where p_t is the heat kernel and the stochastic integral is defined in the sense of Walsh (1986).

Theorem (Dalang (1999))

The non-linear stochastic heat equation (SHE) *has a unique random-field solution such that, for all* $T > 0$,

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}} \mathbb{E}[u_t(x)^2] < \infty.$$

Intermittency and chaos.

We have seen throughout the week that the solution to SHE is a (weakly-) intermittent random-field, provided that:

- $\mathrm{L}_\sigma := \inf_{x\in\mathbb{R}} \frac{|\sigma(x)|}{|x|} > 0.$
- $\sigma(0) = 0$ if $\inf_{x\in\mathbb{R}} u_0(x) = 0.$

Refs:

- Foondun & Khoshnevisan (2009)
- C. & Khoshnevisan (2011).

Weak intermittency implies that the solution develops very high peaks concentrated on some spatial islands for large time t.

For physicists, intermittency is believed to happen in part because the system is chaotic. What happens before the onset of intermittency ?

We have seen throughout the week that the solution to SHE is a (weakly-) intermittent random-field, provided that:

- $\mathrm{L}_\sigma := \inf_{x\in\mathbb{R}} \frac{|\sigma(x)|}{|x|} > 0.$
- $\sigma(0) = 0$ if $\inf_{x\in\mathbb{R}} u_0(x) = 0.$

Refs:

- Foondun & Khoshnevisan (2009)
- C. & Khoshnevisan (2011).

Weak intermittency implies that the solution develops very high peaks concentrated on some spatial islands for large time t.

For physicists, intermittency is believed to happen in part because the system is chaotic. What happens before the onset of intermittency ?

Intermittency and chaos.

We have seen throughout the week that the solution to SHE is a (weakly-) intermittent random-field, provided that:

- $\mathrm{L}_\sigma := \inf_{x \in \mathbb{R}} \frac{|\sigma(x)|}{|x|} > 0.$
- $\sigma(0) = 0$ if $\inf_{x \in \mathbb{R}} u_0(x) = 0.$

Refs:

- Foondun & Khoshnevisan (2009)
- C. & Khoshnevisan (2011).

Weak intermittency implies that the solution develops very high peaks concentrated on some spatial islands for large time t.

For physicists, intermittency is believed to happen in part because the system is chaotic. What happens before the onset of intermittency ?

We have seen throughout the week that the solution to SHE is a (weakly-) intermittent random-field, provided that:

- $\mathrm{L}_\sigma := \inf_{x\in\mathbb{R}} \frac{|\sigma(x)|}{|x|} > 0.$
- $\sigma(0) = 0$ if $\inf_{x\in\mathbb{R}} u_0(x) = 0.$

Refs:

- Foondun & Khoshnevisan (2009)
- C. & Khoshnevisan (2011).

Weak intermittency implies that the solution develops very high peaks concentrated on some spatial islands for large time t.

For physicists, intermittency is believed to happen in part because the system is chaotic. What happens before the onset of intermittency ?

We have seen throughout the week that the solution to SHE is a (weakly-) intermittent random-field, provided that:

- $\mathrm{L}_\sigma := \inf_{x\in\mathbb{R}} \frac{|\sigma(x)|}{|x|} > 0.$
- $\sigma(0) = 0$ if $\inf_{x\in\mathbb{R}} u_0(x) = 0.$

Refs:

- Foondun & Khoshnevisan (2009)
- C. & Khoshnevisan (2011).

Weak intermittency implies that the solution develops very high peaks concentrated on some spatial islands for large time t.

For physicists, intermittency is believed to happen in part because the system is chaotic. What happens before the onset of intermittency ?

We have seen throughout the week that the solution to SHE is a (weakly-) intermittent random-field, provided that:

- $\mathrm{L}_\sigma := \inf_{x\in\mathbb{R}} \frac{|\sigma(x)|}{|x|} > 0.$
- $\sigma(0) = 0$ if $\inf_{x\in\mathbb{R}} u_0(x) = 0.$

Refs:

- Foondun & Khoshnevisan (2009)
- C. & Khoshnevisan (2011).

Weak intermittency implies that the solution develops very high peaks concentrated on some spatial islands for large time t.

For physicists, intermittency is believed to happen in part because the system is chaotic. What happens before the onset of intermittency ?

A couple reminders.

We have seen in Davar's lectures:

Theorem (Foondun-Khoshnevisan (2009))

If $\sigma(0) = 0$ and u_0 has compact support, then for all $t > 0$,

$$\sup_{x\in\mathbb{R}} u_t(x) < \infty \qquad a.s.$$

and

Theorem (C.-Joseph-Khoshnevisan (2011))

If $\sigma(x) = \lambda x$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{2/3}} \asymp C \qquad a.s. \text{ for all } t > 0.$$

Remark: In the deterministic case, the solution remains bounded, whether the initial condition has compact support or is bounded away from 0.

$\Longrightarrow$ the noise induces a chaotic behavior, i.e. a dependence on the initial conditions.

We have seen in Davar's lectures:

Theorem (Foondun-Khoshnevisan (2009))

If $\sigma(0) = 0$ *and* u_0 *has compact support, then for all* $t > 0$,

$$\sup_{x\in\mathbb{R}} u_t(x) < \infty \qquad a.s.$$

and

Theorem (C.-Joseph-Khoshnevisan (2011))

If $\sigma(x) = \lambda x$, *then*

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{2/3}} \asymp C \qquad a.s. \text{ for all } t > 0.$$

Remark: In the deterministic case, the solution remains bounded, whether the initial condition has compact support or is bounded away from 0.

$\Longrightarrow$ the noise induces a chaotic behavior, i.e. a dependence on the initial conditions.

A couple reminders.

We have seen in Davar's lectures:

Theorem (Foondun-Khoshnevisan (2009))

If $\sigma(0) = 0$ and u_0 has compact support, then for all $t > 0$,

$$\sup_{x \in \mathbb{R}} u_t(x) < \infty \qquad a.s.$$

and

Theorem (C.-Joseph-Khoshnevisan (2011))

If $\sigma(x) = \lambda x$, then

$$\limsup_{|x| \to \infty} \frac{\log u_t(x)}{(\log |x|)^{2/3}} \asymp C \qquad a.s. \text{ for all } t > 0.$$

Remark: In the deterministic case, the solution remains bounded, whether the initial condition has compact support or is bounded away from 0.

