Analysis of Stochastic PDEs
CBMS-NSF Course at Michigan State University

Davar Khoshnevisan

Department of Mathematics
University of Utah
http://www.math.utah.edu/~davar
Is the Sun Missing Its Spots?

SUN GAZING These photos show sunspots near solar maximum on July 19, 2000, and near solar minimum on March 18, 2009. Some global warming skeptics speculate that the Sun may be on the verge of an extended slumber.

By KENNETH CHANG
Published: July 20, 2009
Nonlinear noise excitation (Lecture 1)

\[ \partial_t u = \frac{1}{2} \partial_x^2 u + \lambda u \xi \text{ on } [0, 1] \text{ with Dirichlet BC} \]

\[ u_0(x) = \sin(\pi x) \] [K-Kim, 2013]

**Figure:** \(\lambda = 0; \ u_t(x) = \sin(\pi x) \exp(-\pi^2 t/2)\)
Nonlinear noise excitation (Lecture 1)
\[
\partial_t u = \frac{1}{2} \partial_x^2 u + \lambda u \xi \quad \text{on} \quad [0, 1] \quad \text{with Dirichlet BC}
\]
\[
u_0(x) = \sin(\pi x) \quad [\text{K-Kim, 2013}]
\]

Figure: \( \lambda = 0.1; \) max. peak \( \approx 1.4 \)
Nonlinear noise excitation (Lecture 1)

\[ \partial_t u = \frac{1}{2} \partial_x^2 u + \lambda u \xi \text{ on } [0, 1] \text{ with Dirichlet BC} \]

\[ u_0(x) = \sin(\pi x) \text{ [K-Kim, 2013]} \]

**Figure:** \( \lambda = 2; \) max. peak \( \approx 35 \)
Nonlinear noise excitation (Lecture 1)

\[ \partial_t u = \frac{1}{2} \partial_x^2 u + \lambda u \xi \text{ on } [0, 1] \text{ with Dirichlet BC} \]
\[ u_0(x) = \sin(\pi x) \] [K-Kim, 2013]

Figure: \( \lambda = 5; \text{ max. peak } \approx 2.5 \times 10^{19} \)
Nonlinear noise excitation (Lecture 1)

Intermittency

- **Intermittent** (Dictionary.com):
  - Stopping or ceasing for a time; alternately ceasing and beginning again: an intermittent pain;
  - Alternately functioning and not functioning or alternately functioning properly and improperly.

Deep relations to fluid dynamics (Baxendale-Rozovski˘ı, 1993), turbulence (Mandelbrot, 1983; Majda, 1993; Gibbon and Titi, 2005), complex chemical reactions and the large-scale structure of galaxies (Molchanov, 1991; Shandarin-Zel’dovich, 1989; Zel’dovich et al, 1987, 1988, 1990) . . .

Complex problems in random media are associated to intermittency: As the systems feels more noise, it can begin to act erratically.

Many field theories (SPDEs) yield intermittent solutions.
Nonlinear noise excitation (Lecture 1)
Intermittency

- Intermittent (Dictionary.com):
  - stopping or ceasing for a time; alternately ceasing and beginning again: an intermittent pain;
Nonlinear noise excitation (Lecture 1)

Intermittency

- Intermittent (Dictionary.com):
  - stopping or ceasing for a time; alternately ceasing and beginning again: an intermittent pain;
  - alternately functioning and not functioning or alternately functioning properly and improperly.
Intermittency

- **Intermittent (Dictionary.com):**
  - stopping or ceasing for a time; alternately ceasing and beginning again: an intermittent pain;
  - alternately functioning and not functioning or alternately functioning properly and improperly.

- Deep relations to fluid dynamics (Baxendale-Rozovskiĭ, 1993), turbulence (Mandelbrot, 1983; Majda, 1993; Gibbon and Titi, 2005), complex chemical reactions and the large-scale structure of galaxies (Molchanov, 1991; Shandarin-Zel’dovich, 1989; Zel’dovich et al, 1987, 1988, 1990) . . .
Nonlinear noise excitation (Lecture 1)

Intermittency

- **Intermittent** (Dictionary.com):
  - stopping or ceasing for a time; alternately ceasing and beginning again: an intermittent pain;
  - alternately functioning and not functioning or alternately functioning properly and improperly.

- Deep relations to fluid dynamics (Baxendale-Rozovskii, 1993), turbulence (Mandelbrot, 1983; Majda, 1993; Gibbon and Titi, 2005), complex chemical reactions and the large-scale structure of galaxies (Molchanov, 1991; Shandarin-Zel’dovich, 1989; Zel’dovich et al, 1987, 1988, 1990) . . .

- Complex problems in random media are associated to intermittency: As the systems feels more noise, it can begin to act erratically.
Nonlinear noise excitation (Lecture 1)

Intermittency

- **Intermittent** (Dictionary.com):
  - stopping or ceasing for a time; alternately ceasing and beginning again: an intermittent pain;
  - alternately functioning and not functioning or alternately functioning properly and improperly.

- Deep relations to fluid dynamics (Baxendale-Rozovskiĭ, 1993), turbulence (Mandelbrot, 1983; Majda, 1993; Gibbon and Titi, 2005), complex chemical reactions and the large-scale structure of galaxies (Molchanov, 1991; Shandarin-Zel’dovich, 1989; Zel’dovich et al, 1987, 1988, 1990) . . .

- Complex problems in random media are associated to intermittency: As the systems feels more noise, it can begin to act erratically.

- Many field theories (SPDEs) yield intermittent solutions.
Let $\mathcal{L}(\mathbb{R}^m)$ denote the ring of all Borel-measurable subsets of $\mathbb{R}^m$ that have finite Lebesgue measure.
Let $\mathcal{L}(\mathbb{R}^m)$ denote the ring of all Borel-measurable subsets of $\mathbb{R}^m$ that have finite Lebesgue measure.
Let $\mathcal{L}(\mathbb{R}^m)$ denote the ring of all Borel-measurable subsets of $\mathbb{R}^m$ that have finite Lebesgue measure.

**Definition (Wiener, 1923)**

White noise on $\mathbb{R}^m$ is a mean-zero set-indexed Gaussian random field $\{\xi(A)\}_{A \in \mathcal{L}(\mathbb{R}^m)}$ with

$$\text{Cov}(\xi(A_1), \xi(A_2)) = |A_1 \cap A_2| \quad (A_i \in \mathcal{L}(\mathbb{R}^m)),$$

where $|\cdots|$ denotes the $m$-dimensional Lebesgue measure.
Gaussian random fields [GRFs] (Lecture 2)

White noise

- Let $\mathcal{L}(\mathbb{R}^m)$ denote the ring of all Borel-measurable subsets of $\mathbb{R}^m$ that have finite Lebesgue measure.

- **Definition (Wiener, 1923)**

  White noise on $\mathbb{R}^m$ is a mean-zero set-indexed Gaussian random field [GRF] $\{\xi(A)\}_{A \in \mathcal{L}(\mathbb{R}^m)}$ with

  $$\text{Cov}(\xi(A_1), \xi(A_2)) = |A_1 \cap A_2| \quad (A_i \in \mathcal{L}(\mathbb{R}^m)),$$

  where $|\cdots|$ denotes the $m$-dimensional Lebesgue measure.

- Easy fact: White noise exists and is an $L^2(\Omega)$-valued countably-additive measure on $\mathcal{L}(\mathbb{R}^m)$. 
We can identify the \( L^2(\Omega) \)-valued measure \( \xi \) with a \( L^2(\Omega) \)-valued integral:
Gaussian random fields [GRFs] (Lecture 2)
Wiener integrals

- We can identify the $L^2(\Omega)$-valued measure $\xi$ with a $L^2(\Omega)$-valued integral:
  - Suppose $h : \mathbb{R}^m \to \mathbb{R}$ is elementary; i.e.,
    
    \[ h(x) := \sum_{i=1}^n c_i 1_{A_i}(x) \]
    
    where $A_i \in \mathcal{L}(\mathbb{R}^n)$ are disjoint and $c_i \in \mathbb{R}$. Then,
    
    \[ \xi(h) := \int h \, d\xi := \int h(x) \xi(dx) := \sum_{i=1}^n c_i \xi(A_i) \]

    is defined unambiguously; the preceding is: (a) Linear in $h$ [a.s.]; (ii) A GRF indexed by all elementary functions $h$; and (iii)

    \[ E[\xi(h)] = 0 \quad \text{and} \quad E\left(|\xi(h)|^2\right) = \|h\|_{L^2(\mathbb{R}^m)}^2. \]
We can identify the $L^2(\Omega)$-valued measure $\xi$ with a $L^2(\Omega)$-valued integral:

Suppose $h : \mathbb{R}^m \to \mathbb{R}$ is elementary; i.e.,

$$h(x) := \sum_{i=1}^n c_i \mathbf{1}_{A_i}(x)$$

where $A_i \in \mathcal{L}(\mathbb{R}^n)$ are disjoint and $c_i \in \mathbb{R}$. Then,

$$\xi(h) := \int h \, d\xi := \int h(x) \, \xi(dx) := \sum_{i=1}^n c_i \xi(A_i)$$

is defined unambiguously; the preceding is: (a) Linear in $h$ [a.s.]; (ii) A GRF indexed by all elementary functions $h$; and (iii)

$$E[\xi(h)] = 0 \quad \text{and} \quad E\left(|\xi(h)|^2\right) = \|h\|_{L^2(\mathbb{R}^m)}^2.$$ 

This is the \underline{Wiener isometry}: $\|\xi(h)\|_{L^2(\Omega)} = \|h\|_{L^2(\mathbb{R}^m)}$. 

\[\text{Gaussian random fields [GRFs] (Lecture 2)}\]
\[\text{Wiener integrals}\]
We can identify the $L^2(\Omega)$-valued measure $\xi$ with a $L^2(\Omega)$-valued integral:

Suppose $h : \mathbb{R}^m \to \mathbb{R}$ is elementary; i.e.,

$$h(x) := \sum_{i=1}^n c_i \mathbf{1}_{A_i}(x)$$

where $A_i \in \mathcal{L}(\mathbb{R}^n)$ are disjoint and $c_i \in \mathbb{R}$. Then,

$$\xi(h) := \int h \, d\xi := \int h(x) \xi(dx) := \sum_{i=1}^n c_i \xi(A_i)$$

is defined unambiguously; the preceding is: (a) Linear in $h$ [a.s.]; (ii) A GRF indexed by all elementary functions $h$; and (iii)

$$E[\xi(h)] = 0 \quad \text{and} \quad E(\left|\xi(h)\right|^2) = \|h\|^2_{L^2(\mathbb{R}^m)}.$$

This is the Wiener isometry: $\|\xi(h)\|_{L^2(\Omega)} = \|h\|_{L^2(\mathbb{R}^m)}$.

Define $\xi(h) := \int h \, d\xi := \int h(x) \xi(dx)$ for all $h \in L^2(\mathbb{R}^m)$ by density.
We can identify the $L^2(\Omega)$-valued measure $\xi$ with a $L^2(\Omega)$-valued integral:

- Suppose $h : \mathbb{R}^m \to \mathbb{R}$ is elementary; i.e.,
  
  $$h(x) := \sum_{i=1}^n c_i 1_{A_i}(x)$$

  where $A_i \in \mathcal{L}(\mathbb{R}^n)$ are disjoint and $c_i \in \mathbb{R}$. Then,

  $$\xi(h) := \int h \, d\xi := \int h(x) \, \xi(dx) := \sum_{i=1}^n c_i \xi(A_i)$$

  is defined unambiguously; the preceding is: (a) Linear in $h$ [a.s.]; (ii) A GRF indexed by all elementary functions $h$; and (iii)

  $$E[\xi(h)] = 0 \quad \text{and} \quad E\left(\left|\xi(h)\right|^2\right) = \|h\|_{L^2(\mathbb{R}^m)}^2.$$ 

- This is the Wiener isometry:
  $$\|\xi(h)\|_{L^2(\Omega)} = \|h\|_{L^2(\mathbb{R}^m)}.$$ 

- Define $\xi(h) := \int h \, d\xi := \int h(x) \, \xi(dx)$ for all $h \in L^2(\mathbb{R}^m)$ by density.

- $\int_A h \, d\xi := \int h1_A \, d\xi.$
To summarize:

- $L^2(\mathbb{R}^m) \ni h \mapsto \int h \, d\xi \in L^2(\Omega)$ is a linear isometry, as well as a GRF;
- $\mathbb{E} \int h \, d\xi = 0$;
- $\text{Var} \int h \, d\xi = \int |h(x)|^2 \, dx \forall h \in L^2(\mathbb{R}^m)$;
- $\text{Cov}(\int h \, d\xi, \int g \, d\xi) = \int h(x) g(x) \, dx \forall h, g \in L^2(\mathbb{R}^m)$;
- If $h, g \in L^2(\mathbb{R}^m)$ and $h \perp g$ then $\int h \, d\xi$ is independent from $\int g \, d\xi$. 

Gaussian random fields [GRFs] (Lecture 2)
Wiener integrals
To summarize:

\[ L^2(\mathbb{R}^m) \ni h \mapsto \int h \, d\xi \in L^2(\Omega) \text{ is a linear isometry, as well as a GRF}; \]
To summarize:

- $L^2(\mathbb{R}^m) \ni h \mapsto \int h \, d\xi \in L^2(\Omega)$ is a linear isometry, as well as a GRF;
- $\mathbb{E} \int h \, d\xi = 0$; $\text{Var} \int h \, d\xi = \int_{\mathbb{R}^m} |h(x)|^2 \, dx \quad \forall h \in L^2(\mathbb{R}^m)$;
To summarize:

- $L^2(\mathbb{R}^m) \ni h \mapsto \int h \, d\xi \in L^2(\Omega)$ is a linear isometry, as well as a GRF;
- $\mathbb{E} \int h \, d\xi = 0$; $\text{Var} \int h \, d\xi = \int_{\mathbb{R}^m} |h(x)|^2 \, dx$ $\forall h \in L^2(\mathbb{R}^m)$;
- $\text{Cov}(\int h \, d\xi, \int g \, d\xi) = \int_{\mathbb{R}^m} h(x)g(x) \, dx$ $\forall h, g \in L^2(\mathbb{R}^m)$;
To summarize:

- $L^2(\mathbb{R}^m) \ni h \mapsto \int h \, d\xi \in L^2(\Omega)$ is a linear isometry, as well as a GRF;
- $\mathbb{E} \int h \, d\xi = 0$; $\text{Var} \int h \, d\xi = \int_{\mathbb{R}^m} |h(x)|^2 \, dx \quad \forall h \in L^2(\mathbb{R}^m)$;
- $\text{Cov}(\int h \, d\xi, \int g \, d\xi) = \int_{\mathbb{R}^m} h(x)g(x) \, dx \quad \forall h, g \in L^2(\mathbb{R}^m)$;
- If $h, g \in L^2(\mathbb{R}^m)$ and $h \perp g$ then $\int h \, d\xi$ is independent from $\int g \, d\xi$. 
If $f \in L^2(\mathbb{R}^m)$, then define

$$(f \ast \xi)(x) := \int f(x - y) \xi(dy)$$
Gaussian random fields [GRFs] (Lecture 2)
Stochastic Convolutions

- If \( f \in L^2(\mathbb{R}^m) \), then define

\[
(f \ast \xi)(x) := \int f(x - y) \xi(dy)
\]
If $f \in L^2(\mathbb{R}^m)$, then define

$$(f * \xi)(x) := \int f(x - y) \xi(dy)$$

Proposition (A stochastic Young inequality)

If $f \in L^2(\mathbb{R}^m)$, then there is a modification of $f * \xi$ that is measurable [jointly in $(x,\omega)$]. Moreover, for all Borel measures $\mu$ on $\mathbb{R}^m$,

$$E \left( \left| \int_{\mathbb{R}^m} (f * \xi)(x) \mu(dx) \right|^2 \right) \leq \mu(\mathbb{R}^m)^2 \cdot \|f\|_{L^2(\mathbb{R}^m)}^2.$$
Gaussian random fields [GRFs] (Lecture 2)
Outline of proof

- Since $\mathcal{C}_c^\infty(\mathbb{R}^m)$ is dense in $L^2(\mathbb{R}^m)$ it suffices to prove the result for $f \in \mathcal{C}_c^\infty(\mathbb{R}^m)$. 

In that case, the Wiener isometry shows that $E\left|\left(f \ast \xi(x) - f \ast \xi(y)\right)^2\right| = \int_{\mathbb{R}^m} |f(x) - f(y)|^2 dx \leq \text{const} \cdot \|x - y\|^2$.

Kolmogorov continuity thm $\Rightarrow f \ast \xi$ has a cont. modification.

The rest follows from the Cauchy–Schwarz inequality.
Since $C_c^\infty(\mathbb{R}^m)$ is dense in $L^2(\mathbb{R}^m)$ it suffices to prove the result for $f \in C_c^\infty(\mathbb{R}^m)$.

In that case, the Wiener isometry shows that

$$
E \left( |(f \ast \xi)(x) - (f \ast \xi)(y)|^2 \right) = \int_{\mathbb{R}^m} |f(x - z) - f(y - z)|^2 \, dx
\leq \text{const} \cdot ||x - y||^2.
$$
Since $C_c^\infty(\mathbb{R}^m)$ is dense in $L^2(\mathbb{R}^m)$ it suffices to prove the result for $f \in C_c^\infty(\mathbb{R}^m)$.

In that case, the Wiener isometry shows that

$$E\left(\left| (f \ast \xi)(x) - (f \ast \xi)(y) \right|^2 \right) = \int_{\mathbb{R}^m} |f(x-z) - f(y-z)|^2 \, dx \leq const \cdot \|x-y\|^2.$$ 

Kolmogorov continuity thm $\Rightarrow f \ast \xi$ has a cont. modification.
Gaussian random fields [GRFs] (Lecture 2)
Outline of proof

- Since $C_c^\infty(\mathbb{R}^m)$ is dense in $L^2(\mathbb{R}^m)$ it suffices to prove the result for $f \in C_c^\infty(\mathbb{R}^m)$.

- In that case, the Wiener isometry shows that

$$
E \left( |(f \ast \xi)(x) - (f \ast \xi)(y)|^2 \right) = \int_{\mathbb{R}^m} |f(x - z) - f(y - z)|^2 \, dx 
\leq \text{const} \cdot \|x - y\|^2.
$$

- Kolmogorov continuity thm $\Rightarrow f \ast \xi$ has a cont. modification.

- The rest follows from the Cauchy–Schwarz inequality. \hfill \square
Gaussian random fields [GRFs] (Lecture 2)

Example: Brownian sheet

Definition (ˇCencov, 1956)

A Brownian sheet over $\mathbb{R}^m$ is a mean-zero GRF $\{B(x)\}_{x \in \mathbb{R}^m}$ with

$$
\text{Cov}(B(x),B(y)) = m \prod_{j=1}^{\min(|x_j|,|y_j|)} [0,\infty) (x_jy_j) \quad (x,y \in \mathbb{R}^m).
$$

A construction:

$B(x) := \xi(R(x))$ where $R(x)$ denotes the aligned box in $\mathbb{R}^m$ with an extremal vertex at $0 \in \mathbb{R}^m$ and another one at $x \in \mathbb{R}^m$; e.g., $B(x) = \xi([0,x_1] \times \cdots \times [0,x_m])$ when $x \in [0,\infty)^m$.

I.e., “Br. sheet = CDF of white noise.”

When $m = 1$, $B$ is a [two-sided] Brownian motion.

$B$ has a continuous modification [Kolmogorov continuity thm].
Definition (Čencov, 1956)
A Brownian sheet over $\mathbb{R}^m$ is a mean-zero GRF $\{B(x)\}_{x \in \mathbb{R}^m}$ with
\[
\text{Cov}(B(x), B(y)) = \prod_{j=1}^{m} \min(|x_j|, |y_j|) \mathbf{1}_{[0,\infty)}(x_jy_j) \quad (x, y \in \mathbb{R}^m).
\]
Definition (Čencov, 1956)

A Brownian sheet over \( \mathbb{R}^m \) is a mean-zero GRF \( \{B(x)\}_{x \in \mathbb{R}^m} \) with

\[
\text{Cov}(B(x), B(y)) = \prod_{j=1}^{m} \min(|x_j|, |y_j|) \cdot \mathbf{1}_{[0, \infty)}(x_j y_j) \quad (x, y \in \mathbb{R}^m).
\]

A construction: \( B(x) := \xi(R(x)) \) where \( R(x) \) denotes the aligned box in \( \mathbb{R}^m \) with an extremal vertex at \( 0 \in \mathbb{R}^m \) and another one at \( x \in \mathbb{R}^m \); e.g.,

\[
B(x) = \xi([0, x_1] \times \cdots \times [0, x_m]) \quad \text{when} \ x \in [0, \infty)^m.
\]
Definition (Čencov, 1956)
A Brownian sheet over $\mathbb{R}^m$ is a mean-zero GRF $\{B(x)\}_{x \in \mathbb{R}^m}$ with

$$\text{Cov}(B(x), B(y)) = \prod_{j=1}^{m} \min(|x_j|, |y_j|) 1_{[0,\infty)}(x_jy_j) \quad (x, y \in \mathbb{R}^m).$$

A construction: $B(x) := \xi(R(x))$ where $R(x)$ denotes the aligned box in $\mathbb{R}^m$ with an extremal vertex at $0 \in \mathbb{R}^m$ and another one at $x \in \mathbb{R}^m$; e.g.,

$$B(x) = \xi([0, x_1] \times \cdots \times [0, x_m]) \quad \text{when } x \in [0, \infty)^m.$$

I.e., “Br. sheet = CDF of white noise.”
Gaussian random fields \([\text{GRFs}]\) (Lecture 2)
Example: Brownian sheet

- **Definition (Čencov, 1956)**
  A *Brownian sheet* over \(\mathbb{R}^m\) is a mean-zero GRF \(\{B(x)\}_{x \in \mathbb{R}^m}\) with
  \[
  \text{Cov}(B(x), B(y)) = \prod_{j=1}^{m} \min(|x_j|, |y_j|) \cdot 1_{[0,\infty)}(x_j y_j) \quad (x, y \in \mathbb{R}^m).
  \]

- A construction: \(B(x) := \xi(R(x))\) where \(R(x)\) denotes the aligned box in \(\mathbb{R}^m\) with an extremal vertex at 0 ∈ \(\mathbb{R}^m\) and another one at \(x \in \mathbb{R}^m\); e.g.,
  \[
  B(x) = \xi ([0, x_1] \times \cdots \times [0, x_m]) \quad \text{when } x \in [0, \infty)^m.
  \]
  I.e., “Br. sheet = CDF of white noise.”

- When \(m = 1\), \(B\) is a [two-sided] Brownian motion.
Definition (Čencov, 1956)

A Brownian sheet over $\mathbb{R}^m$ is a mean-zero GRF $\{B(x)\}_{x \in \mathbb{R}^m}$ with

$$\text{Cov}(B(x), B(y)) = \prod_{j=1}^{m} \min(|x_j|, |y_j|) \mathbf{1}_{[0,\infty)}(x_jy_j) \quad (x, y \in \mathbb{R}^m).$$

A construction: $B(x) := \xi(R(x))$ where $R(x)$ denotes the aligned box in $\mathbb{R}^m$ with an extremal vertex at $0 \in \mathbb{R}^m$ and another one at $x \in \mathbb{R}^m$; e.g.,

$$B(x) = \xi([0, x_1] \times \cdots \times [0, x_m]) \quad \text{when } x \in [0, \infty)^m.$$

I.e., “Br. sheet = CDF of white noise.”

When $m = 1$, $B$ is a [two-sided] Brownian motion.

$B$ has a continuous modification [Kolmogorov continuity thm].
Gaussian random fields [GRFs] (Lecture 2)

Example: Brownian sheet

▶ Br. sheet = CDF of white noise has the following associated “differentiation theorem”:

\[
\int \phi \, d\xi = (-1)^m \int \left( \frac{\partial^m \phi(x)}{\partial x_1 \cdots \partial x_m} \right) B(x) \, dx \quad \text{a.s.}
\]

In particular, if \( \xi = \xi(t, x) \) is space-time white noise and \( W \) is Br. sheet on \( \mathbb{R}^2 \) then

\[
\frac{\partial^2}{\partial t \partial x} W(t, x) = \xi(t, x).
\]
Gaussian random fields [GRFs] (Lecture 2)

Example: Brownian sheet

- Br. sheet = CDF of white noise has the following associated “differentiation theorem”:

\[
\int \phi \, d\xi = (-1)^m \int \partial^m \phi(x_1) \cdots \partial^m \phi(x_m) W(x) \, dx \quad \text{a.s.}
\]

In particular, if \( \xi = \xi(t,x) \) is space-time white noise and \( W \) is Br. sheet on \( \mathbb{R}^2 \) then

\[
\frac{\partial^2}{\partial t \partial x} W(t,x) = \xi(t,x).
\]
Gaussian random fields [GRFs] (Lecture 2)
Example: Brownian sheet

▶ Br. sheet = CDF of white noise has the following associated “differentiation theorem”:

▶ **Proposition**

For all \( \phi \in C^\infty_c(\mathbb{R}^m) \),

\[
\int \phi \, d\xi = (-1)^m \int_{\mathbb{R}^m} \frac{\partial^m \phi(x)}{\partial x_1 \cdots \partial x_m} B(x) \, dx \quad \text{a.s.}
\]
Gaussian random fields [GRFs] (Lecture 2)
Example: Brownian sheet

- Br. sheet = CDF of white noise has the following associated “differentiation theorem”:

- **Proposition**

  \[
  \int \phi \, d\xi = (-1)^m \int_{\mathbb{R}^m} \frac{\partial^m \phi(x)}{\partial x_1 \cdots \partial x_m} B(x) \, dx \quad \text{a.s.}
  \]

- In particular, if \( \xi = \xi(t, x) \) is space-time white noise and \( W \) is Br. sheet on \( \mathbb{R}^2 \) then

  \[
  \frac{\partial^2}{\partial t \partial x} W(t, x) = \xi(t, x).
  \]
Goal. $\forall \phi \in C^\infty(\mathbb{R})$, supported in $(0,1)$: $\int \phi \, d\xi = -\int_0^1 \phi' B$

[a.s.]
Example: Brownian sheet (Lecture 2)
Proof in the case that $m = 1$

- Goal. $\forall \phi \in C^\infty(\mathbb{R})$, supported in $(0, 1)$: $\int \phi \, d\xi = - \int_0^1 \phi' B$
  [a.s.]
- Define $\phi_n(t) = \phi(\lfloor nt \rfloor / n)$ and note that $\forall n$ large,

\[
\int \phi_n \, d\xi = \sum_{j=0}^{n-1} \phi(j/n) \xi \left( \left[ \frac{j}{n}, \frac{j+1}{n} \right] \right)
\]
Example: Brownian sheet (Lecture 2)
Proof in the case that \( m = 1 \)

- Goal. \( \forall \phi \in C^\infty(\mathbb{R}) \), supported in \((0, 1)\): \( \int \phi \, d\xi = - \int_0^1 \phi' B \) [a.s.]
- Define \( \phi_n(t) = \phi(\lfloor nt \rfloor / n) \) and note that \( \forall n \) large,

\[
\int \phi_n \, d\xi = \sum_{j=0}^{n-1} \phi(j/n) \xi(\left[ \frac{j}{n}, \frac{j+1}{n} \right])
\]
Example: Brownian sheet (Lecture 2)
Proof in the case that \( m = 1 \)

- Goal. \( \forall \phi \in C^\infty(\mathbb{R}), \) supported in \((0, 1): \int \phi \, d\xi = - \int_0^1 \phi' B \) \([\text{a.s.}]\)
- Define \( \phi_n(t) = \phi(\lfloor nt \rfloor / n) \) and note that \( \forall n \) large,

\[
\int \phi_n \, d\xi = \sum_{j=0}^{n-1} \phi(j/n) \xi \left( \left[ \frac{j}{n}, \frac{j+1}{n} \right] \right)
= \sum_{j=0}^{n-1} \phi(j/n) \left\{ B((j+1)/n) - B(j/n) \right\}
\]
Example: Brownian sheet (Lecture 2)
Proof in the case that $m = 1$

- Goal. $\forall \phi \in C^\infty(\mathbb{R})$, supported in $(0,1)$: $\int \phi \, d\xi = - \int_0^1 \phi' B$ [a.s.]
- Define $\phi_n(t) = \phi(\lfloor nt \rfloor / n)$ and note that $\forall n$ large,

$$\int \phi_n \, d\xi = \sum_{j=0}^{n-1} \phi(j/n) \xi \left( \left[ \frac{j}{n}, \frac{j+1}{n} \right] \right)$$

$$= \sum_{j=0}^{n-1} \phi(j/n) \{ B((j+1)/n) - B(j/n) \}$$

$$= - \sum_{j=1}^{n} B(j/n) \left\{ \phi \left( \frac{j}{n} \right) - \phi \left( \frac{j-1}{n} \right) \right\}$$
Example: Brownian sheet (Lecture 2)
Proof in the case that $m = 1$

- Goal. $\forall \phi \in C^\infty(\mathbb{R})$, supported in $(0, 1)$: $\int \phi \, d\xi = -\int_0^1 \phi' B$
  [a.s.]
- Define $\phi_n(t) = \phi([nt]/n)$ and note that $\forall n$ large,

$$
\int \phi_n \, d\xi = \sum_{j=0}^{n-1} \phi(j/n) \xi \left( \left[ \frac{j}{n}, \frac{j+1}{n} \right] \right)
= \sum_{j=0}^{n-1} \phi(j/n) \{ B((j+1)/n) - B(j/n) \}
= - \sum_{j=1}^{n} B(j/n) \left\{ \phi \left( \frac{j}{n} \right) - \phi \left( \frac{j-1}{n} \right) \right\}
\to - \int_0^1 B(t)\phi'(t) \, dt \quad (n \to \infty).
$$
Goal. \( \forall \phi \in C^\infty(\mathbb{R}) \), supported in \((0, 1)\): \( \int \phi \, d\xi = -\int_0^1 \phi' B \) [a.s.]

