Chaotic properties for a family of SPDEs.
(More on Lecture #9 ...)

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This is joint work with:
- Mathew Joseph (Sheffield)
- Davar Khoshnevisan (U of Utah)
- Shang-Yuan Shiu (NCU Taiwan)
We study the non\-linear stochastic heat equation, namely

\[
\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2} \Delta u_t(x) + \sigma(u_t(x)) \dot{W}(t, x), \tag{SHE}
\]

where

- \( t > 0, \ x \in \mathbb{R}; \)
- \( \sigma : \mathbb{R} \to \mathbb{R} \) is a Lipschitz function with constant \( \text{Lip}_\sigma; \)
- \( \dot{W} \) is a noise that is white in time and (possibly) correlated in space, i.e.

\[
\mathbb{E}[\dot{W}(t, x) \dot{W}(s, y)] = \delta_0(t - s) f(x - y),
\]

where \( f \) is a positive definite function (possibly \( \delta_0); \)
- the initial function \( u_0 : \mathbb{R} \to \mathbb{R}_+ \) is bounded.
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Motivation.

(SHE) arises in different settings. For instance,

- With $\sigma(x) = \lambda x$, (SHE) is the continuous version of the Parabolic Anderson Model. It models branching processes in a random environment, when the spatial motion is a Brownian motion.

  \textit{Ref:}\n  \begin{itemize}
  \item Carmona & Molchanov (1994).
  \end{itemize}

- (SHE) is connected to the so-called KPZ equation, modelling growing interfaces: $\log u_t(x)$ ”solves” the KPZ equation.

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  \item Hairer (2012).
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\((u_t(x); t > 0, x \in \mathbb{R})\) satisfying:

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where \(p_t\) is the heat kernel and the stochastic integral is defined in the sense of Walsh (1986).

**Theorem (Dalang (1999))**

*The non-linear stochastic heat equation* (SHE) *has a unique random-field solution such that, for all* \(T > 0,\)

\[
 \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} \mathbb{E}[u_t(x)^2] < \infty.
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We have seen throughout the week that the solution to SHE is a (weakly-) intermittent random-field, provided that:

\[ L_\sigma := \inf_{x \in \mathbb{R}} \frac{|\sigma(x)|}{|x|} > 0. \]

\[ \sigma(0) = 0 \text{ if } \inf_{x \in \mathbb{R}} u_0(x) = 0. \]

Refs:
- Foondun & Khoshnevisan (2009)

Weak intermittency implies that the solution develops very high peaks concentrated on some spatial islands for large time \( t \).

For physicists, intermittency is believed to happen in part because the system is chaotic. What happens before the onset of intermittency?
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A couple reminders.

We have seen in Davar’s lectures:

**Theorem (Foondun-Khoshnevisan (2009))**

If $\sigma(0) = 0$ and $u_0$ has compact support, then for all $t > 0$,

$$\sup_{x \in \mathbb{R}} u_t(x) < \infty \quad \text{a.s.}$$

and

**Theorem (C.-Joseph-Khoshnevisan (2011))**

If $\sigma(x) = \lambda x$, then

$$\limsup_{|x| \to \infty} \frac{\log u_t(x)}{(\log |x|)^{2/3}} \asymp C \quad \text{a.s. for all } t > 0.$$ 

Remark: In the deterministic case, the solution remains bounded, whether the initial condition has compact support or is bounded away from 0.

$\implies$ the noise induces a chaotic behavior, i.e. a dependence on the initial conditions.
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Summary of the techniques.

We essentially need two main ingredients to obtain the result:

- A tail probability estimate on the behavior of $u_t(x)$ for fixed $t$ and $x$. This is obtained with good moment estimates and the Paley-Zygmund inequality.
- A localization result, namely that $u_t(x)$ and $u_t(y)$ are somewhat independent if $x$ and $y$ are sufficiently far away.

Assume from now on (for simplicity) that $u_0(x) \equiv 1$.

**Quizz:** We know that if $f(x) = \delta_0(x)$, then

$$\mathbb{E}[|u_t(x)|^k] \leq C \exp(Ck^\gamma t).$$

What is $\gamma$?
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We also know that $\sigma(u) = \lambda u$ (Parabolic Anderson Model) achieves the upper bound.
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More on the techniques.

Now, we can get a general lower bound on moments, namely

**Proposition**

If \( f(x) = \delta_0(x) \) and \( \inf_{u \in \mathbb{R}} \sigma(u) := a > 0 \), then

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But \( v_t(x) \) is Gaussian, hence \( \mathbb{E}[v_t(x)^{2k}] \sim \sigma^{2k} (2k)! / k! 2^k \). Stirling's formula gives the result.
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We have:
\[ c \exp(ck \log(k)) \leq \mathbb{E}[u_t(x)^k] \leq C \exp(Ck^3) \]

We plug this into the Paley-Zygmund inequality to get

\[ \log P(|u_t(x)| \geq \lambda) \gtrapprox -\lambda^6 \]

A consequence of this is the following general theorem.