$\Longrightarrow$ the noise induces a chaotic behavior, i.e. a dependence on the initial conditions.

A couple reminders.

We have seen in Davar's lectures:

Theorem (Foondun-Khoshnevisan (2009))

If $\sigma(0) = 0$ and u_0 has compact support, then for all $t > 0$,

$$\sup_{x\in\mathbb{R}} u_t(x) < \infty \qquad a.s.$$

and

Theorem (C.-Joseph-Khoshnevisan (2011))

If $\sigma(x) = \lambda x$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{2/3}} \asymp C \qquad a.s.\ \text{for all } t > 0.$$

Remark: In the deterministic case, the solution remains bounded, whether the initial condition has compact support or is bounded away from 0.

$\Longrightarrow$ the noise induces a chaotic behavior, i.e. a dependence on the initial conditions.

We essentially need two main ingredients to obtain the result:

- A tail probability estimate on the behavior of $u_t(x)$ for fixed t and x. This is obtained with good moment estimates and the Paley-Zygmund inequality.
- A localization result, namely that $u_t(x)$ and $u_t(y)$ are somewhat independent if x and y are sufficiently far away.

Assume from now on (for simplicity) that $u_0(x) \equiv 1$.

Quizz: We know that if $f(x) = \delta_0(x)$, then

$$\mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^\gamma t).$$

What is γ ?

We essentially need two main ingredients to obtain the result:

- A tail probability estimate on the behavior of $u_t(x)$ for fixed t and x. This is obtained with good moment estimates and the Paley-Zygmund inequality.
- A localization result, namely that $u_t(x)$ and $u_t(y)$ are somewhat independent if x and y are sufficiently far away.

Assume from now on (for simplicity) that $u_0(x) \equiv 1$.

Quizz: We know that if $f(x) = \delta_0(x)$, then

$$\mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^\gamma t).$$

What is γ ?

Summary of the techniques.

We essentially need two main ingredients to obtain the result:

- A tail probability estimate on the behavior of $u_t(x)$ for fixed t and x. This is obtained with good moment estimates and the Paley-Zygmund inequality.
- A localization result, namely that $u_t(x)$ and $u_t(y)$ are somewhat independent if x and y are sufficiently far away.

Assume from now on (for simplicity) that $u_0(x) \equiv 1$.

Quizz: We know that if $f(x) = \delta_0(x)$, then

$$\mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^\gamma t).$$

What is γ ?

Summary of the techniques.

We essentially need two main ingredients to obtain the result:

- A tail probability estimate on the behavior of $u_t(x)$ for fixed t and x. This is obtained with good moment estimates and the Paley-Zygmund inequality.
- A localization result, namely that $u_t(x)$ and $u_t(y)$ are somewhat independent if x and y are sufficiently far away.

Assume from now on (for simplicity) that $u_0(x) \equiv 1$.

Quizz: We know that if $f(x) = \delta_0(x)$, then

$$\mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^\gamma t).$$

What is γ ?

Summary of the techniques.

We essentially need two main ingredients to obtain the result:

- A tail probability estimate on the behavior of $u_t(x)$ for fixed t and x. This is obtained with good moment estimates and the Paley-Zygmund inequality.
- A localization result, namely that $u_t(x)$ and $u_t(y)$ are somewhat independent if x and y are sufficiently far away.

Assume from now on (for simplicity) that $u_0(x) \equiv 1$.

Quizz: We know that if $f(x) = \delta_0(x)$, then

$$\mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^\gamma t).$$

What is γ ?

We essentially need two main ingredients to obtain the result:

- A tail probability estimate on the behavior of $u_t(x)$ for fixed t and x. This is obtained with good moment estimates and the Paley-Zygmund inequality.
- A localization result, namely that $u_t(x)$ and $u_t(y)$ are somewhat independent if x and y are sufficiently far away.

Assume from now on (for simplicity) that $u_0(x) \equiv 1$.

Proposition

If $f(x) = \delta_0(x)$, then

$$\mathbb{E}[|u_t(x)|^k] \leqslant C\exp(Ck^3 t).$$

We also know that $\sigma(u) = \lambda u$ (Parabolic Anderson Model) achieves the upper bound.

We essentially need two main ingredients to obtain the result:

- A tail probability estimate on the behavior of $u_t(x)$ for fixed t and x. This is obtained with good moment estimates and the Paley-Zygmund inequality.
- A localization result, namely that $u_t(x)$ and $u_t(y)$ are somewhat independent if x and y are sufficiently far away.

Assume from now on (for simplicity) that $u_0(x) \equiv 1$.

Proposition

If $f(x) = \delta_0(x)$, then

$$\mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^3 t).$$

We also know that $\sigma(u) = \lambda u$ (Parabolic Anderson Model) achieves the upper bound.

Now, we can get a general lower bound on moments, namely

Proposition

If $f(x) = \delta_0(x)$ and $\inf_{u\in\mathbb{R}} \sigma(u) := a > 0$, then

$$\mathbb{E}[|u_t(x)|^{2k}] \geqslant c^k t^{k/2} k^k = c\exp(ck\log(k)).$$

We have

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y))W(ds,dy),$$

and, if we set

$$v_t(x) = (p_t * u_0)(x) + a\int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)W(ds,dy),$$

then we can show that $\mathbb{E}[u_t(x)^{2k}] \geqslant \mathbb{E}[v_t(x)^{2k}]$.

But $v_t(x)$ is Gaussian, hence $\mathbb{E}[v_t(x)^{2k}] \simeq \sigma^{2k}\frac{(2k)!}{k!2^k}$. Stirling's formula gives the result.

More on the techniques.