Define \( \phi_n(t) = \phi(\lfloor nt \rfloor / n) \) and note that \( \forall n \) large,

\[
\int \phi_n \, d\xi = \sum_{j=0}^{n-1} \phi(j/n) \xi \left( \left[ \frac{j}{n}, \frac{j+1}{n} \right] \right)
\]

\[
= \sum_{j=0}^{n-1} \phi(j/n) \{ B((j+1)/n) - B(j/n) \}
\]

\[
= -\sum_{j=1}^{n} B(j/n) \left\{ \phi \left( \frac{j}{n} \right) - \phi \left( \frac{j-1}{n} \right) \right\}
\]

\[
\rightarrow -\int_0^1 B(t)\phi'(t) \, dt \quad (n \to \infty).
\]

LHS \( \rightarrow \int \phi \, d\xi \) by definition.
Gaussian random fields [GRFs] (Lecture 2)

Example: Brownian sheet

Corollary (Stochastic Fubini)

If $f \in L^2(\mathbb{R}^m)$ and $\mu$ is a finite Borel measure on $\mathbb{R}^m$, then

$$\int_{\mathbb{R}^m} (f \ast \xi) \, d\mu = \int (\hat{f} \ast \mu)(\xi) \, d\xi \quad (\hat{f}(x) := f(-x)).$$

Step 1: If $f \in C^\infty_c(\mathbb{R}^m)$, then by ordinary Fubini,

$$LHS = (-1)^m \int_{\mathbb{R}^m} \mu(dx) \int_{\mathbb{R}^m} B(y) \frac{\partial^m f(x-y)}{\partial y_1 \cdots \partial y_m} \, dy = (-1)^m \int_{\mathbb{R}^m} B(y) \, dy \int_{\mathbb{R}^m} \mu(dx) f(x-y) = RHS.$$
Corollary (Stochastic Fubini)

If $f \in L^2(\mathbb{R}^m)$ and $\mu$ is a finite Borel measure on $\mathbb{R}^m$, then

$$\int_{\mathbb{R}^m} (f \ast \xi) \, d\mu = \int (\tilde{f} \ast \mu) \, d\xi$$

$(\tilde{f}(x) := f(-x))$. 

Example: Brownian sheet
Example: Brownian sheet

**Corollary (Stochastic Fubini)**

If \( f \in L^2(\mathbb{R}^m) \) and \( \mu \) is a finite Borel measure on \( \mathbb{R}^m \), then

\[
\int_{\mathbb{R}^m} (f \ast \xi) \, d\mu = \int \left( \tilde{f} \ast \mu \right) \, d\xi \quad \quad (\tilde{f}(x) := f(-x)).
\]

**Step 1:** If \( f \in C_\infty(\mathbb{R}^m) \), then by ordinary Fubini,

\[
\text{LHS} = (-1)^m \int_{\mathbb{R}^m} \mu(dx) \int_{\mathbb{R}^m} dy \, B(y) \frac{\partial^m f(x - y)}{\partial y_1 \cdots \partial y_m}.
\]
Corollary (Stochastic Fubini)

If $f \in L^2(\mathbb{R}^m)$ and $\mu$ is a finite Borel measure on $\mathbb{R}^m$, then

$$\int_{\mathbb{R}^m} (f \ast \xi) \, d\mu = \int (\tilde{f} \ast \mu) \, d\xi$$

($\tilde{f}(x) := f(-x)$).

Step 1: If $f \in C^\infty_c(\mathbb{R}^m)$, then by ordinary Fubini,

$$\text{LHS} = (-1)^m \int_{\mathbb{R}^m} \mu(dx) \int_{\mathbb{R}^m} dy \ B(y) \frac{\partial^m f(x-y)}{\partial y_1 \cdots \partial y_m}$$
Corollary (Stochastic Fubini)

If \( f \in L^2(\mathbb{R}^m) \) and \( \mu \) is a finite Borel measure on \( \mathbb{R}^m \), then

\[
\int_{\mathbb{R}^m} (f \ast \xi) \, d\mu = \int \left( \tilde{f} \ast \mu \right) \, d\xi \\
(f(x) := f(-x)).
\]

Step 1: If \( f \in C^\infty_c(\mathbb{R}^m) \), then by ordinary Fubini,

\[
\text{LHS} = (-1)^m \int_{\mathbb{R}^m} \mu(dx) \int_{\mathbb{R}^m} dy \ B(y) \frac{\partial^m f(x-y)}{\partial y_1 \cdots \partial y_m} \\
= (-1)^m \int_{\mathbb{R}^m} B(y) \, dy \frac{\partial^m}{\partial y_1 \cdots \partial y_m} \int_{\mathbb{R}^m} \mu(dx) \ f(x-y)
\]
Gaussian random fields [GRFs] (Lecture 2)

Example: Brownian sheet

▶ Corollary (Stochastic Fubini)

If \( f \in L^2(\mathbb{R}^m) \) and \( \mu \) is a finite Borel measure on \( \mathbb{R}^m \), then

\[
\int_{\mathbb{R}^m} (f \ast \xi) \, d\mu = \int (\tilde{f} \ast \mu) \, d\xi \quad (\tilde{f}(x) := f(-x)).
\]

▶ Step 1: If \( f \in C^\infty_c(\mathbb{R}^m) \), then by ordinary Fubini,

\[
\text{LHS} = (-1)^m \int_{\mathbb{R}^m} \mu(dx) \int_{\mathbb{R}^m} dy \ B(y) \frac{\partial^m f(x - y)}{\partial y_1 \cdots \partial y_m} \\
= (-1)^m \int_{\mathbb{R}^m} B(y) \, dy \ \frac{\partial^m}{\partial y_1 \cdots \partial y_m} \int_{\mathbb{R}^m} \mu(dx) \ f(x - y) \\
= \text{RHS}.
\]
Step 2: If $f \in L^2(\mathbb{R}^m)$ then $\exists f_1, f_2, \ldots \in C^\infty_c(\mathbb{R}^m)$ such that $\lim_{n \to \infty} f_n = f$ in $L^2(\mathbb{R}^m)$. 

By the Young inequality, $\tilde{f}_n \ast \mu \to \tilde{f} \ast \mu$ in $L^2(\mathbb{R}^m)$; therefore, $\int (\tilde{f}_n \ast \mu) d\xi \to \int (\tilde{f} \ast \mu) d\xi$ in $L^2(\Omega)$, by Wiener's isometry.

By Step 1, LHS = $\int (f_n \ast \xi) d\mu$; owing to the stochastic Young inequality, it suffices to prove that $f_n \ast \xi \to f \ast \xi$ in $L^2(\Omega)$, but this holds also by the Wiener isometry.
Gaussian random fields [GRFs] (Lecture 2)
Example: Brownian sheet

Step 2: If $f \in L^2(\mathbb{R}^m)$ then $\exists f_1, f_2, \ldots \in C_c^\infty(\mathbb{R}^m)$ such that $\lim_{n \to \infty} f_n = f$ in $L^2(\mathbb{R}^m)$.

By the Young inequality, $\tilde{f}_n \ast \mu \to \tilde{f} \ast \mu$ in $L^2(\mathbb{R}^m)$; therefore,

$$\int (\tilde{f}_n \ast \mu) \, d\xi \to \int (\tilde{f} \ast \mu) \, d\xi \text{ in } L^2(\Omega),$$

by Wiener’s isometry.
Step 2: If $f \in L^2(\mathbb{R}^m)$ then $\exists f_1, f_2, \ldots \in C_c^\infty(\mathbb{R}^m)$ such that $\lim_{n \to \infty} f_n = f$ in $L^2(\mathbb{R}^m)$.

By the Young inequality, $\tilde{f}_n \ast \mu \to \tilde{f} \ast \mu$ in $L^2(\mathbb{R}^m)$; therefore,

$$\int (\tilde{f}_n \ast \mu) \, d\xi \to \int (\tilde{f} \ast \mu) \, d\xi \text{ in } L^2(\Omega),$$

by Wiener’s isometry.

By Step 1, $\text{LHS} = \int (f_n \ast \xi) \, d\mu$; owing to the stochastic Young inequality, it suffices to prove that $f_n \ast \xi \to f \ast \xi$ in $L^2(\Omega)$, but this holds also by the Wiener isometry.
Gaussian random fields [GRFs] (Lecture 2)
Example: fractional Brownian motion [fBm]

Definition (Mandelbrot–Van Ness, 1968)
An fBm with index $H$ is a centered Gaussian process $\{X_t\}_{t \geq 0}$ with $X_0 = 0$ and $E(\|X_t - X_s\|^2) = |t - s|^{2H}$.

If $fBm \exists$ then $Cov(X_t, X_s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$.

∴ $fBm(H)$ exists iff $H \in (0, 1)$.

$fBm(1/2) = BM$.

$fBm(H)$ has a Hölder-continuous modification [Kolmogorov continuity thm] of index $< H$. 
Definition (Mandelbrot–Van Ness, 1968)

An fBm with index $H$ is a centered Gaussian process $\{X_t\}_{t \geq 0}$ with $X_0 = 0$ and $\mathbb{E} (|X_t - X_s|^2) = |t - s|^{2H}$ ($s, t \geq 0$).
Definition (Mandelbrot–Van Ness, 1968)

An fBm with index $H$ is a centered Gaussian process $\{X_t\}_{t \geq 0}$ with $X_0 = 0$ and $E(|X_t - X_s|^2) = |t - s|^{2H}$ ($s, t \geq 0$).

If fBm $\exists$ then $\text{Cov}(X_t, X_s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right)$. 
Definition (Mandelbrot–Van Ness, 1968)

An fBm with index $H$ is a centered Gaussian process $\{X_t\}_{t \geq 0}$ with $X_0 = 0$ and $E(|X_t - X_s|^2) = |t - s|^{2H}$ ($s, t \geq 0$).

- If fBm exists then $\text{Cov}(X_t, X_s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right)$.
- $\therefore$ fBm$(H)$ exists iff $H \in (0, 1)$.
Definition (Mandelbrot–Van Ness, 1968)

An fBm with index $H$ is a centered Gaussian process $\{X_t\}_{t \geq 0}$ with $X_0 = 0$ and $E(|X_t - X_s|^2) = |t - s|^{2H}$ ($s, t \geq 0$).

- If fBm $\exists$ then $\text{Cov}(X_t, X_s) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H}\right)$.
- $\therefore$ fBm($H$) exists iff $H \in (0, 1)$.
- fBm(1/2) = BM.
Definition (Mandelbrot–Van Ness, 1968)

An \textit{fBm with index} $H$ is a centered Gaussian process $\{X_t\}_{t \geq 0}$ with $X_0 = 0$ and $E(|X_t - X_s|^2) = |t - s|^{2H}$ $(s, t \geq 0)$.

- If fBm $\exists$ then $\text{Cov}(X_t, X_s) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H}\right)$.
- $\therefore$ fBm($H$) exists iff $H \in (0, 1)$.
- fBm($1/2$) = BM.
- fBm($H$) has a Hölder-continuous modification [Kolmogorov continuity thm] of index $< H$. 

Example: fractional Brownian motion [fBm]
Define for all $t \geq 0$ and $H, s \in \mathbb{R}$, $f_H(t, s) := (t - s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}$.
Gaussian random fields [GRFs] (Lecture 2)
Example: fractional Brownian motion [fBm] (some details)

Define for all $t \geq 0$ and $H, s \in \mathbb{R}$,

$$f_H(t, s) := (t - s)^{(H - (1/2))} - (-s)^{(H - (1/2))}$$

Then,

$$\| f_H(t, \bullet) \|^2_{L^2(\mathbb{R})} = \int_0^t (t-s)^{2H-1} \, ds + \int_0^\infty \left[ (t+s)^{(H - (1/2))} - s^{H - (1/2)} \right]^2 \, ds$$

is finite iff $H \in (0, 1)$. And $\forall H \in (0, 1)$:

$$\| f_H(t, \bullet) \|^2_{L^2(\mathbb{R})} = C_H t^{2H}. $$
Gaussian random fields [GRFs] (Lecture 2)
Example: fractional Brownian motion [fBm] (some details)

▶ Define for all \( t \geq 0 \) and \( H, s \in \mathbb{R} \),

\[
f_H(t, s) := (t - s)_+^{H-(1/2)} - (-s)_+^{H-(1/2)}
\]

▶ Then,

\[
\|f_H(t, \bullet)\|_{L^2(\mathbb{R})}^2 = \int_0^t (t-s)^{2H-1} \, ds + \int_0^\infty \left[(t + s)^{H-(1/2)} - s^{H-(1/2)}\right]^2 \, ds
\]

is finite iff \( H \in (0, 1) \). And \( \forall H \in (0, 1) \):

\[
\|f_H(t, \bullet)\|_{L^2(\mathbb{R})}^2 = C_H t^{2H}.
\]

▶ Then we can construct fBm\((H)\) as

\[
X_t := \frac{1}{\sqrt{C_H}} \int f_H(t, s) \xi(ds) \quad (t > 0).
\]
Let \( \{X_t\}_{t \geq 0} \) be a fBm\((H)\).
Let \( \{X_t\}_{t \geq 0} \) be a fBm\((H)\).
Gaussian random fields [GRFs] (Lecture 2)
Example: fractional Brownian motion [fBm] (some background facts)

- Let \( \{X_t\}_{t \geq 0} \) be a fBm\((H)\).

- **Theorem (Marcus, 1968; Shao, 1996; . . .)**

  *With probability one:*

  \[
  \limsup_{\varepsilon \downarrow 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon^H \sqrt{2 \ln \ln(1/\varepsilon)}} = 1 \quad \forall t \geq 0; \quad \text{and}
  \]

  \[
  \lim_{n \to \infty} \sum_{a2^n \leq j \leq b2^n} |X_{(j+1)/2^n} - X_{j/2^n}|^{1/H} = E\left(|\mathcal{N}|^{1/H}\right) \cdot (b - a);
  \]

  \( \forall 0 \leq a < b < \infty, \) where \( \mathcal{N} \) is a standard normal r.v.
A Linear Heat Equation (Lecture 3)

A non-random heat equation \((\partial_t u = (\nu/2)\partial_x^2 u + \mu)\)

- Let \(\mu\) be a finite signed Borel measure on \(\mathbb{R}\). Want to solve the initial-value problem

\[
\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + \mu,
\]

\([u := u_t(x)]\) for \(x \in \mathbb{R}\) with \(t > 0\), subject to a nice initial function \(u_0 : \mathbb{R} \to \mathbb{R}\).
A Linear Heat Equation (Lecture 3)
A non-random heat equation \( \partial_t u = (\nu/2)\partial_x^2 u + \mu \)

- Let \( \mu \) be a finite signed Borel measure on \( \mathbb{R} \). Want to solve the initial-value problem

\[
\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + \mu, \quad \text{(HE)}
\]

\([u := u_t(x)] \) for \( x \in \mathbb{R} \) with \( t > 0 \), subject to a nice initial function \( u_0 : \mathbb{R} \to \mathbb{R} \).
A Linear Heat Equation (Lecture 3)
A non-random heat equation \( (\partial_t u = (\nu/2) \partial_x^2 u + \mu) \)

- Let \( \mu \) be a finite signed Borel measure on \( \mathbb{R} \). Want to solve the initial-value problem

\[
\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + \mu,
\]

\([u := u_t(x)] \) for \( x \in \mathbb{R} \) with \( t > 0 \), subject to a nice initial function \( u_0 : \mathbb{R} \to \mathbb{R} \).

- **Definition**

  We say that \( u = u_t(x) \) is a *weak solution* to (HE) if

  \( u \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}) \) and

  \[- \int_{\mathbb{R}_+ \times \mathbb{R}} u \frac{\partial}{\partial t} \varphi \, dt \, dx = \frac{\nu}{2} \int_{\mathbb{R}_+ \times \mathbb{R}} u \frac{\partial^2}{\partial x^2} \varphi \, dt \, dx + \int \varphi \, d\mu,\]

  for all \( \varphi \in C_c^\infty((0, \infty) \times \mathbb{R}) \).
Define the heat kernel

\[ p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp\left(-\frac{x^2}{2\nu t}\right) \quad (t > 0, x \in \mathbb{R}). \]
A Linear Heat Equation (Lecture 3)
A non-random heat equation \((\partial_t u = (\nu/2)\partial_x^2 u + \mu)\)

- Define the heat kernel

\[
p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp\left(-\frac{x^2}{2\nu t}\right) \quad (t > 0, \ x \in \mathbb{R}).
\]
A Linear Heat Equation (Lecture 3)

A non-random heat equation \((\partial_t u = (\nu/2)\partial_x^2 u + \mu)\)

- Define the heat kernel

\[
p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp \left( -\frac{x^2}{2\nu t} \right) \quad (t > 0, x \in \mathbb{R}).
\]

- Theorem (Duhamel’s principle)

The following is the unique weak solution to (HE):

\[
u_t(x) := (p_t \ast u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x) \mu(ds \, dy).
\]
Define the heat kernel

\[ p_t(x) := \frac{1}{\sqrt{2\nu \pi t}} \exp \left( -\frac{x^2}{2\nu t} \right) \quad (t > 0, \ x \in \mathbb{R}). \]

Theorem (Duhamel’s principle)

The following is the unique weak solution to (HE):

\[ u_t(x) := (p_t \ast u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \mu(ds \ dy). \]

“Mild—a.k.a. integral—solution.”
A Linear Heat Equation (Lecture 3)
A non-random heat equation \( \partial_t u = (\nu/2)\partial_x^2 u + \mu \)

- Define the heat kernel
  \[
  p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp\left(-\frac{x^2}{2\nu t}\right) \quad (t > 0, x \in \mathbb{R}).
  \]

- **Theorem (Duhamel’s principle)**
  The following is the unique weak solution to (HE):
  \[
  u_t(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \mu(ds \, dy).
  \]

- “Mild—a.k.a. integral—solution.”

- “Mild \( \Rightarrow \) weak,” even when \( \mu \) is signed and \( \sigma \)-finite, as long as \( \mu \) has “bounded thermal energy” (Watson, 1974).
A Linear Heat Equation (Lecture 3)

A non-random heat equation \((\partial_t u = (\nu/2)\partial_x^2 u + \mu)\)

- Define the heat kernel

\[ p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp \left( -\frac{x^2}{2\nu t} \right) \quad (t > 0, x \in \mathbb{R}). \]

- **Theorem (Duhamel’s principle)**

The following is the unique weak solution to (HE):

\[ u_t(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \mu(ds \, dy). \]

- “Mild—a.k.a. integral—solution."

- “Mild \(\Rightarrow\) weak,” even when \(\mu\) is signed and \(\sigma\)-finite, as long as \(\mu\) has “bounded thermal energy” (Watson, 1974).

- \(\mu \equiv 0 \Rightarrow u_t(x) = (p_t * u_0)(x)\) solves (HE).
A Linear Heat Equation (Lecture 3)

Comments on the heat kernel

\[ p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp\left(-\frac{x^2}{2\nu t}\right) \quad (t > 0, \, x \in \mathbb{R}). \]
A Linear Heat Equation (Lecture 3)
Comments on the heat kernel

\[ p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp \left( -\frac{x^2}{2\nu t} \right) \quad (t > 0, x \in \mathbb{R}). \]

\[ \hat{p}_t(z) = \exp \left( -\nu tz^2 / 2 \right). \]
A Linear Heat Equation (Lecture 3)
Comments on the heat kernel

- $p_t(x) := \frac{1}{\sqrt{2\nu \pi t}} \exp \left(-\frac{x^2}{2\nu t}\right) \quad (t > 0, \ x \in \mathbb{R}).$
- $\hat{p}_t(z) = \exp \left(-\nu tz^2/2\right).$
- $(p_t * p_s)(x) = p_{t+s}(x) \quad \text{[Chapman–Kolmogorov].}$
A Linear Heat Equation (Lecture 3)
Comments on the heat kernel

\( p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp \left(-\frac{x^2}{2\nu t}\right) \quad (t > 0, \ x \in \mathbb{R}). \)

\( \hat{p}_t(z) = \exp \left(-\nu t z^2 / 2\right). \)

\( (p_t \ast p_s)(x) = p_{t+s}(x) \quad \text{[Chapman–Kolmogorov].} \)

\[ \int_{-\infty}^{\infty} [p_t(x)]^2 \, dx = (p_t \ast p_t)(0) \]
A Linear Heat Equation (Lecture 3)

Comments on the heat kernel

\[ p_t(x) := \frac{1}{\sqrt{2\nu \pi t}} \exp \left( -\frac{x^2}{2\nu t} \right) \quad (t > 0, \; x \in \mathbb{R}). \]

\[ \widehat{p}_t(z) = \exp \left( -\nu tz^2 / 2 \right). \]

\[ (p_t \ast p_s)(x) = p_{t+s}(x) \quad \text{[Chapman–Kolmogorov]} . \]

\[ \int_{-\infty}^{\infty} [p_t(x)]^2 \, dx = (p_t \ast p_t)(0) \]
A Linear Heat Equation (Lecture 3)
Comments on the heat kernel

\( p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp \left( -\frac{x^2}{2\nu t} \right) \quad (t > 0, x \in \mathbb{R}). \)

\( \hat{p}_t(z) = \exp \left( -\nu t z^2 / 2 \right). \)

\((p_t \ast p_s)(x) = p_{t+s}(x) \quad [\text{Chapman–Kolmogorov}].\)

\[ \int_{-\infty}^{\infty} [p_t(x)]^2 \, dx = (p_t \ast p_t)(0) = p_{2t}(0) \]
A Linear Heat Equation (Lecture 3)
Comments on the heat kernel

\[ p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp \left( -\frac{x^2}{2\nu t} \right) \quad (t > 0, \, x \in \mathbb{R}). \]

\[ \hat{p}_t(z) = \exp \left( -\nu tz^2/2 \right). \]

\[ (p_t * p_s)(x) = p_{t+s}(x) \quad \text{[Chapman–Kolmogorov]}. \]

\[ \int_{-\infty}^{\infty} [p_t(x)]^2 \, dx = (p_t * p_t)(0) = p_{2t}(0) = \frac{1}{\sqrt{4\nu\pi t}}. \]

Alternatively, by Plancherel,

\[ \int_{-\infty}^{\infty} [p_t(x)]^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\nu tz^2} \, dz \]
A Linear Heat Equation (Lecture 3)

Comments on the heat kernel

\[ p_t(x) := \frac{1}{\sqrt{2\nu \pi t}} \exp \left( -\frac{x^2}{2\nu t} \right) \quad (t > 0, \ x \in \mathbb{R}). \]

\[ \hat{p}_t(z) = \exp \left( -\nu tz^2 / 2 \right). \]

\[ (p_t \ast p_s)(x) = p_{t+s}(x) \quad \text{[Chapman–Kolmogorov].} \]

\[ \int_{-\infty}^{\infty} [p_t(x)]^2 \, dx = (p_t \ast p_t)(0) = p_{2t}(0) = \frac{1}{\sqrt{4\nu \pi t}}. \]

Alternatively, by Plancherel,

\[ \int_{-\infty}^{\infty} [p_t(x)]^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\nu tz^2} \, dz \]
A Linear Heat Equation (Lecture 3)

Comments on the heat kernel

- \( p_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp \left( -\frac{x^2}{2\nu t} \right) \quad (t > 0, \ x \in \mathbb{R}). \)
- \( \hat{p}_t(z) = \exp \left( -\frac{\nu tz^2}{2} \right). \)
- \( (p_t * p_s)(x) = p_{t+s}(x) \quad \text{[Chapman–Kolmogorov].} \)
- \( \int_{-\infty}^{\infty} [p_t(x)]^2 \, dx = (p_t * p_t)(0) = p_{2t}(0) = \frac{1}{\sqrt{4\nu\pi t}}. \)
- Alternatively, by Plancherel,

\[
\int_{-\infty}^{\infty} [p_t(x)]^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-xz^2} \, dz = \frac{1}{\sqrt{4\nu\pi t}}.
\]
Now we study the “linear stochastic heat equation,”

\[ \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + \xi, \]  

subject to \( u_0 := \) nice and non random; \( \xi := \) space-time white noise.  

\[ [\xi = \partial_t \partial_x W] \]
A Linear Heat Equation (Lecture 3)

A random heat equation \( \partial_t u = (\nu/2) \partial_x^2 u + \xi \)

Now we study the “linear stochastic heat equation,”

\[
\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + \xi,
\]

subject to \( u_0 := \) nice and non random; \( \xi := \) space-time white noise. \( \xi = \partial_t \partial_x W \)

We define the solution as the mild solution,

\[
u_t(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy),
\]

where the integral now is Wiener’s.
Now we study the “linear stochastic heat equation,”

$$\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + \xi,$$

subject to $u_0 :=$ nice and non random; $\xi :=$ space-time white noise.  

We define the solution as the mild solution,

$$u_t(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy),$$

where the integral now is Wiener’s.
A Linear Heat Equation (Lecture 3)

A random heat equation \( (\partial_t u = (\nu/2)\partial_x^2 u + \xi) \)

- Now we study the “linear stochastic heat equation,”

\[
\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + \xi, \quad \text{(SHE)}
\]

subject to \( u_0 := \text{nice and non random}; \xi := \text{space-time white noise}. \quad [\xi = \partial_t \partial_x W] \)

- We define the solution as the mild solution,

\[
u_t(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy),
\]

where the integral now is Wiener’s.

- **Proposition**

  *The mild solution is a weak solution to (SHE).*
A Linear Heat Equation (Lecture 3)

A random heat equation \( \partial_t u = (\nu/2)\partial_x^2 u + \xi \)

- Now we study the “linear stochastic heat equation,”
  \[
  \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + \xi, 
  \]
  \( \text{(SHE)} \)

subject to \( u_0 := \text{nice and non random}; \xi := \text{space-time white noise.} \)

- We define the solution as the mild solution,
  \[
  u_t(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy),
  \]
  where the integral now is Wiener’s.

- **Proposition**
  The mild solution is a weak solution to \( \text{(SHE)} \).

- **Sketch of proof:** Use stochastic Fubini.
A Linear Heat Equation (Lecture 3)
A random heat equation \( (\partial_t u = (\nu/2)\partial_x^2 u + \xi) \)

- Now we study the “linear stochastic heat equation,”

\[
\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + \xi, \tag{SHE}
\]

subject to \( u_0 := \) nice and non random; \( \xi := \) space-time white noise. \([\xi = \partial_t \partial_x W]\)

- We define the solution as the mild solution,

\[
u_t(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy),
\]

where the integral now is Wiener’s.

- **Proposition**

  The mild solution is a weak solution to (SHE).

  - Sketch of proof: Use stochastic Fubini.

  - The solution is a GRF!
Immediate goal: Describe the local behavior of the solution to (SHE). Let

\[ Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy). \]
Immediate goal: Describe the local behavior of the solution to (SHE). Let
\[
Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy).
\]

This solves \( \partial_t Z = (\nu/2) \partial_x^2 Z + \xi \) subject to \( Z_0(x) \equiv 0 \). The general case is \( u_t(x) = (p_t \ast u_0)(x) + Z_t(x) \), so the real issue is to understand the structure of \( Z \).
Immediate goal: Describe the local behavior of the solution to (SHE). Let

\[ Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy). \]

This solves \( \partial_t Z = (\nu/2)\partial_x^2 Z + \xi \) subject to \( Z_0(x) \equiv 0 \). The general case is \( u_t(x) = (p_t * u_0)(x) + Z_t(x) \), so the real issue is to understand the structure of \( Z \).

We are about to see that \( t \mapsto Z_t(x) \) is a smooth perturbation of fBm\((1/4)\) and \( x \mapsto Z_t(x) \) is a smooth perturbation of a two-sided Br. motion. In particular, the local behavior of \( Z \) looks like fBm\((1/4)\) in time and Br. motion in space, viz.
Immediate goal: Describe the local behavior of the solution to (SHE). Let

\[ Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy). \]

This solves \( \partial_t Z = (\nu/2) \partial_x^2 Z + \xi \) subject to \( Z_0(x) \equiv 0 \). The general case is \( u_t(x) = (p_t \ast u_0)(x) + Z_t(x) \), so the real issue is to understand the structure of \( Z \).

We are about to see that \( t \mapsto Z_t(x) \) is a smooth perturbation of fBm\((1/4)\) and \( x \mapsto Z_t(x) \) is a smooth perturbation of a two-sided Br. motion. In particular, the local behavior of \( Z \) looks like fBm\((1/4)\) in time and Br. motion in space, viz.

2-sided BM is any GRF \( \{B(x)\}_{x \in \mathbb{R}} \) with

\[ \mathbb{E}( |B(x) - B(y)|^2 ) \propto |x - y|. \]
A Linear Heat Equation (Lecture 3)
Structure theory (some corollaries; $Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy)$.)