**Theorem (C.-Joseph-Khoshnevisan (2011))**

If \( f(x) = \delta_0(x) \) and \( \inf_{x \in \mathbb{R}} \sigma(x) \geq a > 0 \), then for all \( t > 0 \),

\[ \sup_{x \in [-R,R]} u_t(x) \gtrapprox (\log R)^{1/6} \quad \text{as } R \to \infty. \]

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\[
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If \( f(x) = \delta_0(x) \) and, \( 0 < a < \sigma(x) < b \) for all \( x \in \mathbb{R} \), then

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Spatially colored noise.

We can obtain similar results for colored noise, i.e. when \( f \neq \delta_0 \).

We first assume that \( f(0) < \infty \) with appropriate tail behavior. The latter ensures that localization occurs.

In that case, one can prove that

\[
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This can be directly obtained from the Feynman-Kac formula for moments of the solution (Bertini & Cancrini (1994), Hu & Nualart (2009), C. (2011))

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\mathbb{E}[u_t(x)^k] = \mathbb{E} \left[ \exp \left( \sum_{1 \leq i \neq j \leq k} \int_0^t ds f(\sqrt{\kappa}(B^i_s - B^j_s)) \right) \right]
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The spatially-discrete Parabolic Anderson Model with \( f = \delta_0 \) also satisfies \( \mathbb{E}[u_t(x)^k] \sim \exp(k^2) \). (Carmona-Molchanov (1994))
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*If* $\sigma(x) = \lambda x$, *then*

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We recover the scaling exponents obtained for the KPZ equation.

- Balazs-Quastel-Seppäläinen (2011)

We point out that these results are valid for any $t > 0$. However, if we understand $\kappa \sim 1/t$, then we see that the constant in the result gets small as $t \to 0$. "One needs to go further out in space in order to find the high peaks."
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Extensions and open problems.

Similar results hold for:

- The stochastic non-linear wave equation.
- Higher dimensions.
- The size of the intermittent islands, through a more careful analysis of the proofs (see Davar’s lecture # 10).

Work in progress (with R.Balan):

- Equations driven by a fractional noise in time. (See also Chen, Hu, Song, Xing (2013))

Open problem:

- What happens for generators of Lévy processes instead of Laplacian?
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The wave equation.

We study the stochastic nonlinear wave equation

\[
\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 (\Delta u)_t(x) + \sigma(u_t(x)) \dot{\mathcal{W}}(t, x),
\]

(SWE)

where

- \( t > 0, \ x \in \mathbb{R} \),
- \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) is a Lipschitz function with constant \( \text{Lip}_\sigma \),
- \( \dot{\mathcal{W}} \) is space-time white noise.
- the initial function \( u_0 > 0 \) and derivative \( v_0 \) are constant.

This equation has a unique solution according to Dalang (1999).

If \( L_\sigma > 0 \), then the solution is intermittent. (Dalang & Mueller (2009), C., Joseph, Khoshnevisan & Shiu (2012))
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Case where $\sigma$ is bounded above and below.

We can use a similar argument as for the heat equation and get

$$c \exp(ck \log(k)) \leq \mathbb{E}[|u_t(x)|^k] \leq C \exp(Ck \log(k)).$$

A consequence of this is the following theorem.

**Theorem (C.-Joseph-Khoshnevisan (2011))**

If $f(x) = \delta_0(x)$ and, $0 < a < \sigma(x) < b$ for all $x \in \mathbb{R}$, then

$$\limsup_{|x| \rightarrow \infty} \frac{u_t(x)}{(\log |x|)^{1/2}} \asymp \kappa^{1/2} \quad \text{a.s. for all } t > 0.$$

This shows that if $\sigma$ is bounded above and below, then $u_t(x)$ behaves as a Gaussian process.

Notice the difference with the heat equation in the behavior of $\kappa$. 
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SPDEs: chaotic character. 

Daniel Conus 

NSF/CBMS 

Aug. 22, 2013 

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Case where $\sigma(u) = \lambda u$.

When $\sigma(u) = u$, we obtain

$$c \exp(ck^{3/2}) \leq \mathbb{E}[|u_t(x)|^k] \leq C \exp(Ck^{3/2})$$

A consequence of this is the following estimate:

**Theorem (2013 ??)**

If $f(x) = \delta_0(x)$ and $\sigma(u) = \lambda u$, then

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Thank you for your attention!