Now, we can get a general lower bound on moments, namely

Proposition

If $f(x) = \delta_0(x)$ *and* $\inf_{u\in\mathbb{R}} \sigma(u) := a > 0$*, then*

$$\mathbb{E}[|u_t(x)|^{2k}] \geqslant c^k t^{k/2} k^k = c\exp(ck\log(k)).$$

We have

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y))W(ds,dy),$$

and, if we set

$$v_t(x) = (p_t * u_0)(x) + a\int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)W(ds,dy),$$

then we can show that $\mathbb{E}[u_t(x)^{2k}] \geqslant \mathbb{E}[v_t(x)^{2k}]$.

But $v_t(x)$ is Gaussian, hence $\mathbb{E}[v_t(x)^{2k}] \simeq \sigma^{2k}\frac{(2k)!}{k!2^k}$. Stirling's formula gives the result.

Now, we can get a general lower bound on moments, namely

Proposition

If $f(x) = \delta_0(x)$ *and* $\inf_{u\in\mathbb{R}} \sigma(u) := a > 0$, *then*

$$\mathbb{E}[|u_t(x)|^{2k}] \geqslant c^k t^{k/2} k^k = c \exp(ck \log(k)).$$

We have

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y))W(ds,dy),$$

and, if we set

$$v_t(x) = (p_t * u_0)(x) + a\int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)W(ds,dy),$$

then we can show that $\mathbb{E}[u_t(x)^{2k}] \geqslant \mathbb{E}[v_t(x)^{2k}]$.

But $v_t(x)$ is Gaussian, hence $\mathbb{E}[v_t(x)^{2k}] \simeq \sigma^{2k}\frac{(2k)!}{k!2^k}$. Stirling's formula gives the result.

Now, we can get a general lower bound on moments, namely

Proposition

If $f(x) = \delta_0(x)$ *and* $\inf_{u\in\mathbb{R}} \sigma(u) := a > 0$, *then*

$$\mathbb{E}[|u_t(x)|^{2k}] \geqslant c^k t^{k/2} k^k = c\exp(ck\log(k)).$$

We have

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y))W(ds,dy),$$

and, if we set

$$v_t(x) = (p_t * u_0)(x) + a\int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)W(ds,dy),$$

then we can show that $\mathbb{E}[u_t(x)^{2k}] \geqslant \mathbb{E}[v_t(x)^{2k}]$.

But $v_t(x)$ is Gaussian, hence $\mathbb{E}[v_t(x)^{2k}] \simeq \sigma^{2k}\frac{(2k)!}{k!2^k}$. Stirling's formula gives the result.

Now, we can get a general lower bound on moments, namely

Proposition

If $f(x) = \delta_0(x)$ *and* $\inf_{u\in\mathbb{R}} \sigma(u) := a > 0$*, then*

$$\mathbb{E}[|u_t(x)|^{2k}] \geqslant c^k t^{k/2} k^k = c\exp(ck\log(k)).$$

We have

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y))W(ds,dy),$$

and, if we set

$$v_t(x) = (p_t * u_0)(x) + a\int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)W(ds,dy),$$

then we can show that $\mathbb{E}[u_t(x)^{2k}] \geqslant \mathbb{E}[v_t(x)^{2k}]$.

But $v_t(x)$ is Gaussian, hence $\mathbb{E}[v_t(x)^{2k}] \simeq \sigma^{2k}\frac{(2k)!}{k!2^k}$. Stirling's formula gives the result.

Now, we can get a general lower bound on moments, namely

Proposition

If $f(x) = \delta_0(x)$ *and* $\inf_{u\in\mathbb{R}} \sigma(u) := a > 0$, *then*

$$\mathbb{E}[|u_t(x)|^{2k}] \geqslant c^k t^{k/2} k^k = c\exp(ck\log(k)).$$

We have

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y))W(ds,dy),$$

and, if we set

$$v_t(x) = (p_t * u_0)(x) + a\int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)W(ds,dy),$$

then we can show that $\mathbb{E}[u_t(x)^{2k}] \geqslant \mathbb{E}[v_t(x)^{2k}]$.

But $v_t(x)$ is Gaussian, hence $\mathbb{E}[v_t(x)^{2k}] \simeq \sigma^{2k}\frac{(2k)!}{k!2^k}$. Stirling's formula gives the result.

General result.

We have: $c \exp(ck \log(k)) \leqslant \mathbb{E}[u_t(x)^k] \leqslant C \exp(Ck^3)$

We plug this into the Paley-Zygmund inequality to get

$$\log P(|u_t(x)| \geqslant \lambda) \gtrsim -\lambda^6$$

A consequence of this is the following general theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ *and* $\inf_{x \in \mathbb{R}} \sigma(x) \geqslant a > 0$, *then for all* $t > 0$,

$$\sup_{x \in [-R,R]} u_t(x) \gtrsim (\log R)^{1/6} \qquad \text{as } R \to \infty.$$

The order $(\log R)^{1/6}$ is not sharp.

We will now consider the particular cases, where either

- σ is bounded above and below;
- $\sigma(x) = \lambda x$ (Parabolic Anderson Model).

General result.

We have: $c\exp(ck\log(k)) \leqslant \mathbb{E}[u_t(x)^k] \leqslant C\exp(Ck^3)$

We plug this into the Paley-Zygmund inequality to get

$$\log P(|u_t(x)| \geqslant \lambda) \gtrsim -\lambda^6$$

A consequence of this is the following general theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ and $\inf_{x\in\mathbb{R}} \sigma(x) \geqslant a > 0$, then for all $t > 0$,

$$\sup_{x\in[-R,R]} u_t(x) \gtrsim (\log R)^{1/6} \qquad \text{as } R \to \infty.$$

The order $(\log R)^{1/6}$ is not sharp.

We will now consider the particular cases, where either

- σ is bounded above and below;
- $\sigma(x) = \lambda x$ (Parabolic Anderson Model).

General result.