Corollary (Swanson, 2007; Pospíšil–Tribe, 2007)

With probability one $t \mapsto Z_t(x)$ is Hölder continuous of index $< \frac{1}{4}$ and $x \mapsto Z_t(x)$ is Hölder continuous of index $< \frac{1}{2}$. Moreover:

$$
\limsup_{\varepsilon \downarrow 0} \frac{Z_t(x + \varepsilon) - Z_t(x)}{\sqrt{2\varepsilon \ln \ln(1/\varepsilon)}} = \frac{1}{\sqrt{\nu}} \quad (\forall t > 0, x \in \mathbb{R});
$$
Corollary (Swanson, 2007; Pospìšil–Tribe, 2007)

With probability one $t \mapsto Z_t(x)$ is Hölder continuous of index $< 1/4$
and $x \mapsto Z_t(x)$ is Hölder continuous of index $< 1/2$. Moreover:

$$\limsup_{\varepsilon \downarrow 0} \frac{Z_t(x + \varepsilon) - Z_t(x)}{\sqrt{2\varepsilon \ln \ln(1/\varepsilon)}} = \frac{1}{\sqrt{\nu}} \quad (\forall t > 0, x \in \mathbb{R});$$
Corollary (Swanson, 2007; Pospíšil–Tribe, 2007)

With probability one $t \mapsto Z_t(x)$ is Hölder continuous of index $< \frac{1}{4}$ and $x \mapsto Z_t(x)$ is Hölder continuous of index $< \frac{1}{2}$. Moreover:

$$
\limsup_{\varepsilon \downarrow 0} \frac{Z_t(x + \varepsilon) - Z_t(x)}{\sqrt{2\varepsilon \ln \ln(1/\varepsilon)}} = \frac{1}{\sqrt{\nu}} \quad (\forall t > 0, x \in \mathbb{R});
$$

$$
\limsup_{\varepsilon \downarrow 0} \frac{Z_{t+\varepsilon}(x) - Z_t(x)}{\varepsilon^{1/4} \sqrt{2 \ln \ln(1/\varepsilon)}} = \left(\frac{2}{\nu \pi}\right)^{1/4} \quad (\forall t > 0, x \in \mathbb{R});
$$
A Linear Heat Equation (Lecture 3)
Structure theory (some corollaries; $Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x) \xi(ds \, dy)$.)

**Corollary (Swanson, 2007; Pospíšil–Tribe, 2007)**

*With probability one $t \mapsto Z_t(x)$ is Hölder continuous of index $< 1/4$ and $x \mapsto Z_t(x)$ is Hölder continuous of index $< 1/2$. Moreover:*

$$
\limsup_{\varepsilon \downarrow 0} \frac{Z_t(x + \varepsilon) - Z_t(x)}{\sqrt{2\varepsilon \ln \ln(1/\varepsilon)}} = \frac{1}{\sqrt{\nu}} \quad (\forall t > 0, x \in \mathbb{R});
$$

$$
\limsup_{\varepsilon \downarrow 0} \frac{Z_{t+\varepsilon}(x) - Z_t(x)}{\varepsilon^{1/4} \sqrt{2 \ln \ln(1/\varepsilon)}} = \left( \frac{2}{\nu \pi} \right)^{1/4} \quad (\forall t > 0, x \in \mathbb{R});
$$

$$
\lim_{n \to \infty} \sum_{a2^n \leq j \leq b2^n} \left[ Z_t \left( \frac{j + 1}{2^n} \right) - Z_t \left( \frac{j}{2^n} \right) \right]^2 = \frac{b - a}{\nu} \quad (t > 0, a < b);
$$
Corollary (Swanson, 2007; Pospíšil–Tribe, 2007)

With probability one $t \mapsto Z_t(x)$ is Hölder continuous of index $< 1/4$ and $x \mapsto Z_t(x)$ is Hölder continuous of index $< 1/2$. Moreover:

$$\lim \sup_{\varepsilon \downarrow 0} \frac{Z_t(x + \varepsilon) - Z_t(x)}{\sqrt{2\varepsilon \ln \ln(1/\varepsilon)}} = \frac{1}{\sqrt{\nu}} \quad (\forall t > 0, x \in \mathbb{R});$$

$$\lim \sup_{\varepsilon \downarrow 0} \frac{Z_{t+\varepsilon}(x) - Z_t(x)}{\varepsilon^{1/4} \sqrt{2 \ln \ln(1/\varepsilon)}} = \left(\frac{2}{\nu \pi}\right)^{1/4} \quad (\forall t > 0, x \in \mathbb{R});$$

$$\lim_{n \to \infty} \sum_{a2^n \leq j \leq b2^n} \left[ Z_t \left( \frac{j + 1}{2^n} \right) - Z_t \left( \frac{j}{2^n} \right) \right]^2 = \frac{b - a}{\nu} \quad (t > 0, a < b);$$

$$\lim_{m \to \infty} \sum_{c2^m \leq i \leq d2^m} \left[ Z_{(i+1)/2^m}(x) - Z_{i/2^m}(x) \right]^4 = \frac{6(d - c)}{\nu \pi} \quad (0 < c < d, x \in \mathbb{R}).$$
A Linear Heat Equation (Lecture 3)

Structure theory \( Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy) \)

Theorem (Lei–Nualart, 2009; Foondun–K–Mahboubi, 2013)

(i) \( \forall \) fixed \( x \in \mathbb{R} \), \( \exists \) \( f_{\text{BM}}(1/4) \) \( \{X_t \} \) \( t \geq 0 \) such that \( Z_t(x) - (2\nu\pi)^{1/4} X_t \) \( t \geq 0 \) is a mean-zero continuous Gaussian process that is \( C^\infty \) on \( (0, \infty) \).

(ii) For each fixed \( t \geq 0 \), \( \exists \) 2-sided BM \( \{B(x)\} \) \( x \in \mathbb{R} \) such that \( Z_t(x) - \frac{1}{\sqrt{\nu}} B(x) \) \( x \in \mathbb{R} \) is a mean-zero Gaussian process with \( C^\infty \) sample functions.

▶ A version of (ii) was first found in Walsh (1986).
A Linear Heat Equation (Lecture 3)

Structure theory ($Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x) \xi(ds \, dy)$)

- Theorem (Lei–Nualart, 2009; Foondun–K–Mahboubi, 2013)

-$\forall$ fixed $x \in \mathbb{R}$, $\exists f_{\text{BM}}(1/4) \{X_t\}_{t \geq 0}$ such that $Z_t(x) - (2\nu\pi)^{1/4} X_t(t \geq 0)$ is a mean-zero continuous Gaussian process that is $C_\infty$ on $(0, \infty)$.

-$\forall$ fixed $t \geq 0$, $\exists$ 2-sided BM $\{B(x)\}_{x \in \mathbb{R}}$ such that $Z_t(x) - 1/\sqrt{\nu}B(x)$ ($x \in \mathbb{R}$) is a mean-zero Gaussian process with $C_\infty$ sample functions.

A version of (ii) was first found in Walsh (1986).
structure theory \( Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy) \)

- **Theorem (Lei–Nualart, 2009; Foondun–K–Mahboubi, 2013)**

  (i) \( \forall \) fixed \( x \in \mathbb{R} \), \( \exists \) fBm(\( 1/4 \)) \( \{ X_t \}_{t \geq 0} \) such that

  \[
  Z_t(x) - \left( \frac{2}{\nu \pi} \right)^{1/4} X_t \quad (t \geq 0)
  \]

is a mean-zero continuous Gaussian process that is \( C^\infty \) on \((0, \infty)\).
A Linear Heat Equation (Lecture 3)

Structure theory \( (Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy)) \)

- Theorem (Lei–Nualart, 2009; Foondun–K–Mahboubi, 2013)

  (i) \( \forall \) fixed \( x \in \mathbb{R}, \exists \text{fBm}(1/4) \{X_t\}_{t \geq 0} \) such that

  \[
  Z_t(x) - \left( \frac{2}{\nu \pi} \right)^{1/4} X_t \quad (t \geq 0)
  \]

  is a mean-zero continuous Gaussian process that is \( C^\infty \) on \((0, \infty)\).

  (ii) For each fixed \( t \geq 0 \) \( \exists \) 2-sided BM \( \{B(x)\}_{x \in \mathbb{R}} \) such that

  \[
  Z_t(x) - \frac{1}{\sqrt{\nu}} B(x) \quad (x \in \mathbb{R})
  \]

  is a mean-zero Gaussian process with \( C^\infty \) sample functions.
A Linear Heat Equation (Lecture 3)

Structure theory \( (Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy)) \)

- Theorem (Lei–Nualart, 2009; Foondun–K–Mahboubi, 2013)

  (i) \( \forall \) fixed \( x \in \mathbb{R} \), \( \exists \) fBm\((1/4)\) \( \{X_t\}_{t \geq 0} \) such that

  \[
  Z_t(x) - \left( \frac{2}{\nu \pi} \right)^{1/4} X_t \quad (t \geq 0)
  \]

  is a mean-zero continuous Gaussian process that is \( C^\infty \) on \( (0, \infty) \).

  (ii) For each fixed \( t \geq 0 \) \( \exists \) 2-sided BM \( \{B(x)\}_{x \in \mathbb{R}} \) such that

  \[
  Z_t(x) - \frac{1}{\sqrt{\nu}} B(x) \quad (x \in \mathbb{R})
  \]

  is a mean-zero Gaussian process with \( C^\infty \) sample functions.

- A version of (ii) was first found in Walsh (1986).
Expand

\[ Z_{t+\varepsilon}(x) - Z_t(x) = J_1 + J_2, \]

where

\[ J_1 := \int_{(0,t) \times \mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)] \xi(ds \, dy); \]

\[ J_2 := \int_{(t,t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y-x) \xi(ds \, dy). \]
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; $Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds\,dy)$]

- Expand

$$Z_{t+\varepsilon}(x) - Z_t(x) = J_1 + J_2,$$

where

$$J_1 := \int_{(0,t) \times \mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)] \xi(ds\,dy);$$

$$J_2 := \int_{(t,t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y-x) \xi(ds\,dy).$$

- $J_1$ and $J_2$ are indep’t GRFs.

$[\text{Cov}(\int h\,d\xi, \int g\,d\xi) = \int_{-\infty}^{\infty} h\,g]$
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; $Z_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy)$]

- Expand

$$Z_{t+\varepsilon}(x) - Z_t(x) = J_1 + J_2,$$

where

$$J_1 := \int_{(0,t) \times \mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)] \xi(ds \, dy);$$

$$J_2 := \int_{(t,t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y-x) \xi(ds \, dy).$$

- $J_1$ and $J_2$ are indep’t GRFs.

$$[\text{Cov}(\int h \, d\xi, \int g \, d\xi) = \int_{-\infty}^{\infty} hg]$$

- $\therefore$  

$$E\left(\left|Z_{t+\varepsilon}(x) - Z_t(x)\right|^2\right) = E(J_1^2) + E(J_2^2).$$

We compute these terms separately, using Wiener's isometry.
Expand

\[ Z_{t+\varepsilon}(x) - Z_t(x) = J_1 + J_2, \]

where

\[ J_1 := \int_{(0,t) \times \mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)] \xi(ds\,dy); \]

\[ J_2 := \int_{(t,t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y-x) \xi(ds\,dy). \]

\( J_1 \) and \( J_2 \) are indep’t GRFs.

\[ \text{Cov} (\int h \, d\xi , \int g \, d\xi ) = \int_{-\infty}^{\infty} hg \]

\[ \therefore \quad \mathbb{E} \left( |Z_{t+\varepsilon}(x) - Z_t(x)|^2 \right) = \mathbb{E}(J_1^2) + \mathbb{E}(J_2^2). \]

We compute these terms separately, using Wiener’s isometry.
A Linear Heat Equation (Lecture 3)
Ideas of proof [Lei–Nualart; \( E(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = E(J_1^2) + E(J_2^2) \)]

- Since \( J_2 = \int_{(t,t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y-x) \xi(ds \, dy) \),

\[
E(J_2^2) = \int_t^{t+\varepsilon} ds \int_{-\infty}^{\infty} dy \left[ p_{t+\varepsilon-s}(y-x) \right]^2
\]
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; \( E(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = E(J_1^2) + E(J_2^2) \)]

Since \( J_2 = \int_{(t,t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y-x) \xi(ds \, dy) \),

\[
E(J_2^2) = \int_t^{t+\varepsilon} ds \int_{-\infty}^\infty dy \left[ p_{t+\varepsilon-s}(y-x) \right]^2
\]
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; \( \mathbb{E}(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = \mathbb{E}(J_1^2) + \mathbb{E}(J_2^2) \)]

- Since \( J_2 = \int_{(t, t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y - x) \xi(ds \, dy), \)

\[
\begin{align*}
\mathbb{E}(J_2^2) &= \int_t^{t+\varepsilon} ds \int_{-\infty}^{\infty} dy \, [p_{t+\varepsilon-s}(y - x)]^2 \\
&= \int_0^{\varepsilon} ds \int_{-\infty}^{\infty} dy \, [p_s(y)]^2
\end{align*}
\]
A Linear Heat Equation (Lecture 3)
Ideas of proof [Lei–Nualart; \( E(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = E(J_1^2) + E(J_2^2) \)]

Since \( J_2 = \int_{(t,t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y - x) \xi(ds \, dy) \),

\[
E(J_2^2) = \int_t^{t+\varepsilon} ds \int_{-\infty}^{\infty} dy \ [p_{t+\varepsilon-s}(y - x)]^2
\]

\[
= \int_0^{\varepsilon} ds \int_{-\infty}^{\infty} dy \ [p_s(y)]^2 = \int_0^{\varepsilon} p_{2s}(0) \, ds \quad \text{Chapman–Kolm} \]
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; \( \text{E}(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = \text{E}(J_1^2) + \text{E}(J_2^2) \)]

Since \( J_2 = \int_{(t,t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y-x) \xi(ds \: dy) \),

\[
\text{E}(J_2^2) = \int_t^{t+\varepsilon} ds \int_{-\infty}^{\infty} dy \; [p_{t+\varepsilon-s}(y-x)]^2
\]

\[= \int_0^{\varepsilon} ds \int_{-\infty}^{\infty} dy \; [p_s(y)]^2 = \int_0^{\varepsilon} p_{2s}(0) \: ds \tag{Chapman–Kolm}
\]

\[= \sqrt{\frac{\varepsilon}{\nu \pi}}.\]
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; \(E(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = E(J_1^2) + E(J_2^2)\)]

- Since \(J_2 = \int_{(t,t+\varepsilon) \times \mathbb{R}} \rho_{t+\varepsilon-s}(y - x) \xi(ds \, dy)\),

  \[
  E(J_2^2) = \int_t^{t+\varepsilon} ds \int_{-\infty}^{\infty} dy \left[ \rho_{t+\varepsilon-s}(y - x) \right]^2 \\
  = \int_0^{\varepsilon} ds \int_{-\infty}^{\infty} dy \left[ \rho_s(y) \right]^2 = \int_0^{\varepsilon} \rho_{2s}(0) \, ds \quad \text{(Chapman–Kolm)} \\
  = \sqrt{\frac{\varepsilon}{\nu \pi}}.
  \]

- Since \(J_1 = \int_{(0,t) \times \mathbb{R}} \left[ \rho_{t+\varepsilon-s}(y - x) - \rho_{t-s}(y - x) \right] \xi(ds \, dy)\),

  \[
  E(J_1^2) = \int_0^{t} ds \int_{-\infty}^{\infty} dy \left[ \rho_{t+\varepsilon-s}(y - x) - \rho_{t-s}(y - x) \right]^2
  \]
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; \(E(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = E(J_1^2) + E(J_2^2)\)]

- Since \(J_2 = \int_{(t,t+\varepsilon)\times\mathbb{R}} p_{t+\varepsilon-s}(y-x) \xi(ds \, dy)\),
  
  \[
  E(J_2^2) = \int_t^{t+\varepsilon} ds \int_{-\infty}^{\infty} dy \, [p_{t+\varepsilon-s}(y-x)]^2 
  = \int_0^{\varepsilon} ds \int_{-\infty}^{\infty} dy \, [p_s(y)]^2 = \int_0^{\varepsilon} p_{2s}(0) \, ds 
  = \sqrt{\frac{\varepsilon}{\nu \pi}}. 
  \]

- Since \(J_1 = \int_{(0,t)\times\mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)] \xi(ds \, dy)\),
  
  \[
  E(J_1^2) = \int_0^t ds \int_{-\infty}^{\infty} dy \, [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]^2 
  \]

\(\text{Chapman–Kolmogorov}\)
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; E(\(|Z_{t+\varepsilon}(x) - Z_t(x)|^2\)) = E(J_1^2) + E(J_2^2)]

▶ Since \( J_2 = \int_{(t,t+\varepsilon) \times \mathbb{R}} p_{t+\varepsilon-s}(y-x) \xi(ds \, dy) \),

\[
E(J_2^2) = \int_t^{t+\varepsilon} ds \int_{-\infty}^{\infty} dy \left[ p_{t+\varepsilon-s}(y-x) \right]^2 \\
= \int_0^\varepsilon ds \int_{-\infty}^{\infty} dy \left[ p_s(y) \right]^2 = \int_0^\varepsilon p_{2s}(0) \, ds \tag{Chapman–Kolm}
= \sqrt{\frac{\varepsilon}{\nu \pi}}.
\]

▶ Since \( J_1 = \int_{(0,t) \times \mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)] \xi(ds \, dy) \),

\[
E(J_1^2) = \int_0^t ds \int_{-\infty}^{\infty} dy \left[ p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x) \right]^2 \\
= \int_0^t ds \int_{-\infty}^{\infty} dy \left[ p_{\varepsilon+s}(y) - p_s(y) \right]^2.
\]
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; $E(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = E(J_1^2) + E(J_2^2)$]

$\triangleright$ $E(J_1^2) = \int_0^t ds \int_{-\infty}^{\infty} dy \left[p_{\varepsilon+s}(y) - p_s(y)\right]^2.$
Ideas of proof [Lei–Nualart; $E(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = E(J_1^2) + E(J_2^2)$]

- $E(J_1^2) = \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_{\varepsilon+s}(y) - p_s(y)]^2$.
- By Plancherel’s theorem,

$$\int_0^\infty ds \int_{-\infty}^{\infty} dy \ [p_{\varepsilon+s}(y) - p_s(y)]^2 = \sqrt{\frac{\varepsilon}{\nu \pi}} \left(\sqrt{2} - 1\right) = \sqrt{\frac{2\varepsilon}{\nu \pi}} - E(J_2^2).$$
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; $E(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = E(J_1^2) + E(J_2^2)$]

- $E(J_1^2) = \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_{\varepsilon+s}(y) - p_s(y)]^2$.

- By Plancherel’s theorem,

$$\int_0^\infty ds \int_{-\infty}^{\infty} dy \ [p_{\varepsilon+s}(y) - p_s(y)]^2 = \sqrt{\frac{\varepsilon}{\nu \pi}} (\sqrt{2} - 1) = \sqrt{\frac{2\varepsilon}{\nu \pi}} - E(J_2^2).$$

- Let $\eta$ be an independent white noise on $\mathbb{R}$, and define

$$T_t := (2\nu \pi)^{-1/2} \int_{-\infty}^{\infty} z^{-1}(1 - \exp\{-\nu tz^2/2\}) \eta(dz).$$

Then $T$ is smooth, and similar computations show that

$$E\left(|T_{t+\varepsilon} - T_t|^2\right) = \int_t^\infty ds \int_{-\infty}^{\infty} dy \ [p_{s+\varepsilon}(y) - p_s(y)]^2.$$
A Linear Heat Equation (Lecture 3)

Ideas of proof [Lei–Nualart; $E(|Z_{t+\varepsilon}(x) - Z_t(x)|^2) = E(J_1^2) + E(J_2^2)$]

- $E(J_1^2) = \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_{\varepsilon+s}(y) - p_s(y)]^2$.
- By Plancherel’s theorem,

$$\int_0^\infty ds \int_{-\infty}^{\infty} dy \ [p_{\varepsilon+s}(y) - p_s(y)]^2 = \sqrt{\frac{\varepsilon}{\nu\pi}} \left(\sqrt{2} - 1\right) = \sqrt{\frac{2\varepsilon}{\nu\pi}} - E(J_2^2).$$

- Let $\eta$ be an independent white noise on $\mathbb{R}$, and define $T_t := (2\nu\pi)^{-1/2} \int_{-\infty}^{\infty} z^{-1} (1 - \exp\{-\nu t z^2/2\}) \eta(dz)$. Then $T$ is smooth, and similar computations show that

$$E\left(|T_{t+\varepsilon} - T_t|^2\right) = \int_t^{\infty} ds \int_{-\infty}^{\infty} dy \ [p_{s+\varepsilon}(y) - p_s(y)]^2.$$

- Therefore,

$$E\left(|Z_{t+\varepsilon}(x) + T_{t+\varepsilon} - Z_t(x) + T_t|^2\right) = \sqrt{\frac{2\varepsilon}{\nu\pi}} \quad [\text{fBm}(1/4)].$$
A Linear Heat Equation (Lecture 3)
Ideas of proof [Foondun–K–Mahboubi]

\[ Z_t(x + \varepsilon) - Z_t(x) = \int_{(0,t) \times \mathbb{R}} [p_{t-s}(y-x-\varepsilon) - p_{t-s}(y-x)] \xi(ds \, dy) \Rightarrow \]
A Linear Heat Equation (Lecture 3)
Ideas of proof [Foondun–K–Mahboubi]

\[ Z_t(x+\varepsilon) - Z_t(x) = \int_{(0,t) \times \mathbb{R}} [p_{t-s}(y-x-\varepsilon) - p_{t-s}(y-x)] \xi(ds\,dy) \]

By Wiener+Plancherel,

\[ \mathbb{E}\left(|Z_t(x+\varepsilon) - Z_t(x)|^2\right) = \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_s(y-\varepsilon) - p_s(y)]^2 \]
A Linear Heat Equation (Lecture 3)
Ideas of proof [Fouondun–K–Mahboubi]

1. \( Z_t(x + \varepsilon) - Z_t(x) = \int_{(0,t) \times \mathbb{R}} [p_{t-s}(y-x-\varepsilon) - p_{t-s}(y-x)] \xi(ds \, dy) \Rightarrow \)

2. By Wiener+Plancherel,

\[
\mathbb{E}\left( |Z_t(x + \varepsilon) - Z_t(x)|^2 \right) = \int_0^t ds \int_{-\infty}^\infty dy \ [p_s(y - \varepsilon) - p_s(y)]^2
\]

where \( S(x) := \int_{(t,\infty) \times \mathbb{R}} [p_{t-s}(w-x) - p_{t-s}(w)] \zeta(ds \, dw) \), for an independent space-time white noise \( \zeta \).

So, \( Z_t(x) + S(x) = 2\text{-}sided \: BM \)
A Linear Heat Equation (Lecture 3)
Ideas of proof [Foondun–K–Mahboubi]

- \( Z_t(x+\varepsilon) - Z_t(x) = \int_{(0,t) \times \mathbb{R}} [p_{t-s}(y-x-\varepsilon) - p_{t-s}(y-x)] \xi(ds \, dy) \Rightarrow \)

- By Wiener+Plancherel,

\[
E \left( |Z_t(x + \varepsilon) - Z_t(x)|^2 \right) = \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_s(y-\varepsilon) - p_s(y)]^2 \\
= \frac{\varepsilon}{\nu \pi} \int_{-\infty}^{\infty} \left( 1 - e^{-t\nu (w/\varepsilon)^2} \right) \left( \frac{1 - \cos w}{w^2} \right) dw
\]
A Linear Heat Equation (Lecture 3)
Ideas of proof [Foondun–K–Mahboubi]

\[ Z_t(x+\varepsilon) - Z_t(x) = \int_{0,t} \times \mathbb{R} \left[ p_{t-s}(y-x-\varepsilon) - p_{t-s}(y-x) \right] \xi(ds \, dy) \Rightarrow \]

By Wiener+Plancherel,

\[ \mathbb{E} \left( |Z_t(x+\varepsilon) - Z_t(x)|^2 \right) = \int_0^t ds \int_{-\infty}^{\infty} dy \left[ p_s(y-\varepsilon) - p_s(y) \right]^2 \]

\[ = \frac{\varepsilon}{\nu \pi} \int_{-\infty}^{\infty} \left( 1 - e^{-t \nu (w/\varepsilon)^2} \right) \left( \frac{1 - \cos w}{w^2} \right) dw \]

\[ = \frac{\varepsilon}{\nu} - \frac{\varepsilon}{\nu \pi} \int_{-\infty}^{\infty} e^{-t \nu (w/\varepsilon)^2} \left( \frac{1 - \cos w}{w^2} \right) dw \]
A Linear Heat Equation (Lecture 3)
Ideas of proof [Foondun–K–Mahboubi]

- \( Z_t(x + \varepsilon) - Z_t(x) = \int_{(0,t) \times \mathbb{R}} [p_{t-s}(y-x-\varepsilon) - p_{t-s}(y-x)] \xi(ds \, dy) \Rightarrow \)

- By Wiener+Plancherel,

\[
\begin{align*}
\mathbb{E}\left( |Z_t(x + \varepsilon) - Z_t(x)|^2 \right) &= \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_s(y - \varepsilon) - p_s(y)]^2 \\
&= \frac{\varepsilon}{\nu \pi} \int_{-\infty}^{\infty} \left( 1 - \text{e}^{-\nu(w/\varepsilon)^2} \right) \left( \frac{1 - \cos w}{w^2} \right) dw \\
&= \frac{\varepsilon}{\nu} - \frac{\varepsilon}{\nu \pi} \int_{-\infty}^{\infty} \text{e}^{-\nu(w/\varepsilon)^2} \left( \frac{1 - \cos w}{w^2} \right) dw \\
&= \frac{\varepsilon}{\nu} - \mathbb{E}\left( |S(x + \varepsilon) - S(x)|^2 \right),
\end{align*}
\]

where \( S(x) := \int_{(t, \infty) \times \mathbb{R}} [p_s(w - x) - p_s(w)] \zeta(ds \, dw) \), for an independent space-time white noise \( \zeta \).
A Linear Heat Equation (Lecture 3)
Ideas of proof [Foondun–K–Mahboubi]

- \( Z_t(x+\varepsilon) - Z_t(x) = \int_{(0,t) \times \mathbb{R}} [p_{t-s}(y-x-\varepsilon) - p_{t-s}(y-x)] \xi(ds\,dy) \Rightarrow \)
- By Wiener+Plancherel,

\[
\mathbb{E}\left( |Z_t(x + \varepsilon) - Z_t(x)|^2 \right) = \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_s(y - \varepsilon) - p_s(y)]^2
\]

\[
= \frac{\varepsilon}{\nu\pi} \int_{-\infty}^{\infty} \left( 1 - e^{-t\nu(w/\varepsilon)^2} \right) \left( \frac{1 - \cos w}{w^2} \right) dw
\]

\[
= \frac{\varepsilon}{\nu} - \frac{\varepsilon}{\nu\pi} \int_{-\infty}^{\infty} e^{-t\nu(w/\varepsilon)^2} \left( \frac{1 - \cos w}{w^2} \right) dw
\]

\[
= \frac{\varepsilon}{\nu} - \mathbb{E}\left( |S(x + \varepsilon) - S(x)|^2 \right),
\]

where \( S(x) := \int_{(t,\infty) \times \mathbb{R}} [p_s(w - x) - p_s(w)] \zeta(ds \, dw) \), for an independent space-time white noise \( \zeta \). So, \( Z_t(x) + S(x) = 2\text{-sided BM} \)
Consider a particle system on $\epsilon \mathbb{Z}$ where:

- Every particle moves as BMs;
- The $i$th particle experiences a push, at rate 2, toward the average position of particles $i-\epsilon$ and $i+\epsilon$:

$$dX_{\epsilon}^i(t) = 2(X_{\epsilon}^i(t) + \epsilon) - X_{\epsilon}^i(t) - X_{\epsilon}^i(t) dW^i(t)$$

This means: Given iid BMs $\{B_{\epsilon}^i(t)\}_{i \in \epsilon \mathbb{Z}}$, the dynamics of the particle positions can be described by a semidiscrete linear SPDE.
Consider a particle system on $\epsilon \mathbb{Z}$ where:

- Every particle moves as BMs;
Consider a particle system on $\varepsilon \mathbb{Z}$ where:

- Every particle moves as BMs;
- The $i$th particle experiences a push, at rate 2, toward the average position of particles $i - \varepsilon$ and $i + \varepsilon$: 

\[
\begin{align*}
\frac{dX_t(i)}{dt} &= 2\left(X_t(i + \varepsilon) - X_t(i) + X_t(i - \varepsilon) - X_t(i)\right) + dB_t(i)
\end{align*}
\]
Consider a particle system on $\mathbb{Z}$ where:

- Every particle moves as BMs;
- The $i$th particle experiences a push, at rate 2, toward the average position of particles $i - \varepsilon$ and $i + \varepsilon$;
- This means: Given iid BMs $\{B^{(\varepsilon)}(i)\}_{i \in \varepsilon \mathbb{Z}}$,

$$dX^{(\varepsilon)}_t(i) = 2\left(\frac{X^{(\varepsilon)}_t(i + \varepsilon) + X^{(\varepsilon)}_t(i - \varepsilon)}{2} - X_t(i)\right)dt + dB^{(\varepsilon)}_t(i)$$
Consider a particle system on \( \varepsilon \mathbb{Z} \) where:

- Every particle moves as BMs;
- The \( i \)th particle experiences a push, at rate 2, toward the average position of particles \( i - \varepsilon \) and \( i + \varepsilon \):
- This means: Given iid BMs \( \{B^{(\varepsilon)}(i)\}_{i \in \varepsilon \mathbb{Z}} \),

\[
\frac{dX^{(\varepsilon)}_t(i)}{dt} = 2\left(\frac{X^{(\varepsilon)}_t(i + \varepsilon) + X^{(\varepsilon)}_t(i - \varepsilon)}{2} - X_t(i)\right) dt + dB^{(\varepsilon)}_t(i)
\]
Consider a particle system on $\varepsilon Z$ where:

- Every particle moves as BMs;
- The $i$th particle experiences a push, at rate 2, toward the average position of particles $i - \varepsilon$ and $i + \varepsilon$;
- This means: Given iid BMs $\{B^{(\varepsilon)}(i)\}_{i \in \varepsilon Z}$,

$$
\begin{align*}
\text{d}X^{(\varepsilon)}(i) &= 2 \left( \frac{X^{(\varepsilon)}(i + \varepsilon) + X^{(\varepsilon)}(i - \varepsilon)}{2} - X_t(i) \right) \text{d}t + \text{d}B^{(\varepsilon)}(i) \\
&= (\Delta_{\varepsilon Z}X)_t(i) \text{d}t + \text{d}B^{(\varepsilon)}(i) \quad \text{a semidiscrete linear SPDE}
\end{align*}
$$
A Linear Heat Equation (Lecture 3)
Approximation by interacting BMs

- Speed up the drift \([dt \leftrightarrow \varepsilon^{-2}dt]\) and diffusion
  \([dB_t(i) \leftrightarrow \varepsilon^{-1/2}dB_t(i)]\).
A Linear Heat Equation (Lecture 3)
Approximation by interacting BMs

- Speed up the drift \([dt \leftrightarrow \varepsilon^{-2}dt]\) and diffusion\([dB_t(i) \leftrightarrow \varepsilon^{-1/2}dB_t(i)]\).
- \(dX^{(\varepsilon)} = \varepsilon^{-2}\Delta_\varepsilon Z X^{(\varepsilon)} dt + \varepsilon^{-1/2}dB^{(\varepsilon)} \quad (X_0^{(\varepsilon)}(x) \equiv x_0)\).
Speed up the drift \([dt \leftrightarrow \epsilon^{-2}dt]\) and diffusion
\([dB_t(i) \leftrightarrow \epsilon^{-1/2}dB_t(i)]\).

\[
dX(\epsilon) = \epsilon^{-2} \Delta_{\epsilon} Z X(\epsilon) \, dt + \epsilon^{-1/2} dB^{(\epsilon)} \quad (X_0^{(\epsilon)}(x) \equiv x_0). \]

Theorem (Funaki, 1983; Joseph–Mueller–K, 2013)
As \(\epsilon \to 0^+\), \(X(\epsilon) t(\epsilon \lfloor x/\epsilon \rfloor)\) “converges weakly as a space-time random field” to the solution \(u_t(x)\) to the linear \((SHE)\) with \(\nu = 2\) and \(u_0(x) \equiv x_0\).