We have: $c\exp(ck\log(k)) \leqslant \mathbb{E}[u_t(x)^k] \leqslant C\exp(Ck^3)$

We plug this into the Paley-Zygmund inequality to get

$$\log P(|u_t(x)| \geqslant \lambda) \gtrsim -\lambda^6$$

A consequence of this is the following general theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ *and* $\inf_{x\in\mathbb{R}} \sigma(x) \geqslant a > 0$, *then for all* $t > 0$,

$$\sup_{x\in[-R,R]} u_t(x) \gtrsim (\log R)^{1/6} \qquad \text{as } R \to \infty.$$

The order $(\log R)^{1/6}$ is not sharp.

We will now consider the particular cases, where either

- σ is bounded above and below;
- $\sigma(x) = \lambda x$ (Parabolic Anderson Model).

We have: $c\exp(ck\log(k)) \leqslant \mathbb{E}[u_t(x)^k] \leqslant C\exp(Ck^3)$

We plug this into the Paley-Zygmund inequality to get

$$\log P(|u_t(x)| \geqslant \lambda) \gtrsim -\lambda^6$$

A consequence of this is the following general theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ *and* $\inf_{x\in\mathbb{R}} \sigma(x) \geqslant a > 0$, *then for all* $t > 0$,

$$\sup_{x\in[-R,R]} u_t(x) \gtrsim (\log R)^{1/6} \qquad \text{as } R \to \infty.$$

The order $(\log R)^{1/6}$ is not sharp.

We will now consider the particular cases, where either

- σ is bounded above and below;
- $\sigma(x) = \lambda x$ (Parabolic Anderson Model).

General result.

We have: $c\exp(ck\log(k)) \leqslant \mathbb{E}[u_t(x)^k] \leqslant C\exp(Ck^3)$

We plug this into the Paley-Zygmund inequality to get

$$\log P(|u_t(x)| \geqslant \lambda) \gtrsim -\lambda^6$$

A consequence of this is the following general theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ *and* $\inf_{x\in\mathbb{R}} \sigma(x) \geqslant a > 0$, *then for all* $t > 0$,

$$\sup_{x\in[-R,R]} u_t(x) \gtrsim (\log R)^{1/6} \qquad \text{as } R \to \infty.$$

The order $(\log R)^{1/6}$ is not sharp.

We will now consider the particular cases, where either

- σ is bounded above and below;
- $\sigma(x) = \lambda x$ (Parabolic Anderson Model).

General result.

We have: $$c\exp(ck\log(k)) \leqslant \mathbb{E}[u_t(x)^k] \leqslant C\exp(Ck^3)$$

We plug this into the Paley-Zygmund inequality to get

$$\log P(|u_t(x)| \geqslant \lambda) \gtrsim -\lambda^6$$

A consequence of this is the following general theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ *and* $\inf_{x\in\mathbb{R}} \sigma(x) \geqslant a > 0$, *then for all* $t > 0$,

$$\sup_{x\in[-R,R]} u_t(x) \gtrsim (\log R)^{1/6} \qquad \text{as } R \to \infty.$$

The order $(\log R)^{1/6}$ is not sharp.

We will now consider the particular cases, where either

- σ is bounded above and below;
- $\sigma(x) = \lambda x$ (Parabolic Anderson Model).

Case where σ is bounded above and below.

Similarly, we get a Gaussian upper bound and

$$c\exp(ck\log(k)) \leqslant \mathbb{E}[u_t(x)^{2k}] \leqslant C\exp(Ck\log(k))$$

Using the PZ inequality and a Lemma presented by Davar, we obtain

$$\log P(|u_t(x)| \geqslant \lambda) \asymp -\sqrt{\kappa}\lambda^2$$

A consequence of this is the following theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ and, $0 < a < \sigma(x) < b$ for all $x \in \mathbb{R}$, then

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \asymp \kappa^{-1/4} \qquad \text{a.s. for all } t > 0.$$

This shows that if σ is bounded above and below, then $u_t(x)$ behaves as the solution to the equation with additive noise, i.e. as a Gaussian process.

Case where σ is bounded above and below.

Similarly, we get a Gaussian upper bound and

$$c \exp(ck \log(k)) \leqslant \mathbb{E}[u_t(x)^{2k}] \leqslant C \exp(Ck \log(k))$$

Using the PZ inequality and a Lemma presented by Davar, we obtain

$$\log P(|u_t(x)| \geqslant \lambda) \asymp -\sqrt{\kappa}\lambda^2$$

A consequence of this is the following theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ and, $0 < a < \sigma(x) < b$ for all $x \in \mathbb{R}$, then

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \asymp \kappa^{-1/4} \qquad \text{a.s. for all } t > 0.$$

This shows that if σ is bounded above and below, then $u_t(x)$ behaves as the solution to the equation with additive noise, i.e. as a Gaussian process.

Case where σ is bounded above and below.

Similarly, we get a Gaussian upper bound and

$$c\exp(ck\log(k)) \leqslant \mathbb{E}[u_t(x)^{2k}] \leqslant C\exp(Ck\log(k))$$

Using the PZ inequality and a Lemma presented by Davar, we obtain

$$\log P(|u_t(x)| \geqslant \lambda) \asymp -\sqrt{\kappa}\lambda^2$$

A consequence of this is the following theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ *and,* $0 < a < \sigma(x) < b$ *for all* $x \in \mathbb{R}$, *then*

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \asymp \kappa^{-1/4} \qquad \text{a.s. for all } t > 0.$$

This shows that if σ is bounded above and below, then $u_t(x)$ behaves as the solution to the equation with additive noise, i.e. as a Gaussian process.

Case where σ is bounded above and below.

Similarly, we get a Gaussian upper bound and

$$c \exp(ck \log(k)) \leqslant \mathbb{E}[u_t(x)^{2k}] \leqslant C \exp(Ck \log(k))$$

Using the PZ inequality and a Lemma presented by Davar, we obtain

$$\log P(|u_t(x)| \geqslant \lambda) \asymp -\sqrt{\kappa}\lambda^2$$

A consequence of this is the following theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ and, $0 < a < \sigma(x) < b$ for all $x \in \mathbb{R}$, then

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \asymp \kappa^{-1/4} \qquad a.s. \text{ for all } t > 0.$$

This shows that if σ is bounded above and below, then $u_t(x)$ behaves as the solution to the equation with additive noise, i.e. as a Gaussian process.

Parabolic Anderson Model.

When $\sigma(u) = \lambda u$,

$$c \exp(ck^3) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^3).$$

We have seen that these estimates lead to

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(0) = \delta_0(x)$ and $\sigma(x) = \lambda x$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{2/3}} \asymp \kappa^{-\frac{1}{3}} \qquad a.s.\ \text{for all } t > 0.$$

The power of κ suggests the universality class of random matrices models, unlike the Gaussian case.

When $\sigma(u) = \lambda u$,

$$c\exp(ck^3) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C\exp(Ck^3).$$

We have seen that these estimates lead to

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(0) = \delta_0(x)$ and $\sigma(x) = \lambda x$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{2/3}} \asymp \kappa^{-\frac{1}{3}} \qquad a.s.\ \text{for all } t > 0.$$

The power of κ suggests the universality class of random matrices models, unlike the Gaussian case.

When $\sigma(u) = \lambda u$,

$$c \exp(ck^3) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^3).$$

We have seen that these estimates lead to

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(0) = \delta_0(x)$ and $\sigma(x) = \lambda x$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log |x|)^{2/3}} \asymp \kappa^{-\frac{1}{3}} \qquad a.s. \text{ for all } t > 0.$$

The power of κ suggests the universality class of random matrices models, unlike the Gaussian case.

We can obtain similar results for colored noise, i.e. when $f \neq \delta_0$.

We first assume that $f(0) < \infty$ with appropriate tail behavior. The latter ensures that localization occurs.

In that case, one can prove that

$$c \exp(ck^2) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^2)$$

This can be directly obtained from the Feynman-Kac formula for moments of the solution (Bertini & Cancrini (1994), Hu & Nualart (2009), C. (2011))

$$\mathbb{E}[u_t(x)^k] = \mathbb{E}\left[\exp\left(\sum\sum_{1 \leqslant i \neq j \leqslant k} \int_0^t ds\, f(\sqrt{\kappa}(B_s^i - B_s^j))\right)\right]$$

The spatially-discrete Parabolic Anderson Model with $f = \delta_0$ also satisfies $\mathbb{E}[u_t(x)^k] \sim \exp(k^2)$. (Carmona-Molchanov (1994))

Spatially colored noise.

We can obtain similar results for colored noise, i.e. when $f \neq \delta_0$.

We first assume that $f(0) < \infty$ with appropriate tail behavior. The latter ensures that localization occurs.

In that case, one can prove that

$$c \exp(ck^2) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^2)$$

This can be directly obtained from the Feynman-Kac formula for moments of the solution (Bertini & Cancrini (1994), Hu & Nualart (2009), C. (2011))

$$\mathbb{E}[u_t(x)^k] = \mathbb{E}\left[\exp\left(\sum\sum_{1 \leqslant i \neq j \leqslant k} \int_0^t ds\, f(\sqrt{\kappa}(B_s^i - B_s^j))\right)\right]$$

The spatially-discrete Parabolic Anderson Model with $f = \delta_0$ also satisfies $\mathbb{E}[u_t(x)^k] \sim \exp(k^2)$. (Carmona-Molchanov (1994))

We can obtain similar results for colored noise, i.e. when $f \neq \delta_0$.

We first assume that $f(0) < \infty$ with appropriate tail behavior. The latter ensures that localization occurs.

In that case, one can prove that

$$c \exp(ck^2) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^2)$$

This can be directly obtained from the Feynman-Kac formula for moments of the solution (Bertini & Cancrini (1994), Hu & Nualart (2009), C. (2011))

$$\mathbb{E}[u_t(x)^k] = \mathbb{E}\left[\exp\left(\sum\sum_{1\leqslant i\neq j\leqslant k} \int_0^t ds\, f(\sqrt{\kappa}(B_s^i - B_s^j))\right)\right]$$

The spatially-discrete Parabolic Anderson Model with $f = \delta_0$ also satisfies $\mathbb{E}[u_t(x)^k] \sim \exp(k^2)$. (Carmona-Molchanov (1994))

We can obtain similar results for colored noise, i.e. when $f \neq \delta_0$.

We first assume that $f(0) < \infty$ with appropriate tail behavior. The latter ensures that localization occurs.

In that case, one can prove that

$$c \exp(ck^2) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^2)$$

This can be directly obtained from the Feynman-Kac formula for moments of the solution (Bertini & Cancrini (1994), Hu & Nualart (2009), C. (2011))

$$\mathbb{E}[u_t(x)^k] = \mathbb{E}\left[\exp\left(\sum\sum_{1 \leqslant i \neq j \leqslant k} \int_0^t ds\, f(\sqrt{\kappa}(B_s^i - B_s^j))\right)\right]$$

The spatially-discrete Parabolic Anderson Model with $f = \delta_0$ also satisfies $\mathbb{E}[u_t(x)^k] \sim \exp(k^2)$. (Carmona-Molchanov (1994))

Spatially colored noise.

We can obtain similar results for colored noise, i.e. when $f \neq \delta_0$.

We first assume that $f(0) < \infty$ with appropriate tail behavior. The latter ensures that localization occurs.

In that case, one can prove that

$$c \exp(ck^2) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^2)$$

This can be directly obtained from the Feynman-Kac formula for moments of the solution (Bertini & Cancrini (1994), Hu & Nualart (2009), C. (2011))

$$\mathbb{E}[u_t(x)^k] = \mathbb{E}\left[\exp\left(\sum\sum_{1\leqslant i\neq j\leqslant k} \int_0^t ds\, f(\sqrt{\kappa}(B_s^i - B_s^j))\right)\right]$$

The spatially-discrete Parabolic Anderson Model with $f = \delta_0$ also satisfies $\mathbb{E}[u_t(x)^k] \sim \exp(k^2)$. (Carmona-Molchanov (1994))

Plugging into the usual machinery gives

$$\log P(|u_t(x)| \geqslant \lambda) \asymp -(\log \lambda)^2$$

And,

Theorem (C.-Joseph-Khoshnevisan-Shiu (2011))

If $\sigma(x) = \lambda x$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log |x|)^{1/2}} \asymp 1 \qquad a.s.,$$

for all $t > 0$, and all sufficiently small κ.