There are, more interesting, nonlinear versions as well.
I omit the proof, as it takes us too far afield.

The preceding says that we can think of the linear stochastic heat equation as the infinite-density limit of a system of interacting BMs with nearest-neighbor gravitational attraction.
A Linear Heat Equation (Lecture 3)
Approximation by interacting BMs

- Speed up the drift \([dt \leftrightarrow \varepsilon^{-2} dt]\) and diffusion \([dB_t(i) \leftrightarrow \varepsilon^{-1/2} dB_t(i)]\).
- \(dX^{(\varepsilon)} = \varepsilon^{-2} \Delta_{\varepsilon} X^{(\varepsilon)} dt + \varepsilon^{-1/2} dB^{(\varepsilon)}\) \((X^{(\varepsilon)}_0(x) \equiv x_0)\).

- **Theorem (Funaki, 1983; Joseph–Mueller–K, 2013)**
  As \(\varepsilon \to 0^+\), \(X^{(\varepsilon)}_t(\varepsilon[\cdot/\varepsilon])\) “converges weakly as a space-time random field” to the solution \(u_t(x)\) to the linear (SHE) with \(\nu = 2\) and \(u_0(x) \equiv x_0\).
A Linear Heat Equation (Lecture 3)
Approximation by interacting BMs

- Speed up the drift \([dt \leftrightarrow \epsilon^{-2}dt]\) and diffusion
  \([dB_t(i) \leftrightarrow \epsilon^{-1/2}dB_t(i)]\).
- \(dX^{(\epsilon)} = \epsilon^{-2}\Delta_\epsilon Z X^{(\epsilon)} dt + \epsilon^{-1/2}dB^{(\epsilon)}\) \((X_0^{(\epsilon)}(x) \equiv x_0)\).

- **Theorem (Funaki, 1983; Joseph–Mueller–K, 2013)**
  As \(\epsilon \to 0^+\), \(X_t^{(\epsilon)}(\epsilon[x/\epsilon])\) “converges weakly as a space-time random field” to the solution \(u_t(x)\) to the linear (SHE) with \(\nu = 2\) and \(u_0(x) \equiv x_0\).

- There are, more interesting, nonlinear versions as well.
A Linear Heat Equation (Lecture 3)
Approximation by interacting BMs

- Speed up the drift \([dt \leftrightarrow \varepsilon^{-2}dt]\) and diffusion \([dB_t(i) \leftrightarrow \varepsilon^{-1/2}dB_t(i)]\).

- \[dX^{(\varepsilon)} = \varepsilon^{-2}\Delta\varepsilon X^{(\varepsilon)}\, dt + \varepsilon^{-1/2}dB^{(\varepsilon)} \quad (X^{(\varepsilon)}(0, x) \equiv x_0).\]

- **Theorem (Funaki, 1983; Joseph–Mueller–K, 2013)**
  As \(\varepsilon \to 0^+\), \(X^{(\varepsilon)}_t(\varepsilon [x/\varepsilon])\) “converges weakly as a space-time random field” to the solution \(u_t(x)\) to the linear (SHE) with \(\nu = 2\) and \(u_0(x) \equiv x_0\).

- There are, more interesting, nonlinear versions as well.

- I omit the proof, as it takes us too far afield.
Speed up the drift \([dt \leftrightarrow \varepsilon^{-2} dt]\) and diffusion
\([dB_t(i) \leftrightarrow \varepsilon^{-1/2} dB_t(i)]\).

\[
dX^{(\varepsilon)} = \varepsilon^{-2} \Delta_{\varepsilon} X^{(\varepsilon)} \, dt + \varepsilon^{-1/2} dB^{(\varepsilon)} \quad (X^{(\varepsilon)}_0(x) \equiv x_0).
\]

**Theorem (Funaki, 1983; Joseph–Mueller–K, 2013)**

As \(\varepsilon \to 0^+\), \(X^{(\varepsilon)}_t(\varepsilon \lfloor x/\varepsilon \rfloor)\) “converges weakly as a space-time random field” to the solution \(u_t(x)\) to the linear (SHE) with \(\nu = 2\) and \(u_0(x) \equiv x_0\).

There are, more interesting, nonlinear versions as well.

I omit the proof, as it takes us too far afield.

The preceding says that we can think of the linear stochastic heat equation as the infinite-density limit of a system of interacting BMs with nearest-neighbor gravitational attraction.
So far we have studied the stochastic heat equation (SHE) only when $x \in \mathbb{R}$ is 1-D.
So far we have studied the stochastic heat equation (SHE) only when $x \in \mathbb{R}$ is 1-D.

What if $x \in \mathbb{R}^d$ for $d > 1$?
Curse of dimensionality (Lecture 3)

- So far we have studied the stochastic heat equation (SHE) only when $x \in \mathbb{R}$ is 1-D.
- What if $x \in \mathbb{R}^d$ for $d > 1$?
- The natural candidate for (SHE) in $\mathbb{R}^d$ is \[ \partial_t u = (\nu/2)\Delta u + \xi \quad (t > 0, x \in \mathbb{R}^d), \] subject to a nice initial value such as $u_0(x) \equiv 1$ for all $x \in \mathbb{R}^d$. 

We will soon argue that then $d \geq 2$, the solution is not a random field [i.e., not a function!].

Such SPDEs driven by space-time white noise are inherently equations on $\mathbb{R}^d$ "for $d < 2$.

However, if we "regularize the noise," then the SPDE can sometimes have a function solution in $\mathbb{R}^d$ for $d \geq 2$.

The same sort of remark applies to space-time white noise, as long as we "regularize the Laplacian" [e.g., replace it with $-(\Delta)^{1+\delta}$ for a suitable $\delta > 0$].
Curse of dimensionality (Lecture 3)

- So far we have studied the stochastic heat equation (SHE) only when $x \in \mathbb{R}$ is 1-D.
- What if $x \in \mathbb{R}^d$ for $d > 1$?
- The natural candidate for (SHE) in $\mathbb{R}^d$ is
  \[ \partial_t u = (\nu/2) \Delta u + \xi \quad (t > 0, x \in \mathbb{R}^d), \]
  subject to a nice initial value such as $u_0(x) \equiv 1$ for all $x \in \mathbb{R}^d$.
- We will soon argue that then $d \geq 2$, the solution is not a random field [i.e., not a function!].
Curse of dimensionality (Lecture 3)

- So far we have studied the stochastic heat equation (SHE) only when \( x \in \mathbb{R} \) is 1-D.
- What if \( x \in \mathbb{R}^d \) for \( d > 1 \)?
- The natural candidate for (SHE) in \( \mathbb{R}^d \) is
  \[
  \partial_t u = (\nu/2)\Delta u + \xi \quad (t > 0, x \in \mathbb{R}^d),
  \]
  subject to a nice initial value such as \( u_0(x) \equiv 1 \) for all \( x \in \mathbb{R}^d \).
- We will soon argue that then \( d \geq 2 \), the solution is not a random field [i.e., not a function!].
- Such SPDEs driven by space-time white noise are inherently equations on \( \mathbb{R}^d \) “for \( d < 2 \).”
Curse of dimensionality (Lecture 3)

- So far we have studied the stochastic heat equation (SHE) only when $x \in \mathbb{R}$ is 1-D.
- What if $x \in \mathbb{R}^d$ for $d > 1$?
- The natural candidate for (SHE) in $\mathbb{R}^d$ is
  \[ \partial_t u = \left( \frac{\nu}{2} \right) \Delta u + \xi \quad (t > 0, x \in \mathbb{R}^d), \]
  subject to a nice initial value such as $u_0(x) \equiv 1$ for all $x \in \mathbb{R}^d$.
- We will soon argue that then $d \geq 2$, the solution is not a random field [i.e., not a function!].
- Such SPDEs driven by space-time white noise are inherently equations on $\mathbb{R}^d$ “for $d < 2$.”
- However, if we “regularize the noise,” then the SPDE can sometimes have a function solution in $\mathbb{R}^d$ for $d \geq 2$. 
Curse of dimensionality (Lecture 3)

- So far we have studied the stochastic heat equation (SHE) only when $x \in \mathbb{R}$ is 1-D.

- What if $x \in \mathbb{R}^d$ for $d > 1$?

- The natural candidate for (SHE) in $\mathbb{R}^d$ is\[
\partial_t u = \left(\frac{\nu}{2}\right)\Delta u + \xi \quad (t > 0, x \in \mathbb{R}^d),\]
subject to a nice initial value such as $u_0(x) \equiv 1$ for all $x \in \mathbb{R}^d$.

- We will soon argue that then $d \geq 2$, the solution is not a random field [i.e., not a function!].

- Such SPDEs driven by space-time white noise are inherently equations on $\mathbb{R}^d$ “for $d < 2$.”

- However, if we “regularize the noise,” then the SPDE can sometimes have a function solution in $\mathbb{R}^d$ for $d \geq 2$.

- The same sort of remark applies to space-time white noise, as long as we “regularize the Laplacian” [e.g., replace it with $-(-\Delta)^{1+\delta}$ for a suitable $\delta > 0$].
Now consider the SHE,
\[ \frac{\partial}{\partial t} Z = \frac{\nu}{2} \Delta Z + \xi \quad (t > 0, x \in \mathbb{R}^d, d \geq 2), \]
subject to \( Z_t(x) \equiv 0 \) [say].
Now consider the SHE,
\[
\frac{\partial}{\partial t}Z = \frac{\nu}{2} \Delta Z + \xi \quad (t > 0, x \in \mathbb{R}^d, d \geq 2),
\]
subject to \( Z_t(x) \equiv 0 \) [say].

The weak solution is
\[
Z_t(x) = \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy),
\]
where \( p_t(x) = (2\nu \pi t)^{-d/2} \exp\{-\|x\|^2/(2\nu t)\} \).
Curse of dimensionality (Lecture 3)

Now consider the SHE,

$$\frac{\partial}{\partial t} Z = \frac{\nu}{2} \Delta Z + \xi \quad (t > 0, x \in \mathbb{R}^d, d \geq 2),$$

subject to $Z_t(x) \equiv 0$ [say].

The weak solution is

$$Z_t(x) = \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy),$$

where $p_t(x) = (2\nu \pi t)^{-d/2} \exp\{-\|x\|^2/(2\nu t)\}$.

Not a random function; if it were then it would be a GRF with

$$\mathbb{E}(|Z_t(x)|^2) = \int_0^t ds \int_{\mathbb{R}^d} dy \, [p_s(y)]^2$$
Curse of dimensionality (Lecture 3)

- Now consider the SHE,
  \[ \frac{\partial}{\partial t} Z = \frac{\nu}{2} \Delta Z + \xi \quad (t > 0, x \in \mathbb{R}^d, d \geq 2), \]
  subject to \( Z_t(x) \equiv 0 \) [say].
- The weak solution is
  \[ Z_t(x) = \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy), \]
  where \( p_t(x) = (2\nu \pi t)^{-d/2} \exp\{-\|x\|^2/(2\nu t)\} \).
- Not a random function; if it were then it would be a GRF with
  \[ \mathbb{E} (|Z_t(x)|^2) = \int_0^t ds \int_{\mathbb{R}^d} dy \ [p_s(y)]^2. \]
Now consider the SHE,
\[ \frac{\partial}{\partial t} Z = \frac{\nu}{2} \Delta Z + \xi \quad (t > 0, \ x \in \mathbb{R}^d, \ d \geq 2), \]
subject to \( Z_t(x) \equiv 0 \) [say].

The weak solution is
\[ Z_t(x) = \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \ dy), \]
where \( p_t(x) = (2\nu \pi t)^{-d/2} \exp\{-\|x\|^2/(2\nu t)\}. \)

Not a random function; if it were then it would be a GRF with
\[ E \left( |Z_t(x)|^2 \right) = \int_0^t ds \int_{\mathbb{R}^d} dy \ |p_s(y)|^2 = \int_0^t p_{2s}(0) \ ds. \]
Now consider the SHE,

\[ \frac{\partial}{\partial t} Z = \frac{\nu}{2} \Delta Z + \xi \quad (t > 0, x \in \mathbb{R}^d, d \geq 2), \]

subject to \( Z_t(x) \equiv 0 \) [say].

The weak solution is

\[ Z_t(x) = \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \xi(ds \, dy), \]

where \( p_t(x) = (2\nu \pi t)^{-d/2} \exp\{-\|x\|^2/(2\nu t)\} \).

Not a random function; if it were then it would be a GRF with

\[ E\left( |Z_t(x)|^2 \right) = \int_0^t ds \int_{\mathbb{R}^d} dy \, [p_s(y)]^2 = \int_0^t p_{2s}(0) \, ds \propto \int_0^t s^{-d/2} \, ds = \infty. \]
Non-linear SPDEs (Lecture 3)

\[ \frac{\partial}{\partial t} u = \nu \frac{\partial^2}{2 \partial x^2} u + b(u) + \sigma(u) \xi, \text{ where:} \]

- \(\nu > 0\) is fixed (viscosity coefficient);
- \(b, \sigma: \mathbb{R} \to \mathbb{R}\) are non random, and Lipschitz continuous,
- initial function \(u_0(x)\) non random, as well as measurable and bounded.
- We expect that the mild solution is, by Duhamel's principle,
  \[ u(t)(x) = (p_t \ast u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x) \sigma(u(s)(y)) \xi(ds dy) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x) b(u(s)(y)) ds dy. \]
- We need to make sense of the stochastic integral (next).
- The case \(\sigma \equiv 1\) is similar to \(\sigma \equiv 1, b \equiv 0\) that we just looked at.
- Similar issues arise in the Itô theory of SDEs.
Non-linear SPDEs (Lecture 3)

\[
\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi,
\]
where:

\begin{itemize}
  \item \( \nu > 0 \) is fixed [viscosity coefficient \( \times 2 \)];
\end{itemize}
Non-linear SPDEs (Lecture 3)

\[ \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \]

where:

- \( \nu > 0 \) is fixed [viscosity coefficient \( \times 2 \)];
- \( b, \sigma : \mathbb{R} \rightarrow \mathbb{R} \) are non random, and Lipschitz continuous;
- We need to make sense of the stochastic integral (next).
- The case \( \sigma \equiv 1 \) is similar to \( \sigma \equiv 1, b \equiv 0 \) that we just looked at.
- Similar issues arise in the Itô theory of SDEs.
Non-linear SPDEs (Lecture 3)

\[ \frac{\partial}{\partial t} u = \nu \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \]  

where:

- \( \nu > 0 \) is fixed [viscosity coefficient \( \times 2 \)];
- \( b, \sigma : \mathbb{R} \rightarrow \mathbb{R} \) are non random, and Lipschitz continuous
- initial function \( u_0(x) \) non random, as well as measurable and bounded.
Non-linear SPDEs (Lecture 3)

\[ \frac{\partial}{\partial t} u = \nu \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \]

where:

- \( \nu > 0 \) is fixed [viscosity coefficient \( \times 2 \)];
- \( b, \sigma : \mathbb{R} \to \mathbb{R} \) are non random, and Lipschitz continuous;
- initial function \( u_0(x) \) non random, as well as measurable and bounded.

We expect that the mild solution is, by Duhamel’s principle,

\[
\begin{align*}
\quad u_t(x) &= (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x) \sigma(u_s(y)) \xi(ds \, dy) \\
& \quad + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x) b(u_s(y)) \, ds \, dy.
\end{align*}
\]
Non-linear SPDEs (Lecture 3)

\[
\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \text{ where:}
\]

- \( \nu > 0 \) is fixed [viscosity coefficient \( \times 2 \)];
- \( b, \sigma : \mathbb{R} \rightarrow \mathbb{R} \) are non-random, and Lipschitz continuous
- initial function \( u_0(x) \) non-random, as well as measurable and bounded.

- We expect that the mild solution is, by Duhamel’s principle,

\[
\begin{align*}
  u_t(x) &= \left( p_t \ast u_0 \right)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u_s(y)) \xi(ds \, dy) \\
  & \quad + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) b(u_s(y)) \, ds \, dy.
\end{align*}
\]

- We need to make sense of the stochastic integral (next).

The case \( \sigma \equiv 1 \) is similar to \( \sigma \equiv 1, b \equiv 0 \) that we just looked at. Similar issues arise in the Itô theory of SDEs.
Non-linear SPDEs (Lecture 3)

\[ \frac{\partial}{\partial t} u = \nu \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \]

where:

\[ \nu > 0 \text{ is fixed [viscosity coefficient \times 2]}; \]

\[ b, \sigma: \mathbb{R} \to \mathbb{R} \text{ are non random, and Lipschitz continuous} \]

\[ \text{initial function } u_0(x) \text{ non random, as well as measurable and bounded}. \]

\[ \text{We expect that the mild solution is, by Duhamel’s principle,} \]

\[ u_t(x) = (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u_s(y)) \xi(ds \, dy) \]

\[ + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) b(u_s(y)) \, ds \, dy. \]

\[ \text{We need to make sense of the stochastic integral (next).} \]

\[ \text{The case } \sigma \equiv 1 \text{ is similar to } \sigma \equiv 1, \, b \equiv 0 \text{ that we just looked at}. \]
Non-linear SPDEs (Lecture 3)

\[ \frac{\partial}{\partial t} u = \nu \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \] where:

- \( \nu > 0 \) is fixed [viscosity coefficient \( \times 2 \)];
- \( b, \sigma : \mathbb{R} \to \mathbb{R} \) are non random, and Lipschitz continuous
- initial function \( u_0(x) \) non random, as well as measurable and bounded.

We expect that the mild solution is, by Duhamel’s principle,

\[
\begin{align*}
    u_t(x) &= (p_t \ast u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y)) \xi(ds \, dy) \\
    &\quad + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x)b(u_s(y)) ds \, dy.
\end{align*}
\]

We need to make sense of the stochastic integral (next).

The case \( \sigma \equiv 1 \) is similar to \( \sigma \equiv 1, b \equiv 0 \) that we just looked at.

Similar issues arise in the Itô theory of SDEs.
Wish to construct an Itô-type integral \( \int \Phi_t(x) \xi(dt \, dx) \) when \( \Phi \) is a “predictable” random field. More convenient form: \( \int h_t(x)\Phi_t(x) \xi(dt \, dx) \) for meas. non-random \( h \).
Wish to construct an Itô-type integral $\int \Phi_t(x) \xi(dt\,dx)$ when $\Phi$ is a “predictable” random field. More convenient form: $\int h_t(x)\Phi_t(x) \xi(dt\,dx)$ for meas. non-random $h$.

Need a filtration first . . . Answer: Brownian filtration.
Wish to construct an Itô-type integral \( \int \Phi_t(x) \xi(dt \, dx) \) when \( \Phi \) is a “predictable” random field. More convenient form: \( \int h_t(x)\Phi_t(x) \xi(dt \, dx) \) for meas. non-random \( h \).

Need a filtration first . . . Answer: Brownian filtration.

If \( h \in L^2(\mathbb{R}_+ \times \mathbb{R}) \) then the Gaussian process \( 0 < t \mapsto X_t(h) := \int_{(0,t) \times \mathbb{R}} h \, d\xi \) is a continuous \( L^2(\Omega) \)-martingale, and hence a [nonrandom] time-change of a BM. Let \( \mathcal{F}_{s,x}^0(h) \) denote the sigma-algebra generated by \( \{X_\tau(h) + x\}_{0 \leq \tau \leq s} \).
Wish to construct an Itô-type integral $\int \Phi_t(x) \xi(dt \, dx)$ when $\Phi$ is a “predictable” random field. More convenient form: $\int h_t(x) \Phi_t(x) \xi(dt \, dx)$ for meas. non-random $h$.

Need a filtration first . . . Answer: Brownian filtration.

If $h \in L^2(\mathbb{R}_+ \times \mathbb{R})$ then the Gaussian process $0 < t \mapsto X_t(h) := \int_{(0,t) \times \mathbb{R}} h \, d\xi$ is a continuous $L^2(\Omega)$-martingale, and hence a [nonrandom] time-change of a BM. Let $\mathcal{F}^{0,x}_s(h)$ denote the sigma-algebra generated by $\{X_\tau(h) + x\}_{0 \leq \tau \leq s}$.

$\mathcal{F}_t(h) :=$ the sigma-algebra generated by the completions of all $\mathcal{F}^{0,x}_t(h)$, over all $x \in \mathbb{R}$. 
Wish to construct an Itô-type integral \[ \int \Phi_t(x) \xi(dt \, dx) \]
when \( \Phi \) is a "predictable" random field. More convenient form: \[ \int h_t(x) \Phi_t(x) \xi(dt \, dx) \]
for meas. non-random \( h \).

Need a filtration first . . . Answer: Brownian filtration.

If \( h \in L^2(\mathbb{R}_+ \times \mathbb{R}) \) then the Gaussian process
\[ 0 < t \mapsto X_t(h) := \int_{(0,t) \times \mathbb{R}} h \, d\xi \]
is a continuous \( L^2(\Omega) \)-martingale, and hence a [nonrandom] time-change of a BM. Let \( \mathcal{F}_{s,x}^t(h) \) denote the sigma-algebra generated by \( \{X_\tau(h) + x\}_{0 \leq \tau \leq s} \).

\( \mathcal{F}_t(h) := \) the sigma-algebra generated by the completions of all \( \mathcal{F}_{s,x}^t(h) \), over all \( x \in \mathbb{R} \).

\( \mathcal{F}_t := \) the sigma-algebra generated by \( \mathcal{F}_t(h) \). Intuitively, \( \mathcal{F}_t \) contains all the white noise information by time \( t \).
Walsh–Dalang integrals (Lecture 4)

Stochastic integration

- Wish to construct an Itô-type integral $\int \Phi_t(x) \xi(dt \, dx)$ when $\Phi$ is a “predictable” random field. More convenient form: $\int h_t(x) \Phi_t(x) \xi(dt \, dx)$ for meas. non-random $h$.
- Need a filtration first . . . Answer: Brownian filtration.
- If $h \in L^2(\mathbb{R}_+ \times \mathbb{R})$ then the Gaussian process $0 < t \mapsto X_t(h) := \int_{(0,t) \times \mathbb{R}} h \, d\xi$ is a continuous $L^2(\Omega)$-martingale, and hence a [nonrandom] time-change of a BM. Let $\mathcal{F}_{s,x}(h)$ denote the sigma-algebra generated by $\{X_{\tau}(h) + x\}_{0 \leq \tau \leq s}$.
- $\mathcal{F}_t(h) :=$ the sigma-algebra generated by the completions of all $\mathcal{F}_{s,x}(h)$, over all $x \in \mathbb{R}$.
- $\mathcal{F}_t :=$ the sigma-algebra generated by $\mathcal{F}_t(h)$. Intuitively, $\mathcal{F}_t$ contains all the white noise information by time $t$.
- $\{\mathcal{F}_t\}_{t \geq 0}$ the “Brownian filtration.”
Walsh–Dalang integrals (Lecture 4)

The stochastic integral

\[(t, x) \mapsto \Phi_t(x)\] is an elementary random field when
\[\exists 0 \leq a < b\] and an \(\mathcal{F}_a\)-meas. \(X \in L^2(\Omega)\) and \(\phi \in L^2(\mathbb{R})\) such that

\[\Phi_t(x) = X \mathbf{1}_{[a,b]}(t) \phi(x) \quad (t > 0, x \in \mathbb{R}).\]
The stochastic integral

\[(t, x) \mapsto \Phi_t(x)\] is an elementary random field when \(\exists 0 \leq a < b\) and an \(\mathcal{F}_a\)-meas. \(X \in L^2(\Omega)\) and \(\phi \in L^2(\mathbb{R})\) such that

\[\Phi_t(x) = X 1_{[a,b]}(t) \phi(x) \quad (t > 0, x \in \mathbb{R}).\]

A random field \(\Phi\) is simple if \(\exists\) elementary random fields \(\Phi^{(1)}, \ldots, \Phi^{(n)}\), with disjoint supports, such that
\[\Phi = \sum_{i=1}^{n} \Phi^{(i)}\].
The stochastic integral

\( (t, x) \mapsto \Phi_t(x) \) is an elementary random field when there exist \( 0 \leq a < b \) and an \( \mathcal{F}_a \)-meas. \( X \in L^2(\Omega) \) and \( \phi \in L^2(\mathbb{R}) \) such that

\[
\Phi_t(x) = X 1_{[a,b]}(t) \phi(x) \quad (t > 0, x \in \mathbb{R}).
\]

A random field \( \Phi \) is simple if there exist elementary random fields \( \Phi^{(1)}, \ldots, \Phi^{(n)} \), with disjoint supports, such that

\[
\Phi = \sum_{i=1}^{n} \Phi^{(i)}.
\]

If \( h = h_t(x) \) is nonrandom and \( \Phi \) is elementary, then

\[
\int h\Phi \, d\xi := X \int_{(a,b) \times \mathbb{R}} h_t(x) \phi(x) \xi(dt \, dx).
\]
Walsh–Dalang integrals (Lecture 4)

The stochastic integral

- $(t, x) \mapsto \Phi_t(x)$ is an \textit{elementary} random field when \( \exists 0 \leq a < b \) and an \( \mathcal{F}_a \)-meas. \( X \in L^2(\Omega) \) and \( \phi \in L^2(\mathbb{R}) \) such that

\[
\Phi_t(x) = X \mathbf{1}_{[a,b]}(t) \phi(x) \quad (t > 0, x \in \mathbb{R}).
\]

- A random field \( \Phi \) is \textit{simple} if \( \exists \) elementary random fields \( \Phi^{(1)}, \ldots, \Phi^{(n)} \), with disjoint supports, such that

\[
\Phi = \sum_{i=1}^{n} \Phi^{(i)}.
\]

- If \( h = h_t(x) \) is nonrandom and \( \Phi \) is elementary, then

\[
\int h \Phi \, d\xi := X \int_{(a,b) \times \mathbb{R}} h_t(x) \phi(x) \, \xi(dt \, dx).
\]

- The stochastic integral is Wiener’s; well-defined iff \( h_t(x) \phi(x) \in L^2([a,b] \times \mathbb{R}) \).
If $\Phi$ is a simple random field then

$$\int h\Phi \, d\xi := \sum_{i=1}^{n} \int h\Phi^{(i)} \, d\xi.$$
Walsh–Dalang integrals (Lecture 4)
The stochastic integral

- If $\Phi$ is a simple random field then
  \[
  \int h\Phi \, d\xi := \sum_{i=1}^{n} \int h\Phi^{(i)} \, d\xi.
  \]

- $E \int h\Phi \, d\xi = 0$. 
Walsh–Dalang integrals (Lecture 4)
The stochastic integral

- If $\Phi$ is a simple random field then
  \[ \int h\Phi \, d\xi := \sum_{i=1}^{n} \int h\Phi^{(i)} \, d\xi. \]

- $E \int h\Phi \, d\xi = 0.$

- We have the **Walsh isometry**, 
  \[ E \left( \left| \int h\Phi \, d\xi \right|^2 \right) = \int_0^{\infty} ds \int_{-\infty}^{\infty} dy \, [h_s(y)]^2 E \left( |\Phi_s(y)|^2 \right). \]
Walsh–Dalang integrals (Lecture 4)
The stochastic integral

- If $\Phi$ is a simple random field then
  \[ \int h\Phi \, d\xi := \sum_{i=1}^{n} \int h\Phi^{(i)} \, d\xi. \]

- $E \int h\Phi \, d\xi = 0$.