We notice now that the behavior of the supremum of the solution does not depend on κ.

Plugging into the usual machinery gives

$$\log P(|u_t(x)| \geqslant \lambda) \asymp -(\log \lambda)^2$$

And,

Theorem (C.-Joseph-Khoshnevisan-Shiu (2011))

If $\sigma(x) = \lambda x$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log |x|)^{1/2}} \asymp 1 \qquad a.s.,$$

for all $t > 0$, and all sufficiently small κ.

We notice now that the behavior of the supremum of the solution does not depend on κ.

$f(0) < \infty$

Plugging into the usual machinery gives

$$\log P(|u_t(x)| \geqslant \lambda) \asymp -(\log \lambda)^2$$

And,

Theorem (C.-Joseph-Khoshnevisan-Shiu (2011))

If $\sigma(x) = \lambda x$, then

$$\limsup_{|x| \to \infty} \frac{\log u_t(x)}{(\log |x|)^{1/2}} \asymp 1 \qquad a.s.,$$

for all $t > 0$, and all sufficiently small κ.

We notice now that the behavior of the supremum of the solution does not depend on κ.

We have a similar result in the case of a Riesz kernel covariance function, i.e. $f(x) = |x|^{-\alpha}$.

Then,

$$c \exp(ck^{\frac{4-\alpha}{2-\alpha}}) \leqslant \mathbb{E}[u_t(x)^k] \leqslant C \exp(Ck^{\frac{4-\alpha}{2-\alpha}}),$$

and

Theorem (C.-Joseph-Khoshnevisan-Shiu (2011))

If $\sigma(x) = \lambda x$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{2/(4-\alpha)}} \asymp \kappa^{-\frac{\alpha}{4-\alpha}} \qquad a.s.\ \textit{for all } t > 0.$$

Riesz kernel correlation.

We have a similar result in the case of a Riesz kernel covariance function, i.e. $f(x) = |x|^{-\alpha}$.

Then,

$$c \exp(ck^{\frac{4-\alpha}{2-\alpha}}) \leqslant \mathbb{E}[u_t(x)^k] \leqslant C \exp(Ck^{\frac{4-\alpha}{2-\alpha}}),$$

and

Theorem (C.-Joseph-Khoshnevisan-Shiu (2011))

If $\sigma(x) = \lambda x$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log |x|)^{2/(4-\alpha)}} \asymp \kappa^{-\frac{\alpha}{4-\alpha}} \qquad a.s.\ \text{for all } t > 0.$$

Riesz kernel correlation.

We have a similar result in the case of a Riesz kernel covariance function, i.e. $f(x) = |x|^{-\alpha}$.

Then,

$$c \exp(ck^{\frac{4-\alpha}{2-\alpha}}) \leqslant \mathbb{E}[u_t(x)^k] \leqslant C \exp(Ck^{\frac{4-\alpha}{2-\alpha}}),$$

and

Theorem (C.-Joseph-Khoshnevisan-Shiu (2011))

If $\sigma(x) = \lambda x$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{2/(4-\alpha)}} \asymp \kappa^{-\frac{\alpha}{4-\alpha}} \qquad \text{a.s. for all } t > 0.$$

For (SHE) with $\sigma(x) = \lambda x$ (Parabolic Anderson Model), we have proved both for white and colored noise:

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{\psi}} \asymp \kappa^{-(2\psi-1)} \qquad \text{a.s. for all } t > 0.$$

To summarize, we have:

Noise	Ψ	$2\Psi - 1$
Space-time white noise	$2/3$	$1/3$
Riesz kernel	$2/(4-\alpha)$	$\alpha/(4-\alpha)$.
Colored noise with $f(0) < \infty$	$1/2$	0

Since in dimension $d = 1$, $\alpha \in (0,1)$, Riesz kernels show that we can achieve any exponent between space-time white noise and bounded correlation.

Summary and a comparison.

For (SHE) with $\sigma(x) = \lambda x$ (Parabolic Anderson Model), we have proved both for white and colored noise:

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log |x|)^{\psi}} \asymp \kappa^{-(2\psi-1)} \qquad \text{a.s. for all } t > 0.$$

To summarize, we have:

Noise	Ψ	$2\Psi - 1$
Space-time white noise	$2/3$	$1/3$
Riesz kernel	$2/(4-\alpha)$	$\alpha/(4-\alpha)$.
Colored noise with $f(0) < \infty$	$1/2$	0

Since in dimension $d = 1$, $\alpha \in (0, 1)$, Riesz kernels show that we can achieve any exponent between space-time white noise and bounded correlation.

For (SHE) with $\sigma(x) = \lambda x$ (Parabolic Anderson Model), we have proved both for white and colored noise:

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{\psi}} \asymp \kappa^{-(2\psi-1)} \qquad \text{a.s. for all } t > 0.$$

To summarize, we have:

Noise	Ψ	$2\Psi - 1$
Space-time white noise	$2/3$	$1/3$
Riesz kernel	$2/(4-\alpha)$	$\alpha/(4-\alpha)$.
Colored noise with $f(0) < \infty$	$1/2$	0

Since in dimension $d = 1$, $\alpha \in (0,1)$, Riesz kernels show that we can achieve any exponent between space-time white noise and bounded correlation.

Space-time scaling

We have

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{\psi}} \asymp \kappa^{-(2\psi-1)} \qquad \text{a.s. for all } t>0.$$

Since we can think of κ to intuitively scale like $1/t$, these results can be understood as space-time scaling results.

We recover the scaling exponents obtained for the KPZ equation.

- Balazs-Quastel-Seppäläinen (2011)

We point out that these results are valid for any $t>0$. However, if we understand $\kappa \sim 1/t$, then we see that the constant in the result gets small as $t \to 0$. "One needs to go further out in space in order to find the high peaks."

We have

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{\psi}} \asymp \kappa^{-(2\psi-1)} \qquad \text{a.s. for all } t>0.$$

Since we can think of κ to intuitively scale like $1/t$, these results can be understood as space-time scaling results.

We recover the scaling exponents obtained for the KPZ equation.

- Balazs-Quastel-Seppäläinen (2011)

We point out that these results are valid for any $t > 0$. However, if we understand $\kappa \sim 1/t$, then we see that the constant in the result gets small as $t \to 0$. "One needs to go further out in space in order to find the high peaks."

We have

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{\psi}} \asymp \kappa^{-(2\psi-1)} \qquad \text{a.s. for all } t>0.$$

Since we can think of κ to intuitively scale like $1/t$, these results can be understood as space-time scaling results.

We recover the scaling exponents obtained for the KPZ equation.

- Balazs-Quastel-Seppäläinen (2011)

We point out that these results are valid for any $t > 0$. However, if we understand $\kappa \sim 1/t$, then we see that the constant in the result gets small as $t \to 0$. "One needs to go further out in space in order to find the high peaks."

Space-time scaling

We have

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{\psi}} \asymp \kappa^{-(2\psi-1)} \qquad \text{a.s. for all } t > 0.$$

Since we can think of κ to intuitively scale like $1/t$, these results can be understood as space-time scaling results.

We recover the scaling exponents obtained for the KPZ equation.

- Balazs-Quastel-Seppäläinen (2011)

We point out that these results are valid for any $t > 0$. However, if we understand $\kappa \sim 1/t$, then we see that the constant in the result gets small as $t \to 0$. "One needs to go further out in space in order to find the high peaks."

We have

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log |x|)^{\psi}} \asymp \kappa^{-(2\psi-1)} \qquad \text{a.s. for all } t > 0.$$

Since we can think of κ to intuitively scale like $1/t$, these results can be understood as space-time scaling results.

We recover the scaling exponents obtained for the KPZ equation.

- Balazs-Quastel-Seppäläinen (2011)

We point out that these results are valid for any $t > 0$. However, if we understand $\kappa \sim 1/t$, then we see that the constant in the result gets small as $t \to 0$. "One needs to go further out in space in order to find the high peaks."

Extensions and open problems.

Similar results hold for:

- The stochastic non-linear wave equation.
- Higher dimensions.
- The size of the intermittent islands, through a more careful analysis of the proofs (see Davar's lecture # 10).

Work in progress (with R.Balan):

- Equations driven by a fractional noise in time. (See also Chen, Hu, Song, Xing (2013))

Open problem:

- What happens for generators of Lévy processes instead of Laplacian ?

Extensions and open problems.

Similar results hold for:

- The stochastic non-linear wave equation.
- Higher dimensions.
- The size of the intermittent islands, through a more careful analysis of the proofs (see Davar's lecture # 10).

Work in progress (with R.Balan):

- Equations driven by a fractional noise in time. (See also Chen, Hu, Song, Xing (2013))

Open problem:

- What happens for generators of Lévy processes instead of Laplacian ?

Similar results hold for:

- The stochastic non-linear wave equation.
- Higher dimensions.
- The size of the intermittent islands, through a more careful analysis of the proofs (see Davar's lecture $\#$ 10).

Work in progress (with R.Balan):

- Equations driven by a fractional noise in time. (See also Chen, Hu, Song, Xing (2013))

Open problem:

- What happens for generators of Lévy processes instead of Laplacian ?

Similar results hold for:

- The stochastic non-linear wave equation.
- Higher dimensions.
- The size of the intermittent islands, through a more careful analysis of the proofs (see Davar's lecture # 10).

Work in progress (with R.Balan):

- Equations driven by a fractional noise in time. (See also Chen, Hu, Song, Xing (2013))

Open problem:

- What happens for generators of Lévy processes instead of Laplacian ?

Similar results hold for:

- The stochastic non-linear wave equation.
- Higher dimensions.
- The size of the intermittent islands, through a more careful analysis of the proofs (see Davar's lecture # 10).

Work in progress (with R.Balan):

- Equations driven by a fractional noise in time. (See also Chen, Hu, Song, Xing (2013))

Open problem:

- What happens for generators of Lévy processes instead of Laplacian ?

We study the stochastic nonlinear wave equation

$$\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 (\Delta u)_t(x) + \sigma(u_t(x)) \dot{W}(t,x), \qquad \text{(SWE)}$$

where

- $t > 0$, $x \in \mathbb{R}$,
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ,
- $\dot{W}$ is space-time white noise.
- the initial function $u_0 > 0$ and derivative v_0 are constant.

This equation has a unique solution according to Dalang (1999).

If $L_\sigma > 0$, then the solution is intermittent. (Dalang & Mueller (2009), C., Joseph, Khoshnevisan & Shiu (2012))

We study the stochastic nonlinear wave equation

$$\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 (\Delta u)_t(x) + \sigma(u_t(x)) \dot{W}(t,x), \qquad \text{(SWE)}$$

where

- $t > 0$, $x \in \mathbb{R}$,
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ,
- $\dot{W}$ is space-time white noise.
- the initial function $u_0 > 0$ and derivative v_0 are constant.

This equation has a unique solution according to Dalang (1999).

If $L_\sigma > 0$, then the solution is intermittent. (Dalang & Mueller (2009), C., Joseph, Khoshnevisan & Shiu (2012))

We study the stochastic nonlinear wave equation

$$\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 (\Delta u)_t(x) + \sigma(u_t(x)) \dot{W}(t,x), \tag{SWE}$$

where

- $t > 0$, $x \in \mathbb{R}$,
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ,
- $\dot{W}$ is space-time white noise.
- the initial function $u_0 > 0$ and derivative v_0 are constant.

This equation has a unique solution according to Dalang (1999).

If $L_\sigma > 0$, then the solution is intermittent. (Dalang & Mueller (2009), C., Joseph, Khoshnevisan & Shiu (2012))

The wave equation.