- We have the **Walsh isometry**, 
  \[ E \left( \left| \int h\Phi \, d\xi \right|^2 \right) = \int_{0}^{\infty} ds \int_{-\infty}^{\infty} dy \left[ h_s(y) \right]^2 E \left( |\Phi_s(y)|^2 \right). \]

- $\int h\Phi \, d\xi$ is defined unambiguously, as a result.
Walsh–Dalang integrals (Lecture 4)

The stochastic integral

\[ \|Z\|_p := \{E(|Z|^p)\}^{1/p} \quad L^p(\Omega) \text{ norm} \]
Walsh–Dalang integrals (Lecture 4)
The stochastic integral

\[ \|Z\|_p := \left\{ \mathbb{E}(|Z|^p) \right\}^{1/p} \quad \text{\(L^p(\Omega)\) norm} \]

Choose and fix \( \beta > 0 \) and define for every space-time random field \( v \) the norm,

\[ \mathcal{N}_{\beta,2}(v) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left( e^{-\beta t} \|v_t(x)\|_2 \right) \]
Walsh–Dalang integrals (Lecture 4)

The stochastic integral

\[ \|Z\|_p := \{E(|Z|^p)\}^{1/p} \quad L^p(\Omega) \text{ norm} \]

Choose and fix \( \beta > 0 \) and define for every space-time random field \( v \) the norm,

\[ N_{\beta,2}(v) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left( e^{-\beta t} \|v_t(x)\|_2 \right) \]

Note that if \( \Phi \) is any simple random field, then

\[
E \left( \left| \int h\Phi \, d\xi \right|^2 \right) = \int_0^\infty ds \int_{-\infty}^\infty dy \ [h_s(y)]^2 E \left( |\Phi_s(y)|^2 \right) \\
\leq [N_{\beta,2}(\Phi)]^2 \int_0^\infty e^{\beta s} \, ds \int_{-\infty}^\infty dy \ [h_s(y)]^2.
\]
Walsh–Dalang integrals (Lecture 4)

The stochastic integral

\[ \|Z\|_p := \{E(|Z|^p)\}^{1/p} \quad L^p(\Omega) \text{ norm} \]

- Choose and fix \( \beta > 0 \) and define for every space-time random field \( v \) the norm,

\[ \mathcal{N}_{\beta,2}(v) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left( e^{-\beta t} \|v_t(x)\|_2 \right) \]

- Note that if \( \Phi \) is any simple random field, then

\[
E \left( \left| \int h\Phi \, d\xi \right|^2 \right) = \int_0^\infty ds \int_{-\infty}^\infty dy \left[ h_s(y) \right]^2 E \left( |\Phi_s(y)|^2 \right) \\
\leq \left[ \mathcal{N}_{\beta,2}(\Phi) \right]^2 \int_0^\infty e^{\beta s} \, ds \int_{-\infty}^\infty dy \left[ h_s(y) \right]^2.
\]
Walsh–Dalang integrals (Lecture 4)

The stochastic integral

\[ \| Z \|_p := \{ \mathbb{E}(|Z|^p) \}^{1/p} \]

\( L^p(\Omega) \) norm

Choose and fix \( \beta > 0 \) and define for every space-time random field \( v \) the norm,

\[ N_{\beta,2}(v) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left( e^{-\beta t} \| v_t(x) \|_2 \right) \]

Note that if \( \Phi \) is any simple random field, then

\[ \mathbb{E} \left( \left| \int h\Phi \, d\xi \right|^2 \right) = \int_0^\infty ds \int_{-\infty}^\infty dy \ [h_s(y)]^2 \mathbb{E} (|\Phi_s(y)|^2) \]

\[ \leq \left[ N_{\beta,2}(\Phi) \right]^2 \int_0^\infty e^{\beta s} \, ds \int_{-\infty}^\infty dy \ [h_s(y)]^2. \]

Definition

Let \( L^{\beta,2} := \) the completion of all simple random fields in norm \( N_{\beta,2} \).
Walsh–Dalang integrals (Lecture 4)
Stochastic integration

- If $\Phi \in L^{\beta,2}$, then $I := \int h\Phi \, d\xi$ well-defined, and
  $E(I^2) \leq [N_{\beta,2}(\Phi)]^2 \int_0^\infty e^{\beta s} \, ds \int_{-\infty}^\infty dy \, [h_s(y)]^2$, provided that $h$ is meas. and the preceding integral converges.
Walsh–Dalang integrals (Lecture 4)

Stochastic integration

- If $\Phi \in \mathcal{L}^{\beta,2}$, then $I := \int h\Phi \, d\xi$ well-defined, and 
  $E(I^2) \leq [\mathcal{N}_{\beta,2}(\Phi)]^2 \int_0^\infty e^{\beta s} \, ds \int_{-\infty}^\infty dy \left[ h_s(y) \right]^2$, provided that $h$ is meas. and the preceding integral converges.

- We have the Walsh isometry for all $\Phi \in \mathcal{L}^{\beta,2}$ and $h$ as above: $E(I^2) = \int_0^\infty ds \int_{-\infty}^\infty dx \left[ h_t(x) \right]^2 E(|\Phi_t(x)|^2)$. 

- $\int h\Phi \, d\xi$ is linear in $h$, and linear in $\Phi$.

- If $A \subset \mathbb{R}^+ \times \mathbb{R}$, then $\int_A h\Phi \, d\xi$ is well-defined, finite, etc.

- Remark
  
  $M_t := \int_{(0,t) \times \mathbb{R}} h\Phi \, d\xi$ is a continuous $L_2(\Omega)$-martingale with quadratic variation $\langle M \rangle_t = \int_0^t ds \int_{-\infty}^\infty dy \left[ h_s(y) \right]^2 |\Phi_s(y)|^2$.

- Proof: Check when $\Phi$ is simple; appeal to Doob's maximal inequality when $\Phi \in \mathcal{L}^{\beta,2}$. 

Walsh–Dalang integrals (Lecture 4)
Stochastic integration

- If $\Phi \in \mathcal{L}^{\beta, 2}$, then $I := \int h\Phi \, d\xi$ well-defined, and
  $E(I^2) \leq \left[ N_{\beta, 2}(\Phi) \right]^2 \int_0^\infty e^{\beta s} \, ds \int_{-\infty}^\infty \, dy \left[ h_s(y) \right]^2$, provided that $h$ is meas. and the preceding integral converges.

- We have the Walsh isometry for all $\Phi \in \mathcal{L}^{\beta, 2}$ and $h$ as above: $E(I^2) = \int_0^\infty \, ds \int_{-\infty}^\infty \, dx \left[ h_t(x) \right]^2 E(|\Phi_t(x)|^2)$.

- $\int h\Phi \, d\xi$ is linear in $h$, and linear in $\Phi$. 
If \( \Phi \in \mathcal{L}^{\beta,2} \), then \( I := \int h \Phi \, d\xi \) well-defined, and 
\[
E(I^2) \leq [\mathcal{N}_{\beta,2}(\Phi)]^2 \int_0^\infty e^{\beta s} \, ds \int_{-\infty}^{\infty} dy \, [h_s(y)]^2,
\]
provided that \( h \) is meas. and the preceding integral converges.

We have the Walsh isometry for all \( \Phi \in \mathcal{L}^{\beta,2} \) and \( h \) as above: 
\[
E(I^2) = \int_0^\infty ds \int_{-\infty}^{\infty} dx \, [h_t(x)]^2 E(|\Phi_t(x)|^2).
\]

\( \int h \Phi \, d\xi \) is linear in \( h \), and linear in \( \Phi \).

If \( A \subset \mathbb{R}_+ \times \mathbb{R} \), then \( \int_A h \Phi \, d\xi := \int \mathbf{1}_A h \Phi \, d\xi \) is well-defined, finite, etc.
Walsh–Dalang integrals (Lecture 4)

Stochastic integration

- If $\Phi \in \mathcal{L}^{\beta,2}$, then $I := \int h\Phi \, d\xi$ well-defined, and $E(I^2) \leq [\mathcal{N}_{\beta,2}(\Phi)]^2 \int_0^\infty e^{\beta s} \, ds \int_{-\infty}^\infty dy \, [h_s(y)]^2$, provided that $h$ is meas. and the preceding integral converges.

- We have the Walsh isometry for all $\Phi \in \mathcal{L}^{\beta,2}$ and $h$ as above: $E(I^2) = \int_0^\infty ds \int_{-\infty}^\infty dx \, [h_t(x)]^2 E(|\Phi_t(x)|^2)$.

- $\int h\Phi \, d\xi$ is linear in $h$, and linear in $\Phi$.

- If $A \subset \mathbb{R}_+ \times \mathbb{R}$, then $\int_A h\Phi \, d\xi := \int 1_A h\Phi \, d\xi$ is well-defined, finite, etc.

Remark $M_t := \int_0^t (0,t) \times \mathbb{R} h\Phi \, d\xi$ is a continuous $L^2(\Omega)$-martingale with quadratic variation $\langle M \rangle_t = \int_0^t ds \int_{-\infty}^\infty dy \, [h_s(y)]^2 |\Phi_s(y)|^2$.

Proof: Check when $\Phi$ is simple; appeal to Doob's maximal inequality when $\Phi \in \mathcal{L}^{\beta,2}$.
If $\Phi \in \mathcal{L}^{\beta,2}$, then $I := \int h\Phi \, d\xi$ well-defined, and

\[
E(I^2) \leq [N_{\beta,2}(\Phi)]^2 \int_0^\infty e^{\beta s} \, ds \int_{-\infty}^\infty dy [h_s(y)]^2,
\]
provided that $h$ is meas. and the preceding integral converges.

We have the Walsh isometry for all $\Phi \in \mathcal{L}^{\beta,2}$ and $h$ as above: $E(I^2) = \int_0^\infty ds \int_{-\infty}^\infty dx [h_t(x)]^2 E(|\Phi_t(x)|^2)$.

$\int h\Phi \, d\xi$ is linear in $h$, and linear in $\Phi$.

If $A \subset \mathbb{R}_+ \times \mathbb{R}$, then $\int_A h\Phi \, d\xi := \int 1_A h\Phi \, d\xi$ is well-defined, finite, etc.

**Remark**

$M_t := \int_{(0,t) \times \mathbb{R}} h\Phi \, d\xi$ is a continuous $L^2(\Omega)$-martingale with quadratic variation $\langle M \rangle_t = \int_0^t ds \int_{-\infty}^\infty dy [h_s(y)]^2 |\Phi_s(y)|^2$. 
If \( \Phi \in \mathcal{L}^{\beta,2} \), then \( I := \int h\Phi \, d\xi \) is well-defined, and 
\[
E(I^2) \leq [\mathcal{N}_{\beta,2}(\Phi)]^2 \int_0^\infty e^{\beta s} \, ds \int_{-\infty}^\infty dy \, [h_s(y)]^2,
\]
provided that \( h \) is measurable and the preceding integral converges.

We have the Walsh isometry for all \( \Phi \in \mathcal{L}^{\beta,2} \) and \( h \) as above: 
\[
E(I^2) = \int_0^\infty ds \int_{-\infty}^\infty dx \, [h_t(x)]^2 E(|\Phi_t(x)|^2).
\]
\( \int h\Phi \, d\xi \) is linear in \( h \), and linear in \( \Phi \).

If \( A \subseteq \mathbb{R}_+ \times \mathbb{R} \), then \( \int_A h\Phi \, d\xi := \int 1_A h\Phi \, d\xi \) is well-defined, finite, etc.

**Remark**

\( M_t := \int_{(0,t) \times \mathbb{R}} h\Phi \, d\xi \) is a continuous \( L^2(\Omega) \)-martingale with quadratic variation 
\( \langle M \rangle_t = \int_0^t ds \int_{-\infty}^\infty dy \, [h_s(y)]^2 |\Phi_s(y)|^2 \).

**Proof:** Check when \( \Phi \) is simple; appeal to Doob’s maximal inequality when \( \Phi \in \mathcal{L}^{\beta,2} \).

If $M_t$ is a continuous $L^2(\Omega)$-martingale with quadratic variation $\langle M \rangle_t$, then for all real numbers $k \in [2, \infty)$,

$$\|M_t\|_2^k \leq 4k \|\langle M \rangle_t\|_{2k/2}^2 (t \geq 0).$$

See the lecture notes [Appendix B] for proofs etc.

Equivalently, $E(\|M_t\|_k) \leq (4k)^{1/2} E(\langle M \rangle_{k/2}^t)$. 
Walsh–Dalang integrals (Lecture 4)
BDG inequality


If $M_t$ is a continuous $L^2(\Omega)$-martingale with quadratic variation $\langle M \rangle_t$, then for all real numbers $k \in [2, \infty)$,

$$\|M_t\|_k^2 \leq 4k\|\langle M \rangle_t\|_{k/2}^2 \quad (t \geq 0).$$

If $M_t$ is a continuous $L^2(\Omega)$-martingale with quadratic variation $\langle M \rangle_t$, then for all real numbers $k \in [2, \infty)$,

$$\|M_t\|_k^2 \leq 4k \|\langle M \rangle_t\|_{k/2}^2$$

$(t \geq 0)$.

See the lecture notes [Appendix B] for proofs etc.

If $M_t$ is a continuous $L^2(\Omega)$-martingale with quadratic variation $\langle M \rangle_t$, then for all real numbers $k \in [2, \infty)$,

$$\|M_t\|_k^2 \leq 4k \|\langle M \rangle_t\|_{k/2}^2 \quad (t \geq 0).$$

See the lecture notes [Appendix B] for proofs etc.

Equivalently, $\mathbb{E}(|M_t|^k) \leq (4k)^{1/2} \mathbb{E}(\langle M \rangle_t^{k/2})$. 
Proposition

If $\Phi$ and $h$ are as before, then
\[
\| \int_{(0,t)} \times R h \Phi d\xi \|_2^2 \leq 4 k \int_0^t ds \int_{-\infty}^{\infty} d y \left[ h s (y) \right]^2 \| \Phi s (y) \|^2.
\]

Proof. The quadratic variation of $M_t := \int_{(0,t)} \times R h \Phi d\xi$ is
\[
\langle M \rangle_t = \int_0^t ds \int_{-\infty}^{\infty} d y \left[ h s (y) \right]^2 \| \Phi s (y) \|^2,
\]
whence by BDG,
\[
\| M_t \|_2^2 \leq 2 k \| \int_0^t ds \int_{-\infty}^{\infty} d y \left[ h s (y) \right]^2 \| \Phi s (y) \|^2 \|^{\frac{k}{2}}.
\]

Apply Minkowski's inequality.
Proposition

If $\Phi$ and $h$ are as before, then $\forall t \geq 0$,

$$
\left\| \int_{(0,t) \times \mathbb{R}} h\Phi \, d\xi \right\|_k^2 \leq 4k \int_0^t ds \int_{-\infty}^{\infty} dy \, [h_s(y)]^2 \|\Phi_s(y)\|_k^2.
$$

Proof. The quadratic variation of $M_t := \int_{(0,t) \times \mathbb{R}} h\Phi \, d\xi$ is $\langle M \rangle_t = \int_0^t ds \int_{-\infty}^{\infty} dy \, [h_s(y)]^2 \|\Phi_s(y)\|_k^2$, whence by BDG,

$$
\|M_t\|_k^2 \leq 2k \int_0^t ds \int_{-\infty}^{\infty} dy \, [h_s(y)]^2 \|\Phi_s(y)\|_k^2.
$$

Apply Minkowski's inequality.
Walsh–Dalang integrals (Lecture 4)
BDG inequality

▶ Proposition
If \( \Phi \) and \( h \) are as before, then \( \forall t \geq 0, \)
\[
\left\| \int_{(0,t) \times \mathbb{R}} h \Phi \, d\xi \right\|_k^2 \leq 4k \int_0^t ds \int_{-\infty}^{\infty} dy \left[ h_s(y) \right]^2 \| \Phi_s(y) \|_k^2.
\]

▶ Proof. The quadratic variation of \( M_t := \int_{(0,t) \times \mathbb{R}} h \Phi \, d\xi \) is
\[
\langle M \rangle_t = \int_0^t ds \int_{-\infty}^{\infty} \left[ h_s(y) \right]^2 \left[ \Phi_s(y) \right]^2,
\]
whence by BDG,
\[
\| M_t \|_k^2 \leq 2k \left\| \int_0^t ds \int_{-\infty}^{\infty} \left[ h_s(y) \right]^2 \left[ \Phi_s(y) \right]^2 \right\|_{k/2}.
\]
Apply Minkowski’s inequality.
Good integrands

- Remaining question. When is a random field $\Phi$ in $L^{\beta,2}$?
 WALSH–DALANG INTEGRALS (LECTURE 4) 

GOOD INTEGRANDS

- Remaining question. When is a random field \( \Phi \) in \( L^{\beta,2} \)?
Walsh–Dalang integrals (Lecture 4)
Good integrands

▶ Remaining question. When is a random field $\Phi$ in $L^{\beta,2}$?

▶ Definition
We say that $\Phi$ is a space-time random field [random field, for short] if: (i) $\Phi$ is adapted; i.e., $\Phi_t(x)$ is $\mathcal{F}_t$-meas. for all $t \geq 0$ and $x \in \mathbb{R}$; and
Remaining question. When is a random field $\Phi$ in $\mathcal{L}^{\beta,2}$?

**Definition**

We say that $\Phi$ is a *space-time random field* [random field, for short] if: (i) $\Phi$ is *adapted*; i.e., $\Phi_t(x)$ is $\mathcal{F}_t$-meas. for all $t \geq 0$ and $x \in \mathbb{R}$; and
Walsh–Dalang integrals (Lecture 4)
Good integrands

- Remaining question. When is a random field $\Phi$ in $\mathcal{L}^{\beta,2}$?

- **Definition**

  We say that $\Phi$ is a *space-time random field* [random field, for short] if: (i) $\Phi$ is adapted; i.e., $\Phi_t(x)$ is $\mathcal{F}_t$-meas. for all $t \geq 0$ and $x \in \mathbb{R}$; and

  (ii) $\Phi$ is continuous in $L^2(\Omega)$; i.e., for all $N > 0$,

  $$
  \lim_{n \to \infty} \sup_{(s,y),(t,x) \in [0,N] \times \mathbb{R}} \mathbb{E} \left( |\Phi_s(y) - \Phi_t(x)|^2 \right) = 0.
  $$
Remaining question. When is a random field $\Phi$ in $\mathcal{L}^\beta,2$?

Definition
We say that $\Phi$ is a space-time random field [random field, for short] if: (i) $\Phi$ is adapted; i.e., $\Phi_t(x)$ is $\mathcal{F}_t$-meas. for all $t \geq 0$ and $x \in \mathbb{R}$; and
(ii) $\Phi$ is continuous in $L^2(\Omega)$; i.e., for all $N > 0$,

$$
\lim_{n \to \infty} \sup_{(s,y),(t,x) \in [0,N] \times \mathbb{R}} \frac{1}{|s-t|,|x-y| < 1/n} \mathbb{E} \left( |\Phi_s(y) - \Phi_t(x)|^2 \right) = 0.
$$

Proposition
Suppose $\Phi$ is a space-time random field that is continuous in $L^2(\Omega)$ and $N_{\beta,2}(\Phi) < \infty$, then $\Phi \in \cap_{\alpha > \beta} \mathcal{L}^{\alpha,2}$. 

Approximate $\Phi$ with

$$S_{t}^{n,N}(x) := \Phi_{\lfloor nt \rfloor / n}(\lfloor nx \rfloor / n) \cdot 1_{[0,N] \times \mathbb{R}}(t,x),$$

where $\lfloor -y \rfloor := -\lfloor -y \rfloor$ when $y < 0$. 
 Walsh–Dalang integrals (Lecture 4)  
Good integrands (Idea of proof)

- Approximate $\Phi$ with

$$S_{t}^{n,N}(x) := \Phi_{\lfloor nt \rfloor/n}(\lfloor nx \rfloor/n) \cdot 1_{[0,N] \times \mathbb{R}}(t,x),$$

where $\lfloor -y \rfloor := -\lfloor -y \rfloor$ when $y < 0$.

- We can write ($N \gg n \gg 1$ fixed)

$$S_{t}^{n,N}(x) = \sum_{i,j \in \mathbb{Z}: 0 \leq i < nN} X_i 1_{(i/n,(i+1)/n]}(t) \phi_j(t),$$

where:
Walsh–Dalang integrals (Lecture 4)
Good integrands (Idea of proof)

- Approximate \( \Phi \) with

\[
S_{t}^{n,N}(x) := \Phi_{\lfloor nt \rfloor / n} \left( \lfloor nx \rfloor / n \right) \cdot 1_{[0,N] \times \mathbb{R}}(t, x),
\]

where \( \lfloor -y \rfloor := -\lfloor -y \rfloor \) when \( y < 0 \).

- We can write (\( N \gg n \gg 1 \) fixed)

\[
S_{t}^{n,N}(x) = \sum_{i,j \in \mathbb{Z}: \atop 0 \leq i < nN} X_i 1_{(i/n, (i+1)/n)}(t) \phi_j(t),
\]

where:

- \( X_i := \Phi_{i2^n}(jn) \);
Walsh–Dalang integrals (Lecture 4)
Good integrands (Idea of proof)

- Approximate $\Phi$ with

$$S_{t}^{n,N}(x) := \Phi_{\lfloor nt \rfloor/n} \left( \lfloor nx \rfloor/n \right) \cdot 1_{[0,N] \times \mathbb{R}}(t,x),$$

where $\lfloor -y \rfloor := -\lfloor -y \rfloor$ when $y < 0$.

- We can write ($N \gg n \gg 1$ fixed)

$$S_{t}^{n,N}(x) = \sum_{i,j \in \mathbb{Z}: 0 \leq i < nN} X_i 1_{(i/n,(i+1)/n]}(t) \phi_j(t),$$

where:

- $X_i := \Phi_{i2^n}(jn)$;
- $\phi_j(x) := 1_{[j/n,(j+1)/n]}(x)$. 
Approximate $\Phi$ with

$$S_{t}^{n,N}(x) := \Phi_{\lfloor nt \rfloor /n}(\lfloor nx \rfloor /n) \cdot 1_{[0,N] \times \mathbb{R}}(t, x),$$

where $\lfloor -y \rfloor := -\lceil -y \rceil$ when $y < 0$.

We can write ($N \gg n \gg 1$ fixed)

$$S_{t}^{n,N}(x) = \sum_{i,j \in \mathbb{Z} : \ 0 \leq i < nN} X_{i}1_{(i/n,(i+1)/n]}(t)\phi_{j}(t),$$

where:

- $X_{i} := \Phi_{i2^{n}}(jn)$;
- $\phi_{j}(x) := 1_{[j/n,(j+1)/n]}(x)$.

Prove that $\mathcal{N}_{\alpha,2}(S^{n,N} - \Phi) \to 0$ as $n, N \to \infty$, for all $\alpha > \beta$.\hfill \square
Recall $p := \text{heat kernel.}$
Recall $p :=$ heat kernel.
Recall $p :=$ heat kernel.

Definition

For a random field $\Phi$, define the stochastic convolution of $p$ and $\Phi$ as

$$(p \ast \Phi)(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x) \Phi_s(y) \xi(ds \, dy),$$

whenever this makes sense [as a Walsh–Dalang stochastic integral].
Recall $p :=$ heat kernel.

**Definition**
For a random field $\Phi$, define the *stochastic convolution* of $p$ and $\Phi$ as

$$(p \ast \Phi)_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \Phi_s(y) \xi(ds \, dy),$$

whenever this makes sense [as a Walsh–Dalang stochastic integral].
A Nonlinear Heat Equation (Lecture 5)

Stochastic Convolutions

- Recall \( p := \text{heat kernel}. \)

- **Definition**
  For a random field \( \Phi \), define the **stochastic convolution** of \( p \) and \( \Phi \) as

\[
(p \ast \Phi)_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \Phi_s(y) \xi(ds \, dy),
\]

whenever this makes sense [as a Walsh–Dalang stochastic integral].

- **Theorem**
  If \( \Phi \in L^{\beta,2} \) for some \( \beta > 0 \), then \( p \ast \Phi \) is defined and has a continuous version that is in \( \cap_{\alpha > \beta} L^{\alpha,2} \).
Recall $p :=$ heat kernel.

**Definition**
For a random field $\Phi$, define the *stochastic convolution* of $p$ and $\Phi$ as

$$(p \ast \Phi)_t(x) := \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x)\Phi_s(y) \xi(ds\,dy),$$

whenever this makes sense [as a Walsh–Dalang stochastic integral].

**Theorem**
If $\Phi \in \mathcal{L}^{\beta,2}$ for some $\beta > 0$, then $p \ast \Phi$ is defined and has a continuous version that is in $\cap_{\alpha > \beta} \mathcal{L}^{\alpha,2}$.

$\therefore\quad p \ast \cdot : \cup_{\beta > 0} \mathcal{L}^{\beta,2} \to \cup_{\beta > 0} \mathcal{L}^{\beta,2}$. 
A Nonlinear Heat Equation (Lecture 5)
The key step of the proof [see the lecture notes for the rest]

If $\beta > 0$ and $k \in [2, \infty)$, then

$$\mathcal{N}_{\beta,k}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left( e^{-\beta t} \| \Phi_t(x) \|_k \right).$$
A Nonlinear Heat Equation (Lecture 5)

The key step of the proof [see the lecture notes for the rest]

- If $\beta > 0$ and $k \in [2, \infty)$, then

$$N_{\beta,k}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left( e^{-\beta t} \| \Phi_t(x) \|_k \right).$$

- When $k = 2$ this is a familiar norm.
The key step of the proof [see the lecture notes for the rest]

If \( \beta > 0 \) and \( k \in [2, \infty) \), then

\[
N_{\beta,k}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left( e^{-\beta t} \| \Phi_t(x) \|_k \right).
\]

When \( k = 2 \) this is a familiar norm.
A Nonlinear Heat Equation (Lecture 5)
The key step of the proof [see the lecture notes for the rest]

- If $\beta > 0$ and $k \in [2, \infty)$, then

$$
\mathcal{N}_{\beta,k}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left( e^{-\beta t} \| \Phi_t(x) \|_k \right).
$$

- When $k = 2$ this is a familiar norm.