We study the stochastic nonlinear wave equation

$$\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 (\Delta u)_t(x) + \sigma(u_t(x)) \dot{W}(t,x), \qquad \text{(SWE)}$$

where

- $t > 0$, $x \in \mathbb{R}$,
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ,
- $\dot{W}$ is space-time white noise.
- the initial function $u_0 > 0$ and derivative v_0 are constant.

This equation has a unique solution according to Dalang (1999).

If $L_\sigma > 0$, then the solution is intermittent. (Dalang & Mueller (2009), C., Joseph, Khoshnevisan & Shiu (2012))

We study the stochastic nonlinear wave equation

$$\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 (\Delta u)_t(x) + \sigma(u_t(x)) \dot{W}(t,x), \tag{SWE}$$

where

- $t > 0$, $x \in \mathbb{R}$,
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ,
- $\dot{W}$ is space-time white noise.
- the initial function $u_0 > 0$ and derivative v_0 are constant.

This equation has a unique solution according to Dalang (1999).

If $L_\sigma > 0$, then the solution is intermittent. (Dalang & Mueller (2009), C., Joseph, Khoshnevisan & Shiu (2012))

We study the stochastic nonlinear wave equation

$$\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 (\Delta u)_t(x) + \sigma(u_t(x))\dot{W}(t,x), \tag{SWE}$$

where

- $t > 0$, $x \in \mathbb{R}$,
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ,
- $\dot{W}$ is space-time white noise.
- the initial function $u_0 > 0$ and derivative v_0 are constant.

This equation has a unique solution according to Dalang (1999).

If $L_\sigma > 0$, then the solution is intermittent. (Dalang & Mueller (2009), C., Joseph, Khoshnevisan & Shiu (2012))

We study the stochastic nonlinear wave equation

$$\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 (\Delta u)_t(x) + \sigma(u_t(x))\dot{W}(t,x), \tag{SWE}$$

where

- $t > 0$, $x \in \mathbb{R}$,
- $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant Lip_σ,
- $\dot{W}$ is space-time white noise.
- the initial function $u_0 > 0$ and derivative v_0 are constant.

This equation has a unique solution according to Dalang (1999).

If $L_\sigma > 0$, then the solution is intermittent. (Dalang & Mueller (2009), C., Joseph, Khoshnevisan & Shiu (2012))

We can use a similar argument as for the heat equation and get

$$c\exp(ck\log(k)) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C\exp(Ck\log(k)).$$

A consequence of this is the following theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ and, $0 < a < \sigma(x) < b$ for all $x \in \mathbb{R}$, then

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \asymp \kappa^{1/2} \qquad a.s.\ \text{for all } t > 0.$$

This shows that if σ is bounded above and below, then $u_t(x)$ behaves as a Gaussian process.

Notice the difference with the heat equation in the behavior of κ.

Case where σ is bounded above and below.

We can use a similar argument as for the heat equation and get

$$c\exp(ck\log(k)) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C\exp(Ck\log(k)).$$

A consequence of this is the following theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ and, $0 < a < \sigma(x) < b$ for all $x \in \mathbb{R}$, then

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \asymp \kappa^{1/2} \qquad a.s. \text{ for all } t > 0.$$

This shows that if σ is bounded above and below, then $u_t(x)$ behaves as a Gaussian process.

Notice the difference with the heat equation in the behavior of κ.

We can use a similar argument as for the heat equation and get

$$c\exp(ck\log(k)) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C\exp(Ck\log(k)).$$

A consequence of this is the following theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ and, $0 < a < \sigma(x) < b$ for all $x \in \mathbb{R}$, then

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \asymp \kappa^{1/2} \qquad a.s. \text{ for all } t > 0.$$

This shows that if σ is bounded above and below, then $u_t(x)$ behaves as a Gaussian process.

Notice the difference with the heat equation in the behavior of κ.

We can use a similar argument as for the heat equation and get

$$c \exp(ck \log(k)) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck \log(k)).$$

A consequence of this is the following theorem.

Theorem (C.-Joseph-Khoshnevisan (2011))

If $f(x) = \delta_0(x)$ and, $0 < a < \sigma(x) < b$ for all $x \in \mathbb{R}$, then

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \asymp \kappa^{1/2} \qquad a.s. \text{ for all } t > 0.$$

This shows that if σ is bounded above and below, then $u_t(x)$ behaves as a Gaussian process.

Notice the difference with the heat equation in the behavior of κ.

When $\sigma(u) = u$, we obtain

$$c \exp(ck^{3/2}) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^{3/2})$$

A consequence of this is the following estimate:

Theorem (2013 ??)

If $f(x) = \delta_0(x)$ and $\sigma(u) = \lambda u$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log |x|)^{1/3}} \asymp \kappa^{1/3} \qquad a.s. \text{ for all } t > 0.$$

This result would also show a drastically different behavior in the intermittent vs. non-intermittent case. It relies on a yet to be completed proof.

When $\sigma(u) = u$, we obtain

$$c\exp(ck^{3/2}) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C\exp(Ck^{3/2})$$

A consequence of this is the following estimate:

Theorem (2013 ??)

If $f(x) = \delta_0(x)$ and $\sigma(u) = \lambda u$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log|x|)^{1/3}} \asymp \kappa^{1/3} \qquad a.s.\ \text{for all } t > 0.$$

This result would also show a drastically different behavior in the intermittent vs. non-intermittent case. It relies on a yet to be completed proof.

When $\sigma(u) = u$, we obtain

$$c \exp(ck^{3/2}) \leqslant \mathbb{E}[|u_t(x)|^k] \leqslant C \exp(Ck^{3/2})$$

A consequence of this is the following estimate:

Theorem (2013 ??)

If $f(x) = \delta_0(x)$ and $\sigma(u) = \lambda u$, then

$$\limsup_{|x|\to\infty} \frac{\log u_t(x)}{(\log |x|)^{1/3}} \asymp \kappa^{1/3} \qquad a.s.\ \textit{for all } t > 0.$$

This result would also show a drastically different behavior in the intermittent vs. non-intermittent case. It relies on a yet to be completed proof.

Thank you for your attention!