- Proposition (Foondun–K, 2009; Conus–K, 2010)

  For all $\beta > 0$, $k \in [2, \infty)$, and $\Phi \in \mathcal{L}^{\beta,2}$,

  $$
  \mathcal{N}_{\beta,k}(p \ast \Phi) \leq \frac{k^{1/2}}{(\nu \beta / 2)^{1/4}} \cdot \mathcal{N}_{\beta,k}(\Phi).
  $$
A Nonlinear Heat Equation (Lecture 5)

Proof of the stochastic Young inequality \[ \mathcal{N}_{\beta,k}(p \otimes \Phi) \leq \frac{k^{1/2}}{(\nu \beta/2)^{1/4}} \cdot \mathcal{N}_{\beta,k}(\Phi) \]

- BDG inequality yields

\[
\|p \otimes \Phi\|_k^2 \leq 4k \int_0^t ds \int_{-\infty}^{\infty} dy \left[ p_{t-s}(y-x) \right]^2 \|\Phi_s(y)\|_k^2
\]
A Nonlinear Heat Equation (Lecture 5)

Proof of the stochastic Young inequality \([\mathcal{N}_{\beta,k}(p \odot \Phi) \leq \frac{k^{1/2}}{(\nu \beta/2)^{1/4}} \cdot \mathcal{N}_{\beta,k}(\Phi)}\]

- BDG inequality yields

\[
\|p \odot \Phi\|_k^2 \leq 4k \int_0^t ds \int_{-\infty}^{\infty} dy \left[ p_{t-s}(y - x) \right]^2 \|\Phi_s(y)\|_k^2
\]
A Nonlinear Heat Equation (Lecture 5)

Proof of the stochastic Young inequality

\[ \mathcal{N}_{\beta,k}(p \ast \Phi) \leq \frac{k^{1/2}}{(\nu_{\beta}/2)^{1/4}} \cdot \mathcal{N}_{\beta,k}(\Phi) \]

- BDG inequality yields

\[
\|p \ast \Phi\|^2_k \leq 4k \int_0^t ds \int_{-\infty}^{\infty} dy \left[ p_{t-s}(y-x) \right]^2 \|\Phi_s(y)\|^2_k \\
\leq 4k [\mathcal{N}_{\beta,k}(\Phi)]^2 \int_0^t e^{2\beta s} ds \int_{-\infty}^{\infty} dy \left[ p_{t-s}(y-x) \right]^2
\]

Do the remaining arithmetic.
A Nonlinear Heat Equation (Lecture 5)

Proof of the stochastic Young inequality \([\mathcal{N}_{\beta,k}(p \otimes \Phi) \leq \frac{k^{1/2}}{(\nu^2/2)^{1/4}} \cdot \mathcal{N}_{\beta,k}(\Phi)}\]

BDG inequality yields

\[
\|p \otimes \Phi\|_k^2 \leq 4k \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_{t-s}(y-x)]^2 \|\Phi_s(y)\|_k^2
\]

\[
\leq 4k[\mathcal{N}_{\beta,k}(\Phi)]^2 \int_0^t e^{2\beta s} ds \int_{-\infty}^{\infty} dy \ [p_{t-s}(y-x)]^2
\]

\[
= 4k[\mathcal{N}_{\beta,k}(\Phi)]^2 e^{2\beta t} \int_0^t e^{-2\beta r} dr \left(\frac{\nu \pi r}{4}\right)^{1/2}.
\]

Do the remaining arithmetic.
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \partial_t u = \left( \frac{\nu}{2} \right) \partial_x^2 u + b(u) + \sigma(u) \xi; \]
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \partial_t u = (\nu/2) \partial_x^2 u + b(u) + \sigma(u)\xi; \]

\[ \text{subject to } u_0(x) \text{ being non-random, measurable, and bounded; } \]
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

- \( \partial_t u = (\nu/2)\partial_x^2 u + b(u) + \sigma(u)\xi; \)
- subject to \( u_0(x) \) being non-random, measurable, and bounded;
- \( \nu > 0 \) fixed;
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \partial_t u = (\nu/2) \partial_x^2 u + b(u) + \sigma(u) \xi; \]
\[ \text{subject to } u_0(x) \text{ being non-random, measurable, and bounded}; \]
\[ \nu > 0 \text{ fixed}; \]
\[ \sigma, b : \mathbb{R} \to \mathbb{R} \text{ Lipschitz. That is, } \exists \text{Lip} > 0 \text{ such that} \]
\[ |\sigma(x) - \sigma(y)| \lor |b(x) - b(y)| \leq \text{Lip}|x - y| \quad \forall x, y \in \mathbb{R}. \]
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \partial_t u = \left( \frac{\nu}{2} \right) \partial_x^2 u + b(u) + \sigma(u)\xi; \]

\[ \text{subject to } u_0(x) \text{ being non-random, measurable, and bounded;} \]

\[ \nu > 0 \text{ fixed;} \]

\[ \sigma, b : \mathbb{R} \rightarrow \mathbb{R} \text{ Lipschitz. That is, } \exists \text{Lip} > 0 \text{ such that} \]

\[ |\sigma(x) - \sigma(y)| \vee |b(x) - b(y)| \leq \text{Lip}|x - y| \quad \forall x, y \in \mathbb{R}. \]

\[ \text{WLOG, Lip} \geq \max\{|\sigma(0)|, |b(0)|\}. \]
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \partial_t u = \left( \frac{\nu}{2} \right) \partial_x^2 u + b(u) + \sigma(u) \xi; \]

subject to \( u_0(x) \) being non-random, measurable, and bounded;

\( \nu > 0 \) fixed;

\( \sigma, b : \mathbb{R} \rightarrow \mathbb{R} \) Lipschitz. That is, \( \exists \text{Lip} > 0 \) such that

\[ |\sigma(x) - \sigma(y)| \vee |b(x) - b(y)| \leq \text{Lip}|x - y| \quad \forall x, y \in \mathbb{R}. \]

WLOG, \( \text{Lip} \geq \max\{|\sigma(0)|, |b(0)|\}. \)

Because \( |\sigma(x)| \leq |\sigma(0)| + \text{Lip}|x| \) and \( |b(x)| \leq |b(0)| + \text{Lip}|x|, \)

\[ |\sigma(x)| \vee |b(x)| \leq \text{Lip}(1 + |x|) \quad \forall x \in \mathbb{R}. \]
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \quad u_0 \in L^\infty(\mathbb{R}) \text{ non-random.} \]
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \quad u_0 \in L^\infty(\mathbb{R}) \text{ non-random.} \]
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \quad u_0 \in L^\infty(\mathbb{R}) \text{ non-random.} \]

\[ \text{Theorem} \]
There exists a random field \( u \in \bigcup_{\beta > 0} L^{\beta, 2} \) that solves this initial value problem. Moreover, it is [a.s.] the only solution for which there exists a positive and finite \( L \) such that

\[ \sup_{x \in \mathbb{R}} \mathbb{E} \left( \left| u_t(x) \right|^k \right) \leq L^k \exp \left\{ Lk^3 t \right\} \quad \forall k \in [1, \infty), \ t > 0. \]
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \quad u_0 \in L^\infty(\mathbb{R}) \text{ non-random}. \]

**Theorem**
There exists a random field \( u \in \bigcup_{\beta > 0} L^{\beta,2} \) that solves this initial value problem. Moreover, it is [a.s.] the only solution for which there exists a positive and finite \( L \) such that

\[ \sup_{x \in \mathbb{R}} \mathbb{E} \left( |u_t(x)|^k \right) \leq L^k \exp \{ Lk^3 t \} \quad \forall k \in [1, \infty), \quad t > 0. \]
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \quad u_0 \in L^\infty(\mathbb{R}) \text{ non-random.} \]

**Theorem**

There exists a random field \( u \in \bigcup_{\beta > 0} \mathcal{L}^{\beta,2} \) that solves this initial value problem. Moreover, it is [a.s.] the only solution for which there exists a positive and finite \( L \) such that

\[ \sup_{x \in \mathbb{R}} \mathbb{E} \left( |u_t(x)|^k \right) \leq L^k \exp \{ Lk^3 t \} \quad \forall k \in [1, \infty), \ t > 0. \]

**Remark**

For uniqueness, \( k = 2 \) suffices. The proof will show more: \( t \mapsto u_t(x) \) is Hölder continuous of index \(< 1/4 \) and \( x \mapsto v_t(x) \) is Hölder continuous of index \(< 1/2 \).

We will see soon that the exponent bound of \( k^3 \) is not artificial.
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \quad u_0 \in L^\infty(\mathbb{R}) \text{ non-random.} \]

▶ Theorem
There exists a random field \( u \in \bigcup_{\beta > 0} L^{\beta,2} \) that solves this initial value problem. Moreover, it is [a.s.] the only solution for which there exists a positive and finite \( L \) such that

\[ \sup_{x \in \mathbb{R}} E \left( |u_t(x)|^k \right) \leq L^k \exp \left\{ Lk^3 t \right\} \quad \forall k \in [1, \infty), \; t > 0. \]

▶ Remark
For uniqueness, \( k = 2 \) suffices.
A Nonlinear Heat Equation (Lecture 5)
Existence and uniqueness

\[ \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \quad u_0 \in L^\infty(\mathbb{R}) \text{ non-random.} \]

\[ \text{Theorem} \]
There exists a random field \( u \in \bigcup_{\beta > 0} \mathcal{L}^{\beta,2} \) that solves this initial value problem. Moreover, it is \([a.s.]\) the only solution for which there exists a positive and finite \( L \) such that

\[ \sup_{x \in \mathbb{R}} E \left( |u_t(x)|^k \right) \leq L^k \exp \{ Lk^3 t \} \quad \forall k \in [1, \infty), \ t > 0. \]

\[ \text{Remark} \]

\[ \text{For uniqueness, } k = 2 \text{ suffices.} \]

\[ \text{The proof will show more: } t \mapsto u_t(x) \text{ is Hölder continuous of index } < \frac{1}{4} \text{ and } x \mapsto v_t(x) \text{ is Hölder continuous of index } < \frac{1}{2}. \]
A Nonlinear Heat Equation (Lecture 5)

Existence and uniqueness

\[ \frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + b(u) + \sigma(u) \xi, \quad u_0 \in L^\infty(\mathbb{R}) \text{ non-random.} \]

**Theorem**

There exists a random field \( u \in \bigcup_{\beta > 0} L^{\beta,2} \) that solves this initial value problem. Moreover, it is [a.s.] the only solution for which there exists a positive and finite \( L \) such that

\[ \sup_{x \in \mathbb{R}} E \left( |u_t(x)|^k \right) \leq L^k \exp \left\{ Lk^3 t \right\} \quad \forall k \in [1, \infty), \ t > 0. \]

**Remark**

- For uniqueness, \( k = 2 \) suffices.
- The proof will show more: \( t \mapsto u_t(x) \) is Hölder continuous of index \( < \frac{1}{4} \) and \( x \mapsto v_t(x) \) is Hölder continuous of index \( < \frac{1}{2} \).
- We will see soon that the exponent bound of \( k^3 \) is not artificial.
We consider the more important contribution of diffusion; thus, \( b \equiv 0 \) from now on; it is easy to adapt the proof to work for general drift function \( b \).
We consider the more important contribution of diffusion; thus, \( b \equiv 0 \) from now on; it is easy to adapt the proof to work for general drift function \( b \).

Recall that \( u \) is a mild solution if

\[
    u_t(x) = (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x) \sigma(u_s(y)) \xi(ds \, dy) \\
    = (p_t * u_0)(x) + (p \otimes \sigma(u))_t(x).
\]
A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof

- We consider the more important contribution of diffusion; thus, \( b \equiv 0 \) from now on; it is easy to adapt the proof to work for general drift function \( b \).
- Recall that \( u \) is a mild solution if

\[
    u_t(x) = (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u_s(y)) \xi(ds \, dy)
\]

\[
    = (p_t * u_0)(x) + (p \otimes \sigma(u))_t(x).
\]

- Proceed by Picard iteration.
A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof

We consider the more important contribution of diffusion; thus, $b \equiv 0$ from now on; it is easy to adapt the proof to work for general drift function $b$.

Recall that $u$ is a mild solution if

$$u_t(x) = (p_t \ast u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y)) \xi(ds \, dy)$$

$$= (p_t \ast u_0)(x) + (p \ast \sigma(u))_t(x).$$

Proceed by Picard iteration.

$u^{(0)}_t(x) := u_0(x);$
A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof

- We consider the more important contribution of diffusion; thus, $b \equiv 0$ from now on; it is easy to adapt the proof to work for general drift function $b$.
- Recall that $u$ is a mild solution if

$$u_t(x) = (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x) \sigma(u_s(y)) \xi(ds \, dy)$$

$$= (p_t * u_0)(x) + (p \ast \sigma(u))_t(x).$$

- Proceed by Picard iteration.
- $u^{(0)}_t(x) := u_0(x)$;
- $u^{(n+1)}_t(x) :=$

$$ (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x) \sigma(u^{(n)}_s(y)) \xi(ds \, dy).$$
A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof

- We consider the more important contribution of diffusion; thus, $b \equiv 0$ from now on; it is easy to adapt the proof to work for general drift function $b$.
- Recall that $u$ is a mild solution if
  \[
  u_t(x) = (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x)\sigma(u_s(y)) \xi(ds \, dy)
  = (p_t * u_0)(x) + (p \otimes \sigma(u))_t(x).
  \]
- Proceed by Picard iteration.
  - $u_t^{(0)}(x) := u_0(x)$;
  - $u_t^{(n+1)}(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x)\sigma(u_s^{(n)}(y)) \xi(ds \, dy)$.
- We will have to prove \textit{a priori} that $u^{(n)}$’s are all well defined etc.
A Nonlinear Heat Equation (Lecture 5)

Existence; sketch of proof

\[ u^{(n+1)}_t(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u^{(n)}_s(y)) \xi(ds \, dy). \]

▶ A priori bound #1:

\[ \mathcal{N}_{\beta,k} \left( u^{(n+1)} \right) \leq \mathcal{N}_{\beta,k} (p_\bullet * u_0) + \mathcal{N}_{\beta,k} \left( p * \sigma(u^{(n)}) \right) \]
A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof

\[ u^{(n+1)}_t(x) := (p_t \ast u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u_s^{(n)}(y)) \xi(ds\,dy). \]

\begin{itemize}
  \item A priori bound \#1:
  \[ N_{\beta,k}(u^{(n+1)}) \leq N_{\beta,k}(p \ast u_0) + N_{\beta,k}(p \otimes \sigma(u^{(n)})) \]
\end{itemize}
A Nonlinear Heat Equation (Lecture 5)

Existence; sketch of proof

\[ u^{(n+1)}_t(x) := (p_t \ast u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u^{(n)}_s(y)) \xi(ds \, dy). \]

A priori bound #1:

\[
N_{\beta,k} \left( u^{(n+1)} \right) \leq N_{\beta,k} \left( p \ast \ast u_0 \right) + N_{\beta,k} \left( p \ast \sigma(u^{(n)}) \right) \\
\leq \|u_0\|_{L^\infty(\mathbb{R})} + \frac{k^{1/2}}{(\nu \beta/2)^{1/4}} N_{\beta,k} \left( \sigma \circ u^{(n)} \right)
\]
A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof

\[ u_t^{(n+1)}(x) := (p_t \ast u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x)\sigma(u_s^{(n)}(y)) \xi(ds\,dy). \]

- **A priori bound #1**: 

\[
\mathcal{N}_{\beta,k}(u^{(n+1)}) \leq \mathcal{N}_{\beta,k}(p \ast u_0) + \mathcal{N}_{\beta,k}(p \otimes \sigma(u^{(n)})) \\
\leq \|u_0\|_{L^\infty(\mathbb{R})} + \frac{k^{1/2}}{(\nu \beta/2)^{1/4}} \mathcal{N}_{\beta,k}(\sigma \circ u^{(n)}) \\
\leq \text{const} \cdot \left[ 1 + \frac{k^{1/2}}{\beta^{1/4}} \left( 1 + \mathcal{N}_{\beta,k}(u^{(n)}) \right) \right].
\]
A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof
\[ u^{(n+1)}(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u^{(n)}_s)(y) \xi(ds \, dy). \]

- **A priori bound #1:**

  \[
  N_{\beta,k}(u^{(n+1)}) \leq N_{\beta,k}(p \ast \bullet \cdot * u_0) + N_{\beta,k}(p \ast \sigma(u^{(n)})) \\
  \leq \|u_0\|_{L^{\infty}({\mathbb R})} + \frac{k^{1/2}}{(\nu \beta/2)^{1/4}} N_{\beta,k}(\sigma \circ u^{(n)}) \\
  \leq \text{const} \cdot \left[ 1 + \frac{k^{1/2}}{\beta^{1/4}} \left( 1 + N_{\beta,k}(u^{(n)}) \right) \right].
  \]

- For the special choice, \( \beta = 16k^2 \): \( \exists C < \infty \)--independent of \( k, n \)—such that \( N_{16k^2,k}(u^{(n)}) \leq C \).
A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof

\[ u_t^{(n+1)}(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x)\sigma(u_s^{(n)}(y)) \xi(ds \, dy). \]

- **A priori bound #1:**

\[ \mathcal{N}_{\beta,k}(u^{(n+1)}) \leq \mathcal{N}_{\beta,k}(p \ast u_0) + \mathcal{N}_{\beta,k}(p \ast \sigma(u^{(n)})) \]

\[ \leq \|u_0\|_{L^\infty(\mathbb{R})} + \frac{k^{1/2}}{(\nu \beta/2)^{1/4}} \mathcal{N}_{\beta,k}(\sigma \circ u^{(n)}) \]

\[ \leq \text{const} \cdot \left[ 1 + \frac{k^{1/2}}{\beta^{1/4}} \left( 1 + \mathcal{N}_{\beta,k}(u^{(n)}) \right) \right]. \]

- For the special choice, $\beta = 16k^2$: $\exists C < \infty$—independent of $k, n$—such that $\mathcal{N}_{16k^2,k}(u^{(n)}) \leq C$. 

A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof

\[ u_{t}^{(n+1)}(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u_s^{(n)}(y)) \xi(ds \, dy). \]

- **A priori bound #1:**

\[
\mathcal{N}_{\beta,k} \left( u^{(n+1)} \right) \leq \mathcal{N}_{\beta,k} (p \bullet * u_0) + \mathcal{N}_{\beta,k} \left( p \otimes \sigma(u^{(n)}) \right) \\
\leq \|u_0\|_{L^\infty(\mathbb{R})} + \frac{k^{1/2}}{(\nu \beta/2)^{1/4}} \mathcal{N}_{\beta,k} \left( \sigma \circ u^{(n)} \right) \\
\leq \text{const} \cdot \left[ 1 + \frac{k^{1/2}}{\beta^{1/4}} \left( 1 + \mathcal{N}_{\beta,k} \left( u^{(n)} \right) \right) \right].
\]

- For the special choice, \( \beta = 16k^2 \): \( \exists C < \infty \) —independent of \( k, n \)—such that \( \mathcal{N}_{16k^2,k} (u^{(n)}) \leq C \).

- \( \mathbb{E} \left( \left| u_t^{(n)}(x) \right|^k \right) \leq C^k \exp \{16k^3t\} \).
A Nonlinear Heat Equation (Lecture 5)

Existence; sketch of proof

\[ u_t^{(n+1)}(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u_s^{(n)}(y)) \xi(ds \, dy). \]

▶ A priori bound #2 (similar): \( \exists D < \infty \) —independently of \( k, n \)—such that \( N_{16k^2, k} \left( u^{(n+1)} - u^{(n)} \right) \leq De^{-n/D} \).

\( \{u^{(n)}\}_{n \geq 0} = \text{Cauchy} \)
A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof
\[ u^{(n+1)}_t(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u^{(n)}_s(y)) \xi(ds \, dy). \]

- **A priori bound #2** (similar): \( \exists D < \infty \) — independently of \( k, n \) — such that \( \mathcal{N}_{16k^2} \left( u^{(n+1)} - u^{(n)} \right) \leq De^{-n/D} \).

\[ \{ u^{(n)} \}_{n \geq 0} = \text{Cauchy} \]

- **Easier bounds:** \( \forall T, k > 0 \ \exists \tilde{\mathcal{C}}_{k,T} < \infty \) — independently of \( n \) — s.t.

\[
E \left( \left| u^{(n)}_t(x) - u^{(n)}_{t'}(x') \right|^k \right) \leq \tilde{\mathcal{C}}_{k,T} \left( |x - x'|^{k/2} + |t - t'|^{k/4} \right),
\]

uniformly for \( x, x' \in \mathbb{R} \) and \( t, t' \in [0, T] \).
A Nonlinear Heat Equation (Lecture 5)
Existence; sketch of proof

\[ u^{(n+1)}(x) := (p_t \ast u_0)(x) + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u_s^{(n)}(y)) \xi(ds \, dy). \]

- **A priori bound #2** (similar): \( \exists D < \infty \) independently of \( k, n \) such that \( N_{16k^2,k}(u^{(n+1)} - u^{(n)}) \leq De^{-n/D} \).

\( \{u^{(n)}\}_{n \geq 0} = \text{Cauchy} \)

- Easier bounds: \( \forall T, k > 0 \ \exists \tilde{C}_{k,T} < \infty \) independently of \( n \) s.t.

\[
E \left( \left| u^{(n)}_t(x) - u^{(n)}_t(x') \right|^k \right) \leq \tilde{C}_{k,T} \left( |x - x'|^{k/2} + |t - t'|^{k/4} \right),
\]

uniformly for \( x, x' \in \mathbb{R} \) and \( t, t' \in [0, T] \).

- \( N_{16k^2,k}(u^{(n)}) < \infty \) and continuity in \( L^2(\Omega) \) \( \Rightarrow u^{(n)} \in \mathcal{L}^{64,2} \) \( \forall n \), and existence works, as in PDEs, by taking Cauchy limits.
A Nonlinear Heat Equation (Lecture 5)
Uniqueness; sketch of proof

► Suppose ⋄α > 0 and u, v ∈ L^{α,2}, so that
  \[ \sup_{x \in \mathbb{R}} \{ E(|u_t(x)|^2) \lor E(|v_t(x)|^2) \} \leq L \exp{\alpha t} \quad \forall t > 0 \]
and
  \[ u_t(x) = (p_t * u_0)(x) + (p \ast \sigma(u))_t(x), \]
  \[ v_t(x) = (p_t * u_0)(x) + (p \ast \sigma(v))_t(x). \]
Suppose \( \exists \alpha > 0 \) and \( u, v \in L^{\alpha,2} \), so that
\[
\sup_{x \in \mathbb{R}} \{ E(|u_t(x)|^2) \vee E(|v_t(x)|^2) \} \leq L \exp\{\alpha t\} \ \forall \ t > 0 \ 	ext{and}
\]
\[
\begin{align*}
    u_t(x) &= (p_t * u_0)(x) + (p \otimes \sigma(u))_t(x), \\
    v_t(x) &= (p_t * u_0)(x) + (p \otimes \sigma(v))_t(x).
\end{align*}
\]

\( \forall \beta > \alpha \),
\[
N_{\beta,2}(u - v) \leq N_{\beta,2}(p \otimes [\sigma(u) - \sigma(v)])
\]
Suppose $\exists \alpha > 0$ and $u, v \in \mathcal{L}^{\alpha, 2}$, so that
\[
\sup_{x \in \mathbb{R}} \{\mathbb{E}(|u_t(x)|^2) \vee \mathbb{E}(|v_t(x)|^2)\} \leq L \exp\{\alpha t\} \quad \forall t > 0 \text{ and}
\]
\[
u_t(x) = (p_t \ast u_0)(x) + (p \ast \sigma(u))_t(x),
\]
\[
u_t(x) = (p_t \ast u_0)(x) + (p \ast \sigma(v))_t(x).
\]

\forall \beta > \alpha,
\[
\mathcal{N}_{\beta, 2} (u - v) \leq \mathcal{N}_{\beta, 2} (p \ast [\sigma(u) - \sigma(v)])
\]
A Nonlinear Heat Equation (Lecture 5)
Uniqueness; sketch of proof

Suppose $\exists \alpha > 0$ and $u, v \in L^{\alpha, 2}$, so that

$$\sup_{x \in \mathbb{R}} \left\{ \mathbb{E}(|u_t(x)|^2) \vee \mathbb{E}(|v_t(x)|^2) \right\} \leq L \exp\{\alpha t\} \ \forall t > 0$$

and

$$u_t(x) = (p_t * u_0)(x) + (p * \sigma(u))_t(x),$$
$$v_t(x) = (p_t * u_0)(x) + (p * \sigma(v))_t(x).$$

∀$\beta > \alpha$,

$$\mathcal{N}_{\beta, 2}(u - v) \leq \mathcal{N}_{\beta, 2}(p \boxtimes [\sigma(u) - \sigma(v)])$$

$$\leq \frac{\text{Lip} \sqrt{2}}{(\nu/\beta)^{1/4}} \mathcal{N}_{\beta, 2}(u - v).$$
A Nonlinear Heat Equation (Lecture 5)
Uniqueness; sketch of proof

★ Suppose $\exists \alpha > 0$ and $u, v \in L^{\alpha, 2}$, so that
\[
\sup_{x \in \mathbb{R}} \{\mathbb{E}(|u_t(x)|^2) \lor \mathbb{E}(|v_t(x)|^2)\} \leq L \exp\{\alpha t\} \forall t > 0 \text{ and }
\]
\[
\begin{align*}
    u_t(x) &= (p_t * u_0)(x) + (p \ast \sigma(u))_t(x), \\
    v_t(x) &= (p_t * u_0)(x) + (p \ast \sigma(v))_t(x).
\end{align*}
\]

★ $\forall \beta > \alpha$,
\[
\mathcal{N}_{\beta, 2}(u - v) \leq \mathcal{N}_{\beta, 2}(p \ast [\sigma(u) - \sigma(v)])
\leq \frac{\text{Lip}\sqrt{2}}{(\nu \beta / 2)^{1/4}} \mathcal{N}_{\beta, 2}(u - v).
\]

★ Choose $\beta$ large to see that $\mathcal{N}_{\beta, 2}(u - v) = 0$. \qed
Mild solution = weak solution (see lecture notes); proof requires a stochastic Fubini theorem, as in the linear case.
Mild solution = weak solution (see lecture notes); proof requires a stochastic Fubini theorem, as in the linear case.

“Stopping-time arguments” can be used to replace the Lipschitz-continuity hypotheses on \( \sigma \) and \( b \) to the following:

\[
\sigma \text{ and } b \text{ are locally Lipschitz continuous;}
\]
\[
\sigma \text{ and } b \text{ have at-most-linear growth.}
\]
Mild solution = weak solution (see lecture notes); proof requires a stochastic Fubini theorem, as in the linear case.

“Stopping-time arguments” can be used to replace the Lipschitz-continuity hypotheses on $\sigma$ and $b$ to the following:

- $\sigma$ and $b$ are locally Lipschitz continuous;

Theorem (Foondun–Parshad, 2013)

Suppose $b \geq 0$, $\inf u_0 > 0$, and $|\sigma(y)| \geq C|y|^{1+\delta}$ for all $y$. Then there is no solution that satisfies

$$\sup_{x \in \mathbb{R}} E(|u_t(x)|^2) \leq L \exp\{Lt\}$$

for all $t > 0$.

See lecture notes for proof.

Similarly for $b$, as we will see next (easier).
Mild solution = weak solution (see lecture notes); proof requires a stochastic Fubini theorem, as in the linear case.

“Stopping-time arguments” can be used to replace the Lipschitz-continuity hypotheses on $\sigma$ and $b$ to the following:

- $\sigma$ and $b$ are locally Lipschitz continuous;
- $\sigma$ and $b$ have at-most-linear growth.

Theorem (Foondun–Parshad, 2013)

Suppose $b \geq 0$, $\inf u_0 > 0$, and $|\sigma(y)| \geq C|y|^{1+\delta}$ for all $y$. Then there is no solution that satisfies $\sup_{x \in \mathbb{R}} E(|u_t(x)|^2) \leq L \exp\{Lt\}$ for all $t > 0$.

See lecture notes for proof.

Similarly for $b$, as we will see next (easier).
Mild solution = weak solution (see lecture notes); proof requires a stochastic Fubini theorem, as in the linear case.

“Stopping-time arguments” can be used to replace the Lipschitz-continuity hypotheses on $\sigma$ and $b$ to the following:

- $\sigma$ and $b$ are locally Lipschitz continuous;
- $\sigma$ and $b$ have at-most-linear growth.

At-most-linear growth cannot be improved upon. For example:

\[ \text{Theorem (Foondun–Parshad, 2013)} \]

Suppose $b \geq 0$, $\inf u_0 > 0$, and $|\sigma(y)| \geq C|y|^{1+\delta}$ for all $y$. Then there is no solution that satisfies \[ \sup_{x \in \mathbb{R}} E(|u_t(x)|^2) \leq L \exp\{Lt\} \text{ for all } t > 0. \]

See lecture notes for proof.

Similarly for $b$, as we will see next (easier).
Mild solution = weak solution (see lecture notes); proof requires a stochastic Fubini theorem, as in the linear case.

“Stopping-time arguments” can be used to replace the Lipschitz-continuity hypotheses on $\sigma$ and $b$ to the following:

- $\sigma$ and $b$ are locally Lipschitz continuous;
- $\sigma$ and $b$ have at-most-linear growth.

At-most-linear growth cannot be improved upon. For example:

**Theorem (Foondun–Parshad, 2013)**

Suppose $b \geq 0$, $\inf u_0 > 0$, and $|\sigma(y)| \geq C|y|^{1+\delta}$ for all $y$. Then there is no solution that satisfies

$$\sup_{x \in \mathbb{R}} \mathbb{E}(|u_t(x)|^2) \leq L \exp\{Lt\} \text{ for all } t > 0.$$
A Nonlinear Heat Equation (Lecture 5)
Addenda

- Mild solution = weak solution (see lecture notes); proof requires a stochastic Fubini theorem, as in the linear case.
- “Stopping-time arguments” can be used to replace the Lipschitz-continuity hypotheses on $\sigma$ and $b$ to the following:
  - $\sigma$ and $b$ are locally Lipschitz continuous;
  - $\sigma$ and $b$ have at-most-linear growth.
- At-most-linear growth cannot be improved upon. For example:

**Theorem (Foondun–Parshad, 2013)**

*Suppose $b \geq 0$, $\inf u_0 > 0$, and $|\sigma(y)| \geq C|y|^{1+\delta}$ for all $y$. Then there is no solution that satisfies $
\sup_{x \in \mathbb{R}} E(|u_t(x)|^2) \leq L \exp\{Lt\}$ for all $t > 0$.*

- See lecture notes for proof.
Mild solution = weak solution (see lecture notes); proof requires a stochastic Fubini theorem, as in the linear case.

“Stopping-time arguments” can be used to replace the Lipschitz-continuity hypotheses on $\sigma$ and $b$ to the following:

- $\sigma$ and $b$ are locally Lipschitz continuous;
- $\sigma$ and $b$ have at-most-linear growth.

At-most-linear growth cannot be improved upon. For example:

**Theorem (Foondun–Parshad, 2013)**

Suppose $b \geq 0$, $\inf u_0 > 0$, and $|\sigma(y)| \geq C|y|^{1+\delta}$ for all $y$. Then there is no solution that satisfies

$$\sup_{x \in \mathbb{R}} \mathbb{E}(|u_t(x)|^2) \leq L \exp\{Lt\} \text{ for all } t > 0.$$  

- See lecture notes for proof.
- Similarly for $b$, as we will see next (easier).
Goal [Foondun–Parshad]: If \(\ell := \inf u_0 > 0\) and 
\(q := \inf_y [b(y)/|y|^{1+\delta}] > 0\) for some \(\delta > 0\), then there is no “finite energy solution.”
Goal [Foondun–Parshad]: If \( \ell := \inf u_0 > 0 \) and
\( q := \inf_y [b(y)/|y|^{1+\delta}] > 0 \) for some \( \delta > 0 \), then there is no “finite energy solution.”

Note that

\[
E(|u_t(x)|) \geq E_{u_t}(x) \\
= (p_t * u_0)(x) + \int_0^t ds \int_{-\infty}^{\infty} dy \ p_{t-s}(y-x)E[b(u_s(y))]
\]
Goal [Foondun–Parshad]: If \( \ell := \inf u_0 > 0 \) and \( q := \inf_y [b(y)/|y|^{1+\delta}] > 0 \) for some \( \delta > 0 \), then there is no “finite energy solution.”

Note that
\[
E(\|u_t(x)\|) \geq E u_t(x) = (p_t \ast u_0)(x) + \int_0^t ds \int_{-\infty}^{\infty} dy \, p_{t-s}(y-x) E[b(u_s(y))]
\]
Goal [Foondun–Parshad]: If \( \ell := \inf u_0 > 0 \) and 
\( q := \inf_y [b(y)/|y|^{1+\delta}] > 0 \) for some \( \delta > 0 \), then there is no "finite energy solution."

Note that 
\[
E(|u_t(x)|) \geq E u_t(x) \\
= (p_t \ast u_0)(x) + \int_0^t ds \int_{-\infty}^{\infty} dy \ p_{t-s}(y-x) E [b(u_s(y))] \\
\geq \ell + q \int_0^t ds \int_{-\infty}^{\infty} dy \ p_{t-s}(y-x) E (|u_s(y)|^{1+\delta})
\]
Goal [Foondun–Parshad]: If $\ell := \inf u_0 > 0$ and $q := \inf_y [b(y)/|y|^{1+\delta}] > 0$ for some $\delta > 0$, then there is no “finite energy solution.”

Note that

$$E(|u_t(x)|) \geq E u_t(x)$$

$$= (p_t \ast u_0)(x) + \int_0^t ds \int_{-\infty}^{\infty} dy \, p_{t-s}(y-x)E[b(u_s(y))]$$

$$\geq \ell + q \int_0^t ds \int_{-\infty}^{\infty} dy \, p_{t-s}(y-x)E[|u_s(y)|^{1+\delta}]$$

$$\geq \ell + q \int_0^t ds \int_{-\infty}^{\infty} dy \, p_{t-s}(y-x)[E(|u_s(y)|)]^{1+\delta}.$$
Goal [Foondun–Parshad]: If $\ell := \inf u_0 > 0$ and $q := \inf_y [b(y)/|y|^{1+\delta}] > 0$ for some $\delta > 0$, then there is no “finite energy solution.”

Note that

$$E(|u_t(x)|) \geq E(u_t(x))$$

$$= (p_t * u_0)(x) + \int_0^t ds \int_{-\infty}^{\infty} dy \, p_{t-s}(y-x) E[b(u_s(y))]$$

$$\geq \ell + q \int_0^t ds \int_{-\infty}^{\infty} dy \, p_{t-s}(y-x) \left( E\left(|u_s(y)|^{1+\delta}\right) \right)$$

$$\geq \ell + q \int_0^t ds \int_{-\infty}^{\infty} dy \, p_{t-s}(y-x) \left[ E\left(|u_s(y)|\right)^{1+\delta} \right].$$

$\therefore J(t) := \inf_y E(|u_t(y)|)$ is a supersolution to the ODE $f'(t) = q[f(t)]^{1+\delta}$ s.t. $f(0) = \ell$. 
Goal [Foondun–Parshad]: If \( \ell := \inf u_0 > 0 \) and 
\( q := \inf_y [b(y)/|y|^{1+\delta}] > 0 \) for some \( \delta > 0 \), then there is no “finite energy solution.”

Note that

\[
E(|u_t(x)|) \geq E u_t(x)
\]

\[
= (p_t \ast u_0)(x) + \int_0^t ds \int_{-\infty}^{\infty} dy \ p_{t-s}(y-x)E[b(u_s(y))]
\]

\[
\geq \ell + q \int_0^t ds \int_{-\infty}^{\infty} dy \ p_{t-s}(y-x)E(|u_s(y)|^{1+\delta})
\]

\[
\geq \ell + q \int_0^t ds \int_{-\infty}^{\infty} dy \ p_{t-s}(y-x) [E(|u_s(y)|)]^{1+\delta}.
\]

\[
\therefore J(t) := \inf_y E(|u_t(y)|) \text{ is a supersolution to the ODE } f'(t) = q[f(t)]^{1+\delta} \text{ s.t. } f(0) = \ell.
\]
Goal [Foondun–Parshad]: If \( \ell := \inf u_0 > 0 \) and \( q := \inf_y [b(y)/|y|^{1+\delta}] > 0 \) for some \( \delta > 0 \), then there is no “finite energy solution.”

Note that

\[
\mathbb{E}(|u_t(x)|) \geq \mathbb{E}u_t(x)
\]

\[
= (p_t \ast u_0)(x) + \int_0^t ds \int_{-\infty}^{\infty} dy \ p_{t-s}(y-x) \mathbb{E}[b(u_s(y))]
\]

\[
\geq \ell + q \int_0^t ds \int_{-\infty}^{\infty} dy \ p_{t-s}(y-x) \mathbb{E}(|u_s(y)|^{1+\delta})
\]

\[
\geq \ell + q \int_0^t ds \int_{-\infty}^{\infty} dy \ p_{t-s}(y-x) \mathbb{E}(|u_s(y)|)^{1+\delta}.
\]

\[
J(t) := \inf_y \mathbb{E}(|u_t(y)|)
\]

is a supersolution to the ODE

\[
f'(t) = q[f(t)]^{1+\delta} \text{ s.t. } f(0) = \ell.
\]

Easy: \( f(t) = \infty \) for all \( t \geq (\delta q \ell^\delta)^{-1} \) \( \Rightarrow J(t) = \infty \) for all \( t \geq (\delta q \ell^\delta)^{-1} \).
Let \( \{X_t\}_{t \geq 0} := \text{BM with speed } \kappa > 0. \)

\[
\text{Var } X_t = \langle X \rangle_t = \kappa t
\]
Let $\{X_t\}_{t \geq 0} := \text{BM with speed } \kappa > 0$.
$\text{Var } X_t = \langle X \rangle_t = \kappa t$

$A(x) := |x|$ satisfies $A'(x) = \text{sgn}(x)$ and $A''(x) = 2\delta_0(x)$. 

$L_x t(X) := \int_{0}^{t} \delta_x (X_s) \, ds$ is a continuous random field.
Let \( \{X_t\}_{t \geq 0} \) := BM with speed \( \kappa > 0 \).
\[
\text{Var } X_t = \langle X \rangle_t = \kappa t
\]

\( A(x) := |x| \) satisfies \( A'(x) = \text{sgn}(x) \) and \( A''(x) = 2\delta_0(x) \).

Apply Itô’s formula—à la Tanaka (1962/1963)—to the function \( f(z) = |z - x| \) to see that

\[
|X_t - x| = |x| + \int_0^t \text{sgn}(X_s - x) \, dX_s + \kappa \int_0^t \delta_x(X_s) \, ds.
\]
Let \( \{X_t\}_{t \geq 0} := \text{BM with speed } \kappa > 0. \)
\[
\text{Var } X_t = \langle X \rangle_t = \kappa t
\]
\( A(x) := |x| \) satisfies \( A'(x) = \text{sgn}(x) \) and \( A''(x) = 2\delta_0(x). \)

Apply Itô’s formula—à la Tanaka (1962/1963)—to the function \( f(z) = |z - x| \) to see that
\[
|X_t - x| = |x| + \int_0^t \text{sgn}(X_s - x) \, dX_s + \kappa \int_0^t \delta_x(X_s) \, ds.
\]

\( L^x_t(X) := \int_0^t \delta_x(X_s) \, ds \) is a continuous random field. local times
Consider the following \textit{parabolic Anderson model}:
\[
\frac{\partial}{\partial t} u = \nu \frac{\partial^2}{2 \partial x^2} u + u \xi, \quad \text{(PAM)}
\]
subject to \( u_0(x) := 1 \).
Intermezzo: PAM (Lecture 6)

Consider the following *parabolic Anderson model*:

\[
\frac{\partial}{\partial t} u = \nu \frac{\partial^2}{\partial x^2} u + u \xi, \quad \text{(PAM)}
\]

subject to \( u_0(x) := 1 \).

The solution exists, is unique, and satisfies

\[
\sup_{x \in \mathbb{R}} E \left( |u_t(x)|^k \right) \leq L \exp \left\{ L k^3 t \right\} \quad \forall k \in [2, \infty), \ t \geq 0.
\]
Consider the following *parabolic Anderson model*:

\[ \frac{\partial}{\partial t} u = \nu \frac{\partial^2}{\partial x^2} u + u \xi, \quad (PAM) \]

subject to \( u_0(x) := 1 \).

The solution exists, is unique, and satisfies

\[ \sup_{x \in \mathbb{R}} E \left( |u_t(x)|^k \right) \leq L \exp \left\{ Lk^3 t \right\} \quad \forall k \in [2, \infty), \ t \geq 0. \]
Consider the following parabolic Anderson model:

$$\frac{\partial}{\partial t} u = \frac{\nu}{2} \frac{\partial^2}{\partial x^2} u + u \xi,$$

subject to $u_0(x) := 1$.

The solution exists, is unique, and satisfies

$$\sup_{x \in \mathbb{R}} E \left( |u_t(x)|^k \right) \leq L \exp \{ Lk^3 t \} \quad \forall k \in [2, \infty), \ t \geq 0.$$

Theorem (Bertini–Cancrini, 1995)

For all integers $k \geq 1$, and all reals $t \geq 0$ and $x \in \mathbb{R}$,

$$E \left( |u_t(x)|^k \right) \geq \exp \left( \frac{k(k^2 - 1)t}{24\nu} \right).$$
Because $\left(p_t \ast 1\right)(x) = 1$ and $u$ is mild, $u = 1 + (p \ast u)$, whence Feynman $u = 1 + (p \ast 1) + (p \ast 1 \ast 1) + \cdots$. 
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = (\nu/2)\partial_x^2 u + u\xi$, $u_0 \equiv 1$

- Because $(p_t \ast 1)(x) = 1$ and $u$ is mild, $u = 1 + (p \ast u)$, whence Feynman $u = 1 + (p \ast 1) + (p \ast 1 \ast 1) + \cdots$.
- We seek a different representation. Feynman paths.
Because \((pt * 1)(x) = 1\) and \(u\) is mild, \(u = 1 + (p \otimes u)\), whence Feynman \(u = 1 + (p \otimes 1) + (p \otimes 1 \otimes 1) + \cdots\).

We seek a different representation. Feynman paths

Let us first solve a related [bona fide] PDE with random forcing:

\[
\partial_t U_t(x) = \frac{\nu}{2} \partial_x^2 U_t(x) + U_t(x)G_t(x),
\]

where \(U_0 \equiv 1\) and \(G\) is a smooth function of \((t, x)\) and doesn’t grow too fast.
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = \left( \frac{\nu}{2} \right) \partial_x^2 u + u \xi$, $u_0 \equiv 1$

- Because $(p_t \ast 1)(x) = 1$ and $u$ is mild, $u = 1 + (p \ast u)$, whence Feynman $u = 1 + (p \ast 1) + (p \ast 1 \ast 1) + \cdots$.
- We seek a different representation. Feynman paths
- Let us first solve a related [bona fide] PDE with random forcing:
  $$\partial_t U_t(x) = \frac{\nu}{2} \partial_x^2 U_t(x) + U_t(x) G_t(x),$$

  where $U_0 \equiv 1$ and $G$ is a smooth function of $(t, x)$ and doesn’t grow too fast.
- By the Feynman–Kac formula,
  $$U_t(x) = \mathbb{E} \left[ \exp \left( \int_0^t G_{t-s}(B_s + x) \, ds \right) \right],$$

  where $\{G_t\}_{t \geq 0}$ is BM with speed $\nu$. 
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi, \ u_0 \equiv 1 \)

\[ \therefore \text{if } x_1, \ldots, x_k \in \mathbb{R}, \text{ then} \]

\[ \prod_{j=1}^{k} U_t(x_j) = \mathbb{E} \left( e^{\sum_{j=1}^{k} \Gamma^{(j)}} \right), \]

where \( \Gamma^{(j)} := \int_0^t G_{t-s}(B_s^{(j)} + x_j) \, ds \) for i.i.d. BMs \( B^{(1)}, \ldots, B^{(k)} \) of speed \( \nu \).
Intermezzo: PAM (Lecture 6)

PAM: $\frac{\partial}{\partial t} u = (\nu/2) \frac{\partial^2}{\partial x^2} u + u \xi$, $u_0 \equiv 1$

- \[ \therefore \text{if } x_1, \ldots, x_k \in \mathbb{R}, \text{ then} \]

\[ \prod_{j=1}^{k} U_t(x_j) = E \left( e^{\sum_{j=1}^{k} \Gamma^{(j)}} \right), \]

where $\Gamma^{(j)} := \int_0^t G_{t-s} (B_s^{(j)} + x_j) \, ds$ for i.i.d. BMs $B^{(1)}, \ldots, B^{(k)}$ of speed $\nu$.

- Intuitively, $\xi$ is a centered GRF with

\[ \text{Cov}(\xi_t(x), \xi_s(y)) = \delta_0(t-s)\delta_0(x-y). \]

[Formally integrate both sides w.r.t. $\varphi_t(x)\psi_s(y)dt \, dx \, ds \, dy$]


Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2) \partial^2_x u + u \xi, \ u_0 \equiv 1 \)

\[ \therefore \text{if } x_1, \ldots, x_k \in \mathbb{R}, \text{ then} \]

\[ \prod_{j=1}^{k} U_t(x_j) = \mathbb{E} \left( e^{\sum_{j=1}^{k} \Gamma(j)} \right), \]

where \( \Gamma(j) := \int_0^t G_{t-s}(B_s^{(j)} + x_j) \, ds \) for i.i.d. BMs \( B^{(1)}, \ldots, B^{(k)} \) of speed \( \nu \).

\[ \text{Intuitively, } \xi \text{ is a centered GRF with} \]

\[ \text{Cov}(\xi_t(x), \xi_s(y)) = \delta_0(t - s)\delta_0(x - y). \]

[Formally integrate both sides w.r.t. \( \varphi_t(x)\psi_s(y)dt \, dx \, ds \, dy \)]

\[ \text{Now consider the case where } G \text{ is a smooth approximation to } \xi: \text{ A GRF with } \mathbb{E}G_t(x) = 0 \text{ and} \]

\[ \text{Cov}(G_t(x), G_s(y)) = p_\varepsilon(s - t)p_\eta(x - y), \text{ where } \varepsilon, \eta \approx 0 \text{ are positive.} \]
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = (\nu/2)\partial_x^2 u + u\xi$, $u_0 \equiv 1$

$\therefore \quad \Gamma^{(j)} := \int_0^t G_s(B_s^{(j)} + x_j) \, ds$

$$E \left( \prod_{j=1}^{k} U_t(x_j) \right) = E \left( e^{\sum_{j=1}^{k} \Gamma^{(j)}} \right)$$
Intermezzo: PAM (Lecture 6)
PAM: $\partial_t u = (\nu/2) \partial_x^2 u + u \xi$, $u_0 \equiv 1$

$\therefore \quad [\Gamma^{(j)} := \int_0^t G_s (B_s^{(j)} + x_j) \, ds]$

$$E \left( \prod_{j=1}^k U_t(x_j) \right) = E \left( e^{\sum_{j=1}^k \Gamma^{(j)}} \right)$$
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = (\nu/2)\partial^2_x u + u\xi$, $u_0 \equiv 1$

$\therefore \quad \Gamma(j) := \int_0^t G_s(B_s^{(j)} + x_j) \, ds$

\[
E\left(\prod_{j=1}^k U_t(x_j)\right) = E\left(e^{\sum_{j=1}^k \Gamma(j)}\right) = E\left[ E\left(e^{\sum_{j=1}^k \Gamma(j)} \bigg| B(1), \ldots, B(k)\right)\right]
\]
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi, \ u_0 \equiv 1 \)

\[ \therefore \quad \left[ \Gamma^{(j)} := \int_0^t G_s(B^{(j)}_s + x_j) \, ds \right] \]

\[
\begin{align*}
E \left( \prod_{j=1}^k U_t(x_j) \right) &= E \left( e^{\sum_{j=1}^k \Gamma^{(j)}} \right) \\
&= E \left[ E \left( e^{\sum_{j=1}^k \Gamma^{(j)}} \mid B^{(1)}, \ldots, B^{(k)} \right) \right] \\
&= E \exp \left( \frac{1}{2} \text{Var} \left( \sum_{j=1}^k \Gamma^{(j)} \mid B^{(1)}, \ldots, B^{(k)} \right) \right)
\end{align*}
\]
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi, \ u_0 \equiv 1 \)

\[
\therefore \quad [\Gamma^{(j)} := \int_0^t G_s(B_s^{(j)} + x_j) \, ds]
\]

\[
E \left( \prod_{j=1}^k U_t(x_j) \right) = E \left( e^{\sum_{j=1}^k \Gamma^{(j)}} \right) = E \left[ E \left( e^{\sum_{j=1}^k \Gamma^{(j)}} \bigg| B^{(1)}, \ldots, B^{(k)} \right) \right] \\
= E \exp \left( \frac{1}{2} \text{Var} \left( \sum_{j=1}^k \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right) \right)
\]

\[
\text{Compute directly:}
\]

\[
\text{Var} \left( \sum_{j=1}^k \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right)
\]
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = (\nu/2)\partial_x^2 u + u \xi$, $u_0 \equiv 1$

- \[ [\Gamma^{(j)} := \int_0^t G_s (B_s^{(j)} + x_j) \, ds] \]

\[
E \left( \prod_{j=1}^{k} U_t (x_j) \right) = E \left( e^{\sum_{j=1}^{k} \Gamma^{(j)}} \right) = E \left[ E \left( e^{\sum_{j=1}^{k} \Gamma^{(j)}} \bigg| B^{(1)}, \ldots, B^{(k)} \right) \right] \\
= E \exp \left( \frac{1}{2} \text{Var} \left( \sum_{j=1}^{k} \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right) \right)
\]

- Compute directly:

\[
\text{Var} \left( \sum_{j=1}^{k} \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right)
\]
Intermezzo: PAM (Lecture 6)
PAM: $\partial_t u = (\nu/2) \partial^2_x u + u \xi$, $u_0 \equiv 1$

\[ \therefore \quad [\Gamma^{(j)} := \int_0^t G_s(B^{(j)}_s + x_j) \, ds] \]

\[ \begin{align*}
\mathbb{E} \left( \prod_{j=1}^k U_t(x_j) \right) &= \mathbb{E} \left( e^{\sum_{j=1}^k \Gamma^{(j)}} \right) = \mathbb{E} \left[ \mathbb{E} \left( e^{\sum_{j=1}^k \Gamma^{(j)}} \bigg| B^{(1)}, \ldots, B^{(k)} \right) \right] \\
&= \mathbb{E} \exp \left( \frac{1}{2} \text{Var} \left( \sum_{j=1}^k \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right) \right)
\end{align*} \]

\[ \text{Compute directly:} \]

\[ \begin{align*}
\text{Var} \left( \sum_{j=1}^k \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right) \\
&= \sum \int_0^t ds \int_0^t dr \, p_\varepsilon(s-t)p_\eta \left( B^{(j)}_s - B^{(i)}_r - x_i + x_j \right).
\end{align*} \]
PAM: $\partial_t u = (\nu/2) \partial_x^2 u + u \xi$, $u_0 \equiv 1$

- Set $x_1, \ldots, x_k := x \in \mathbb{R}$ to see that

\[
\text{Var} \left( \sum_{j=1}^{k} \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right) = \sum_{1 \leq i, j \leq k} \int_0^t \int_0^t ds \int_0^t dr \ p_\varepsilon(s - r) p_\eta \left( B_{s}^{(j)} - B_{r}^{(i)} \right)
\]
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2)\partial_x^2 u + u \xi, \ u_0 \equiv 1 \)

- Set \( x_1, \ldots, x_k : = x \in \mathbb{R} \) to see that

\[
\text{Var} \left( \sum_{j=1}^{k} \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right) = \sum_{1 \leq i, j \leq k} \int_0^t ds \int_0^t dr \ p_\varepsilon(s-r)p_\eta \left( B^{(j)}_s - B^{(i)}_r \right)
\]
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = (\nu/2) \partial_x^2 u + u \xi$, $u_0 \equiv 1$

- Set $x_1, \ldots, x_k := x \in \mathbb{R}$ to see that

$$
\text{Var} \left( \sum_{j=1}^{k} \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right)
= \sum_{1 \leq i, j \leq k} \int_0^t ds \int_0^t dr \ p_\varepsilon(s-r) p_\eta \left( B_s^{(j)} - B_r^{(i)} \right)
\approx tk p_\eta(0) + \sum_{1 \leq i \neq j \leq k} \int_0^t p_\eta \left( B_s^{(j)} - B_s^{(i)} \right) ds \quad [\varepsilon \approx 0]
$$
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2) \partial^2_x u + u \xi, \ u_0 \equiv 1 \)

Set \( x_1, \ldots, x_k := x \in \mathbb{R} \) to see that

\[
\text{Var} \left( \sum_{j=1}^{k} \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right) \\
= \sum_{1 \leq i,j \leq k} \int_0^t ds \int_0^t dr \ p_\varepsilon(s-r) p_\eta \left( B_s^{(j)} - B_r^{(i)} \right) \\
\approx tkp_\eta(0) + \sum_{1 \leq i \neq j \leq k} \int_0^t p_\eta \left( B_s^{(j)} - B_s^{(i)} \right) ds \quad [\varepsilon \approx 0] \\
\approx \frac{tk}{\sqrt{2\pi \eta \nu}} + \sum_{1 \leq i \neq j \leq k} L_t^0 \left( B^{(j)} - B^{(i)} \right) \quad [\eta \approx 0].
\]
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = (\nu/2) \partial_x^2 u + u \xi$, $u_0 \equiv 1$

- Set $x_1, \ldots, x_k := x \in \mathbb{R}$ to see that

\[
\text{Var} \left( \sum_{j=1}^{k} \Gamma^{(j)} \bigg| B^{(1)}, \ldots, B^{(k)} \right) = \sum_{1 \leq i, j \leq k} \int_0^t ds \int_0^t dr \ p_\varepsilon(s-r) p_\eta \left( B_s^{(j)} - B_r^{(i)} \right)
\]

\[
\approx tk p_\eta(0) + \sum_{1 \leq i \neq j \leq k} \int_0^t p_\eta \left( B_s^{(j)} - B_s^{(i)} \right) ds \quad [\varepsilon \approx 0]
\]

\[
\approx \frac{tk}{\sqrt{2\pi \eta \nu}} + \sum_{1 \leq i \neq j \leq k} L_t^0 \left( B^{(j)} - B^{(i)} \right) \quad [\eta \approx 0].
\]
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = (\nu/2) \partial_x^2 u + u \xi$, $u_0 \equiv 1$

▶ Combine to see that

$$E \left( |U_t(x)|^k \right) \approx E \exp \left( \frac{1}{2} \left\{ \frac{tk}{\sqrt{2\pi \eta \nu}} + \sum_{1 \leq i \neq j \leq k} L_t^0 \left( B^j - B^i \right) \right\} \right).$$
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = (\nu/2) \partial^2_x u + u \xi$, $u_0 \equiv 1$

- Combine to see that

$$E\left( |U_t(x)|^k \right) \approx E\exp\left( \frac{1}{2} \left\{ \frac{tk}{\sqrt{2\pi\eta\nu}} + \sum_{1 \leq i \neq j \leq k} L^0_t \left( B^{(j)} - B^{(i)} \right) \right\} \right).$$

- $X_{t}^{(i,j)} := B_{t}^{(j)} - B_{t}^{(i)}$ is a BM with speed $2\nu$. 
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = (\nu/2) \partial_x^2 u + u \xi$, $u_0 \equiv 1$

- Combine to see that

\[
\mathbb{E}\left(|U_t(x)|^k\right) \approx \mathbb{E}\exp\left(\frac{1}{2} \left\{ \frac{tk}{\sqrt{2\pi\eta\nu}} + \sum_{1 \leq i \neq j \leq k} L_t^0 \left( B^j - B^i \right) \right\} \right).
\]

- $X_{t}^{(i,j)} := B_{t}^{(j)} - B_{t}^{(i)}$ is a BM with speed $2\nu$.

- This and Brownian scaling together motivate [Itô vs Stratonovich]:
Intermezzo: PAM (Lecture 6)

PAM: \[ \partial_t u = (\nu/2)\partial^2_x u + u \xi, \quad u_0 \equiv 1 \]

- Combine to see that
  \[
  E \left( |U_t(x)|^k \right) \approx E \exp \left( \frac{1}{2} \left\{ \frac{tk}{\sqrt{2\pi \eta \nu}} + \sum_{1 \leq i \neq j \leq k} L^0_t \left( B^{(j)} - B^{(i)} \right) \right\} \right).
  \]

- \( X^{(i,j)}_t := B^{(j)}_t - B^{(i)}_t \) is a BM with speed 2\( \nu \).

- This and Brownian scaling together motivate [Itô vs Stratonovich]:

**Theorem (Bertini–Cancrini, 1995; Hu–Nualart, 2009; Conus, 2011)**

For all integers \( k \geq 2 \) and all reals \( t \geq 0 \) and \( x \in \mathbb{R} \),

\[
E \left( |u_t(x)|^k \right) = E \exp \left( \frac{1}{\nu} \sum_{1 \leq i < j \leq k} L^0_{\nu t} \left( b^{(j)} - b^{(i)} \right) \right),
\]

where the \( b^{(i)} \)'s are i.i.d. BMs with speed one.
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi, \ u_0 \equiv 1 \)

\[ \Rightarrow \quad E(\left| u_t(x) \right|^k) = E \exp\{\nu^{-1} \mathcal{L}_t\}, \text{ where} \]

\[ \mathcal{L}_t = \sum_{1 \leq i < j \leq k} L_{\nu t}^0 \left( b^{(j)} - b^{(i)} \right). \]
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = \left( \nu / 2 \right) \partial_x^2 u + u \xi, \ u_0 \equiv 1 \)

\( \triangleright \) \( \mathbb{E}(|u_t(x)|^k) = \mathbb{E} \exp\{\nu^{-1} \mathcal{L}_t\} \), where

\[
\mathcal{L}_t = \sum_{1 \leq i < j \leq k} L^0_{\nu t} \left( b^{(j)} - b^{(i)} \right).
\]

\( \triangleright \) Note: \( b^{(j)} - b^{(i)} \)'s are BMs with speed two. Therefore, by Tanaka's formula, \( \mathcal{L}_t \geq -\frac{1}{2} M_{\nu t} \), where

\[
M_t = \sum_{1 \leq i < j \leq t} W^{(i,j)}_t \quad \text{for} \quad W^{(i,j)}_t := \int_0^t \text{sgn} \left( b^{(j)}_s - b^{(i)}_s \right) d \left( b^{(j)} - b^{(i)} \right)_s
\]
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi, \ u_0 \equiv 1 \)

\[
E(|u_t(x)|^k) = E \exp\{\nu^{-1} \mathcal{L}_t\}, \text{ where}
\]
\[
\mathcal{L}_t = \sum_{1 \leq i < j \leq k} L_{\nu t}^0 \left( b^{(j)} - b^{(i)} \right).
\]

Note: \( b^{(j)} - b^{(i)} \)'s are BMs with speed two. Therefore, by Tanaka’s formula, \( \mathcal{L}_t \geq -\frac{1}{2} M_{\nu t} \), where

\[
M_t = \sum_{1 \leq i < j \leq t} W^{(i,j)}_t \quad \text{for} \quad W^{(i,j)}_t := \int_0^t \text{sgn} \left( b^{(j)}_s - b^{(i)}_s \right) d \left( b^{(j)} - b^{(i)} \right)_s
\]

But
\[
M_t = \frac{1}{2} \sum_{1 \leq i, j \leq k} W^{(i,j)}_t = \sum_{j=1}^k \int_0^t \sum_{i=1}^k \text{sgn} \left( b^{(j)}_s - b^{(i)}_s \right) db^{(j)}_s.
\]
**Intermezzo: PAM (Lecture 6)**

**PAM:** \( \partial_t u = (\nu/2) \partial_{xx}^2 u + u \xi, \ u_0 \equiv 1 \)

- \( E(|u_t(x)|^k) = E \exp\{\nu^{-1} L_t\} \), where

\[
L_t = \sum_{1 \leq i < j \leq k} L^0_{\nu t} \left( b^{(j)} - b^{(i)} \right).
\]

- Note: \( b^{(j)} - b^{(i)} \)'s are BMs with speed two. Therefore, by Tanaka's formula, \( L_t \geq -\frac{1}{2} M_{\nu t} \), where

\[
M_t = \sum_{1 \leq i < j \leq t} W_{t}^{(i,j)} \quad \text{for} \quad W_{t}^{(i,j)} := \int_0^t \text{sgn} \left( b_s^{(j)} - b_s^{(i)} \right) \, d \left( b^{(j)} - b^{(i)} \right)_s
\]

- But

\[
M_t = \frac{1}{2} \sum_{1 \leq i, j \leq k} W_{t}^{(i,j)} = \sum_{j=1}^k \int_0^t \sum_{i=1}^k \text{sgn} \left( b_s^{(j)} - b_s^{(i)} \right) \, db_s^{(j)}.
\]

- Thus, \( \langle M \rangle_t = \sum_{j=1}^k \int_0^t \left[ \sum_{i=1}^k \text{sgn} \left( b_s^{(j)} - b_s^{(i)} \right) \right]^2 \, ds \)
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi, \ u_0 \equiv 1 \)

- After a little bookkeeping, \( \langle M \rangle_t = k(k^2-1) \frac{k}{3} \times t \).
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi, \ u_0 \equiv 1 \)

- After a little bookkeeping, \( \langle M \rangle_t = \frac{k(k^2-1)}{3} \times t \).
- Therefore, \( M \) is a BM with speed \( \sigma^2 = \frac{k(k^2 - 1)}{3} \)  
  Lévy’s theorem
Intermezzo: PAM (Lecture 6)

PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi \), \( u_0 \equiv 1 \)

▶ After a little bookkeeping, \( \langle M \rangle_t = \frac{k(k^2 - 1)}{3} \times t \).

▶ Therefore, \( M \) is a BM with speed \( \sigma^2 = k(k^2 - 1)/3 \) Lévy’s theorem

▶ Combine to finish:
After a little bookkeeping, \( \langle M \rangle_t = \frac{k(k^2 - 1)}{3} \times t \).

Therefore, \( M \) is a BM with speed \( \sigma^2 = \frac{k(k^2 - 1)}{3} \) Lévy’s theorem

Combine to finish:

\[
E(|u_t(x)|^k) = E \exp\{\nu^{-1} \mathcal{L}_t\};
\]
PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi, \ u_0 \equiv 1 \)

After a little bookkeeping, \( \langle M \rangle_t = \frac{k(k^2-1)}{3} \times t. \)

Therefore, \( M \) is a BM with speed \( \sigma^2 = k(k^2 - 1)/3 \)

Combine to finish:

\[ E(|u_t(x)|^k) = E \exp\{\nu^{-1} \mathcal{L}_t\}; \]
\[ \mathcal{L}_t \geq \frac{1}{2} M_{\nu t}; \]

Lévy’s theorem
Intermezzo: PAM (Lecture 6)

PAM: $\partial_t u = (\nu/2) \partial_x^2 u + u \xi, \ u_0 \equiv 1$

- After a little bookkeeping, $\langle M \rangle_t = \frac{k(k^2-1)}{3} \times t$.
- Therefore, $M$ is a BM with speed $\sigma^2 = \frac{k(k^2 - 1)}{3}$ Lévy’s theorem

- Combine to finish:
  - $E(|u_t(x)|^k) = E \exp\{\nu^{-1} \mathcal{L}_t\}$;
  - $\mathcal{L}_t \geq \frac{1}{2} M_{\nu t}$;
  - $E(|u_t(x)|^k) \geq E \exp\{(2\nu)^{-1} M_{\nu t}\} = \exp\{\sigma^2 t/(8\nu)\}$. □
Let $\psi := \{\psi_\ell(z)\}_{\ell \geq 0, z \in \mathbb{Z}}$ be a [discrete] space- [discrete] time non negative random field and $z \mapsto \psi_\ell(z)$ is i.i.d. mean one $\forall \ell \geq 0$. 

If $2 \leq k \leq K$, then $\gamma(k) \leq \gamma(K)$, by Jensen's inequality. The issue is with strict inequalities.
Intermittency (Lecture 7)

Some motivation

- Let $\psi := \{\psi_\ell(z)\}_{\ell \geq 0, z \in \mathbb{Z}}$ be a [discrete] space- [discrete] time non negative random field and $z \mapsto \psi_\ell(z)$ is i.i.d. mean one $\forall \ell \geq 0$.

- Suppose the \textit{kth moment Lyapunov exponent} $\gamma(k) := \lim_{\ell \to \infty} \frac{1}{\ell} \log \mathbb{E} \left( [\psi_\ell(z)]^k \right)$ exists and is positive and finite.
Intermittency (Lecture 7)

Some motivation

Let $\psi := \{\psi_\ell(z)\}_{\ell \geq 0, z \in \mathbb{Z}}$ be a [discrete] space- [discrete] time non negative random field and $z \mapsto \psi_\ell(z)$ is i.i.d. mean one $\forall \ell \geq 0$.

Suppose the $kth$ moment Lyapunov exponent

$$\gamma(k) := \lim_{\ell \to \infty} \frac{1}{\ell} \log \mathbb{E} \left( [\psi_\ell(z)]^k \right)$$

exists and is positive and finite.
Intermittency (Lecture 7)

Some motivation

- Let $\psi := \{\psi_\ell(z)\}_{\ell \geq 0, z \in \mathbb{Z}}$ be a [discrete] space- [discrete] time non negative random field and $z \mapsto \psi_\ell(z)$ is i.i.d. mean one $\forall \ell \geq 0$.

- Suppose the $k$th moment Lyapunov exponent
  \[ \gamma(k) := \lim_{\ell \to \infty} \frac{1}{\ell} \log \mathbb{E}\left(\left[\psi_\ell(z)\right]^k\right) \]
  exists and is positive and finite.

- **Definition (Zel'dovich–Molchanov–Ruzmaikin–Sokoloff, 1988; Zel'dovich–Ruzmaikin–Sokoloff, 1990; Molchanov, 1991; Gibbon–Titi, 2005; . . . )**

  We say that $\psi$ is **intermittent** when $k \mapsto \frac{\gamma(k)}{k}$ is strictly increasing for $k \in [2, \infty)$.
Intermittency (Lecture 7)

Some motivation

- Let $\psi := \{\psi_\ell(z)\}_{\ell \geq 0, z \in \mathbb{Z}}$ be a [discrete] space-[discrete] time non negative random field and $z \mapsto \psi_\ell(z)$ is i.i.d. mean one $\forall \ell \geq 0$.

- Suppose the $k$th moment Lyapunov exponent

$$\gamma(k) := \lim_{\ell \to \infty} \frac{1}{\ell} \log \mathbb{E} \left( [\psi_\ell(z)]^k \right)$$

exists and is positive and finite.

- **Definition (Zel’dovich–Molchanov–Ruzmaikin–Sokoloff, 1988; Zel’dovich–Ruzmaikin–Sokoloff, 1990; Molchanov, 1991; Gibbon–Titi, 2005; ...)**

  We say that $\psi$ is intermittent when $k \mapsto \frac{\gamma(k)}{k}$ is strictly increasing for $k \in [2, \infty)$. 

- If $2 \leq k \leq K$, then $\frac{\gamma(k)}{k} \leq \frac{\gamma(K)}{K}$, by Jensen’s inequality. The issue is with strict inequalities.

  $$k^{-1} \log \mathbb{E}([\psi_\ell(z)]^k) \leq K^{-1} \log \mathbb{E}([\psi_\ell(z)]^K) \text{ when } k < K$$
Intermittency (Lecture 7)
Some motivation

- The following shows $\psi$ is intermittent, as long as $\gamma(2) > 0$.
  dynamo effect (Baxendale–Rozovskiï, 1993)
Intermittency (Lecture 7)
Some motivation

- The following shows $\psi$ is intermittent, as long as $\gamma(2) > 0$.
  dynamo effect (Baxendale–Rozovskiĭ, 1993)
Intermittency (Lecture 7)

Some motivation

- The following shows \( \psi \) is intermittent, as long as \( \gamma(2) > 0 \).
  
  dynamo effect (Baxendale–Rozovskii, 1993)

- Proposition (Carmona–Molchanov, 1994)

  The function \( k \mapsto k^{-1} \gamma(k) \) is well-defined and convex on \((0, \infty)\).
  Moreover, if \( \gamma(k_0) > 0 \) for some \( k_0 > 1 \), then \( k \mapsto k^{-1} \gamma(k) \) is strictly increasing on \([k_0, \infty)\).
Intermittency (Lecture 7)
Some motivation

- The following shows $\psi$ is intermittent, as long as $\gamma(2) > 0$.
  dynamo effect (Baxendale–Rozovskiï, 1993)

- **Proposition (Carmona–Molchanov, 1994)**
  
  The function $k \mapsto k^{-1}\gamma(k)$ is well-defined and convex on $(0, \infty)$. Moreover, if $\gamma(k_0) > 0$ for some $k_0 > 1$, then $k \mapsto k^{-1}\gamma(k)$ is strictly increasing on $[k_0, \infty)$.

  - Proof. Convexity follows from Jensen’s inequality. If $K > k \geq k_0 > 1$, then write $k = \alpha K + (1 - \alpha)$ for $\alpha := (k - 1)/(K - 1)$. 


Intermittency (Lecture 7)

Some motivation

- The following shows \( \psi \) is intermittent, as long as \( \gamma(2) > 0 \).

  dynamo effect (Baxendale–Rozovskiĭ, 1993)

- Proposition (Carmona–Molchanov, 1994)

  The function \( k \mapsto k^{-1} \gamma(k) \) is well-defined and convex on \((0, \infty)\).
  Moreover, if \( \gamma(k_0) > 0 \) for some \( k_0 > 1 \), then \( k \mapsto k^{-1} \gamma(k) \) is strictly increasing on \([k_0, \infty)\).

  Proof. Convexity follows from Jensen’s inequality. If \( K > k \geq k_0 > 1 \), then write \( k = \alpha K + (1 - \alpha) \) for \( \alpha := (k - 1)/(K - 1) \).

  Because \( \gamma(1) = 0 \), convexity yields

  \[
  \gamma(k) \leq \alpha \gamma(K) + (1 - \alpha) \gamma(1) = \frac{k - 1}{K - 1} \gamma(K).
  \]

  Rearrange, using the facts that: (i) \( \gamma(k) > 0 \) for all \( k \geq k_0 \); and (ii) \( (k - 1)/(K - 1) < k/K \).
Lemma (Paley–Zygmund, 1932)

Fix reals $n > m \geq 2$, and let $X \in L^n(\Omega)$ be non-negative with $P\{X > 0\} > 0$. Then $\forall \delta \in (0, 1)$,

$$P\{X \geq \delta \|X\|^m_m\} \geq (1 - \delta^{m/n})^{n/(n-m) \cdot [E(X^m)]^{n/(n-m)} [E(X^n)]^{m/(n-m)}}.$$ 

Proof. Apply Hölder’s inequality:

$$E(X^m) \leq \delta^m E(X^m) + E(X^m; X > \delta \|X\|^m_m) \leq \delta^m E(X^m) + (E(X^n))^{m/n} (P\{X > \delta \|X\|^m_m\})^{(n-m)/n}.$$ 

Solve to finish.
Lemma (Paley–Zygmund, 1932)

Fix reals $n > m \geq 2$, and let $X \in L^n(\Omega)$ be non negative with $P\{X > 0\} > 0$. Then $\forall \delta \in (0, 1)$,

$$P \{X \geq \delta \|X\|_m\} \geq (1 - \delta^m)^{n/(n-m)} \cdot \frac{[E(X^m)]^{n/(n-m)}}{[E(X^n)]^{m/(n-m)}}.$$
Lemma (Paley–Zygmund, 1932)

Fix reals $n > m \geq 2$, and let $X \in L^n(\Omega)$ be non negative with $P\{X > 0\} > 0$. Then $\forall \delta \in (0, 1)$,

$$P \{X \geq \delta \|X\|_m\} \geq (1 - \delta^m)^{n/(n-m)} \cdot \frac{[E(X^m)]^{n/(n-m)}}{[E(X^n)]^{m/(n-m)}}.$$  

Proof. Apply Hölder’s inequality:

$$E(X^m) \leq \delta^m E(X^m) + E(X^m; X > \delta \|X\|_m)$$
Lemma (Paley–Zygmund, 1932)

Fix reals \( n > m \geq 2 \), and let \( X \in L^n(\Omega) \) be non negative with \( \mathbb{P}\{X > 0\} > 0 \). Then \( \forall \delta \in (0, 1) \),

\[
\mathbb{P}\{X \geq \delta \|X\|_m\} \geq (1 - \delta^m)^{n/(n-m)} \cdot \frac{[\mathbb{E}(X^m)]^{n/(n-m)}}{[\mathbb{E}(X^n)]^{m/(n-m)}}.
\]

Proof. Apply Hölder’s inequality:

\[
\mathbb{E}(X^m) \leq \delta^m \mathbb{E}(X^m) + \mathbb{E}(X^m; X > \delta \|X\|_m)
\]
Lemma (Paley–Zygmund, 1932)

Fix reals \( n > m \geq 2 \), and let \( X \in L^n(\Omega) \) be non negative with \( \Pr\{X > 0\} > 0 \). Then \( \forall \delta \in (0, 1) \),

\[
\Pr\{X \geq \delta \|X\|_m\} \geq (1 - \delta^m)^{n/(n-m)} \cdot \frac{\left[\mathbb{E}(X^m)\right]^{n/(n-m)}}{\left[\mathbb{E}(X^n)\right]^{m/(n-m)}}.
\]

Proof. Apply Hölder’s inequality:

\[
\mathbb{E}(X^m) \leq \delta^m \mathbb{E}(X^m) + \mathbb{E}(X^m; X > \delta \|X\|_m)
\leq \delta^m \mathbb{E}(X^m) + \left(\mathbb{E}(X^n)\right)^{m/n} \left(\Pr\{X > \delta \|X\|_m\}\right)^{(n-m)/n}.
\]
Lemma (Paley–Zygmund, 1932)

Fix reals $n > m \geq 2$, and let $X \in L^n(\Omega)$ be non negative with $P\{X > 0\} > 0$. Then $\forall \delta \in (0, 1)$,

$$P\{X \geq \delta \|X\|_m\} \geq (1 - \delta^m)^{n/(n-m)} \cdot \frac{[E(X^m)]^{n/(n-m)}}{[E(X^n)]^{m/(n-m)}}.$$

Proof. Apply Hölder’s inequality:

$$E(X^m) \leq \delta^m E(X^m) + E(X^m; X > \delta \|X\|_m)$$

$$\leq \delta^m E(X^m) + (E(X^n))^{m/n} (P\{X > \delta \|X\|_m\})^{(n-m)/n}.$$

Solve to finish.
Lemma

∀ \in [2, \infty) and \in (0, 1),
\lim\inf_{\to\infty} \frac{1}{\log P\{\psi_{\gamma}(z) \geq \delta \|\psi_{\gamma}(0)\| \gamma\}} \geq -\inf_{n>m} (m\gamma(n) - n\gamma(m) \gamma(n) - m\gamma(n) + o(\ell)).

Proof. By Paley–Zygmund inequality, for all \n>m, 
P\{\psi_{\gamma}(z) \geq \delta \|\psi_{\gamma}(0)\| \gamma\} \geq (1 - \delta \gamma) \frac{n}{n - m} \cdot \frac{\mathbb{E}(\psi_{\gamma}(0))}{m} \gamma(m)} \frac{\mathbb{E}(\psi_{\gamma}(n))}{n - m} \gamma(n)} = \exp(\ell \cdot (m\gamma(n) - n\gamma(m) \gamma(n) - m\gamma(n) + o(\ell))).

Take logs etc.
Lemma

\[ \forall m \in [2, \infty) \text{ and } \delta \in (0, 1), \]
\[ \liminf_{\ell \to \infty} \frac{1}{\ell} \log P \{ \psi_\ell(z) \geq \delta \| \psi_\ell(0) \|_m \} \geq - \inf_{n > m} \left( \frac{m \gamma(n) - n \gamma(m)}{n - m} \right). \]
Lemma

∀m ∈ [2, ∞) and δ ∈ (0, 1),

\[ \liminf_{\ell \to \infty} \frac{1}{\ell} \log P \{ \psi_\ell(z) \geq \delta \| \psi_\ell(0) \|_m \} \geq - \inf_{n > m} \left( \frac{m \gamma(n) - n \gamma(m)}{n - m} \right). \]

Proof. By Paley–Zygmund inequality, for all n > m,

\[ P \{ \psi_\ell(z) \geq \delta \| \psi_\ell(0) \|_m \} \geq (1 - \delta^m)^{n/(n - m)} \cdot \frac{[E(\| \psi_\ell(0) \|^m)]^{n/(n - m)}}{[E(\| \psi_\ell(0) \|^n)]^{m/(n - m)}}. \]
Lemma

\[ \forall m \in [2, \infty) \text{ and } \delta \in (0, 1), \]
\[ \liminf_{\ell \to \infty} \frac{1}{\ell} \log P \{ \psi_\ell(z) \geq \delta \| \psi_\ell(0) \|_m \} \geq - \inf_{n > m} \left( \frac{m\gamma(n) - n\gamma(m)}{n - m} \right). \]

Proof. By Paley–Zygmund inequality, for all \( n > m, \)
\[ P \{ \psi_\ell(z) \geq \delta \| \psi_\ell(0) \|_m \} \geq (1 - \delta^m)^{n/(n-m)} \cdot \frac{[E([\psi_\ell(0)]^m)]^{n/(n-m)}}{[E([\psi_\ell(0)]^n)]^{m/(n-m)}}. \]
Lemma

∀m ∈ [2, ∞) and δ ∈ (0, 1),

\[ \liminf_{\ell \to \infty} \frac{1}{\ell} \log P \{ \psi_\ell(z) \geq \delta \| \psi_\ell(0) \|_m \} \geq - \inf_{n>m} \left( \frac{m \gamma(n) - n \gamma(m)}{n - m} \right). \]

Proof. By Paley–Zygmund inequality, for all \( n > m \),

\[ P \{ \psi_\ell(z) \geq \delta \| \psi_\ell(0) \|_m \} \geq (1 - \delta^m)^{n/(n-m)} \cdot \frac{[E([\psi_\ell(0)]^m)]^{n/(n-m)}}{[E([\psi_\ell(0)]^n)]^{m/(n-m)}} \]

\[ = \exp \left( \ell \cdot \frac{n \gamma(m) - m \gamma(n)}{n - m} + o(\ell) \right). \]
Lemma

\forall m \in [2, \infty) \text{ and } \delta \in (0, 1),

\liminf_{\ell \to \infty} \frac{1}{\ell} \log P \{ \psi_\ell(z) \geq \delta \| \psi_\ell(0) \|_m \} \geq - \inf_{n > m} \left( \frac{m \gamma(n) - n \gamma(m)}{n - m} \right).

Proof. By Paley–Zygmund inequality, for all \( n > m \),

\[ P \{ \psi_\ell(z) \geq \delta \| \psi_\ell(0) \|_m \} \geq (1 - \delta^m)^{n/(n-m)} \cdot \frac{[E([\psi_\ell(0)]^m)]^{n/(n-m)}}{[E([\psi_\ell(0)]^n)]^{m/(n-m)}} \]

\[ = \exp \left( \ell \cdot \frac{n \gamma(m) - m \gamma(n)}{n - m} + o(\ell) \right). \]

Take logs etc.
There is an easy corresponding upper bound too:

\[
P \left\{ \max_{1 \leq z \leq \exp(\theta N)} \psi_N(z) \geq \|\psi_N(0)\|_m \right\} \\
\leq \sum_{1 \leq z \leq \exp(\theta N)} P \left\{ \psi_N(0) \geq \|\psi_N(0)\|_m \right\}
\]
There is an easy corresponding upper bound too:

\[
P \left\{ \max_{1 \leq z \leq \exp(\theta N)} \psi_N(z) \geq \|\psi_N(0)\|_m \right\} \\
\leq \sum_{1 \leq z \leq \exp(\theta N)} P \{ \psi_N(0) \geq \|\psi_N(0)\|_m \}
\]
There is an easy corresponding upper bound too:

\[
P \left\{ \max_{1 \leq z \leq \exp(\theta N)} \psi_N(z) \geq \|\psi_N(0)\|_m \right\} \\
\leq \sum_{1 \leq z \leq \exp(\theta N)} P \{\psi_N(0) \geq \|\psi_N(0)\|_m\} \leq (1 + [e^{\theta N}]) \frac{E(\psi_N(0))}{\|\psi_N(0)\|_m}
\]
There is an easy corresponding upper bound too:

\[
P \left\{ \max_{1 \leq z \leq \exp(\theta N)} \psi_N(z) \geq \|\psi_N(0)\|_m \right\} \\
\leq \sum_{1 \leq z \leq \exp(\theta N)} P \{\psi_N(0) \geq \|\psi_N(0)\|_m\} \leq (1 + \lfloor e^{\theta N} \rfloor) \frac{E(\psi_N(0))}{\|\psi_N(0)\|_m} \\
= \exp \left( -N \left[ \frac{\gamma(m)}{m} - \theta \right] + o(N) \right).
\]
There is an easy corresponding upper bound too:

\[
P \left\{ \max_{1 \leq z \leq \exp(\theta N)} \psi_N(z) \geq \|\psi_N(0)\|_m \right\} \\
\leq \sum_{1 \leq z \leq \exp(\theta N)} P \left\{ \psi_N(0) \geq \|\psi_N(0)\|_m \right\} \leq (1 + \lceil e^{\theta N} \rceil) \frac{E(\psi_N(0))}{\|\psi_N(0)\|_m} \\
= \exp \left( -N \left[ \frac{\gamma(m)}{m} - \theta \right] + o(N) \right).
\]

Borel–Cantelli: \( \exists 0 < \theta_1 < \theta_2 < \cdots \) such that a.s. \( \forall i \),

\[
0 < \limsup_{N \to \infty} \frac{1}{N} \max_{1 \leq z \leq \exp(\theta_i N)} \log \psi_N(z) \\
< \liminf_{N \to \infty} \frac{1}{N} \max_{1 \leq z \leq \exp(\theta_{i+1} N)} \log \psi_N(z) < \infty.
\]
Consider the drift-free SHE $[b \equiv 0]$ \( \partial_t u = (\nu/2)\partial_x^2 u + \sigma(u)\xi \), \( u_0 \in L^\infty(\mathbb{R}) \) non random — all as before.
Consider the drift-free SHE \( b \equiv 0 \) \( \partial_t u = (\nu/2) \partial_x^2 u + \sigma(u) \xi \),
\( u_0 \in L^\infty(\mathbb{R}) \) non random — all as before.
Consider the drift-free SHE $[b \equiv 0]$ \[ \partial_t u = \left( \frac{\nu}{2} \right) \partial^2_x u + \sigma(u) \xi, \]
\[ u_0 \in L^\infty(\mathbb{R}) \text{ non random — all as before.} \]

Definition

The lower and upper Lyapunov exponents:

\[
\gamma_k(x) := \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left( |u_t(x)|^k \right),
\]
Consider the drift-free SHE \([b \equiv 0]\) \(\partial_t u = (\nu/2)\partial_x^2 u + \sigma(u)\xi\), \(u_0 \in L^\infty(\mathbb{R})\) non random — all as before.

Definition

The **lower** and **upper Lyapunov** exponents:

\[
\gamma_k(x) := \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left( |u_t(x)|^k \right),
\]

Fact. If \(\gamma_2(x) > 0\) \(\forall x\) then \(k \mapsto \gamma_k(x) - \gamma_{k-1}(x)\) is strictly increasing; same for upper L. exponents.
Consider the drift-free SHE \([b \equiv 0]\) \(\partial_t u = (\nu/2)\partial_x^2 u + \sigma(u)\xi\), \(u_0 \in L^\infty(\mathbb{R})\) non random — all as before.

**Definition**

The *lower* and *upper Lyapunov* exponents:

\[
\gamma_k(x) := \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left( |u_t(x)|^k \right),
\]

\[
\overline{\gamma}_k(x) := \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left( |u_t(x)|^k \right).
\]

**Fact.** If \(\gamma_2(x) > 0 \ \forall x\) then \(k \mapsto k^{-1}\gamma_k(x)\) is strictly increasing; same for upper L. exponents.
Theorem (Foondun–K, 2009; see also Döring–Savov, 2010)

If \( \inf |u_0| > 0 \) then

\[
\inf_{x \in \mathbb{R}} \gamma^2(x) \geq (4\nu)^{-1} \inf_{z \in \mathbb{R} \setminus \{0\}} \frac{|\sigma(z)|}{|z|} \geq c
\]

⇒ "weak intermittency."

Proof works by direct estimation, and a comparison argument:

\[
E(|u_t(x)|^2) = |(p_t^* u_0)(x)|^2 + \int_0^t ds \int_{\mathbb{R}} \int_{\mathbb{R}} [p_{t-s}(y-x)]^2 E(\sigma^2(u_s(y))) \geq \inf |u_0|^2 + c \int_0^t I(s) ds \int_{\mathbb{R}} \int_{\mathbb{R}} [p_{t-s}(y-x)]^2 = \inf |u_0|^2 + c' \int_0^t I(s) \sqrt{t-s} ds,
\]

where \( I(t) := \inf_{y \in \mathbb{R}} E(|u_s(y)|^2) \Rightarrow I(t) \geq c_1 + c_2 \int_0^t I(s) \sqrt{t-s} ds.\)
Theorem (Foondun–K, 2009; see also Döring–Savov, 2010)

If \( \inf |u_0| > 0 \) then \( \inf_{x \in \mathbb{R}} \gamma_2(x) \geq (4\nu)^{-1} \inf_{z \in \mathbb{R}\setminus\{0\}} \left| \frac{\sigma(z)}{z} \right|^4 \).

\( \inf |u_0| > 0 \) and \( \left| \frac{\sigma(z)}{z} \right| \geq c \) \( \Rightarrow \) “weak intermittency.”
**Theorem** (Foondun–K, 2009; see also Döring–Savov, 2010)

If \( \inf |u_0| > 0 \) then \( \inf_{x \in \mathbb{R}} \gamma_2(x) \geq (4\nu)^{-1} \inf_{z \in \mathbb{R} \setminus \{0\}} \left| \frac{\sigma(z)}{z} \right|^4 \).

\( \inf |u_0| > 0 \) and \( |\sigma(z)/z| \geq c \) \( \Rightarrow \) “weak intermittency.”

**Proof** works by direct estimation, and a comparison argument:

\[
E(|u_t(x)|^2) = |(p_t \ast u_0)(x)|^2 + \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_{t-s}(y-x)]^2 E(\sigma^2(u_s(y)))
\]
Intermittency (Lecture 7)

Back to SPDEs

- **Theorem (Foondun–K, 2009; see also Döring–Savov, 2010)**

  If \( \inf |u_0| > 0 \) then \( \inf_{x \in \mathbb{R}} \gamma_2(x) \geq (4 \nu)^{-1} \inf_{z \in \mathbb{R} \setminus \{0\}} \left| \frac{\sigma(z)}{z} \right|^4 \).

  \( \inf |u_0| > 0 \) and \( |\sigma(z)/z| \geq c \Rightarrow \) “weak intermittency.”

- Proof works by direct estimation, and a comparison argument:

  \[
  E(|u_t(x)|^2) = |(p_t * u_0)(x)|^2 + \int_0^t ds \int_{-\infty}^\infty dy \ [p_{t-s}(y - x)]^2 E(\sigma^2(u_s(y)))
  \]
Theorem (Foondun–K, 2009; see also Döring–Savov, 2010)

If \( \inf |u_0| > 0 \) then
\[
\inf_{x \in \mathbb{R}} \gamma_2(x) \geq (4\nu)^{-1} \inf_{z \in \mathbb{R} \setminus \{0\}} \left| \frac{\sigma(z)}{z} \right|^4.
\]

\( \inf |u_0| > 0 \) and \( |\sigma(z)/z| \geq c \) \( \Rightarrow \) “weak intermittency.”

Proof works by direct estimation, and a comparison argument:

\[
E(|u_t(x)|^2) = |(p_t * u_0)(x)|^2 + \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_{t-s}(y-x)]^2 E(\sigma^2(u_s(y)))
\]

\[
\geq \inf |u_0|^2 + c \int_0^t I(s) \ ds \int_{-\infty}^{\infty} [p_{t-s}(y-x)]^2.
\]
Theorem (Foondun–K, 2009; see also Döring–Savov, 2010)

If \( \inf |u_0| > 0 \) then
\[
\inf_{x \in \mathbb{R}} \gamma_2(x) \geq (4\nu)^{-1} \inf_{z \in \mathbb{R}\setminus\{0\}} \left| \frac{\sigma(z)}{z} \right|^4.
\]

\( \inf |u_0| > 0 \) and \( |\sigma(z)/z| \geq c \) \( \Rightarrow \) “weak intermittency.”

Proof works by direct estimation, and a comparison argument:

\[
E(|u_t(x)|^2) = |(p_t \ast u_0)(x)|^2 + \int_0^t ds \int_{-\infty}^{\infty} dy \left[ p_{t-s}(y-x) \right]^2 E(\sigma^2(u_s(y)))
\]

\[
\geq \inf |u_0|^2 + c \int_0^t I(s) \, ds \int_{-\infty}^{\infty} \left[ p_{t-s}(y-x) \right]^2
\]

\[
= \inf |u_0|^2 + c' \int_0^t \frac{I(s)}{\sqrt{t-s}} \, ds,
\]

where \( I(s) := \inf_{y \in \mathbb{R}} E(|u_s(y)|^2) \).
Theorem (Foondun–K, 2009; see also Döring–Savov, 2010)

If \( \inf |u_0| > 0 \) then \( \inf_{x \in \mathbb{R}} \gamma_2(x) \geq (4\nu)^{-1} \inf_{z \in \mathbb{R}\setminus\{0\}} \frac{\sigma(z)}{|z|}^4 \).

\( \inf |u_0| > 0 \) and \( |\sigma(z)/z| \geq c \Rightarrow \) “weak intermittency.”

Proof works by direct estimation, and a comparison argument:

\[
\mathbb{E}(\|u_t(x)\|^2) = |(p_t * u_0)(x)|^2 + \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_{t-s}(y-x)]^2 \mathbb{E}(\sigma^2(u_s(y))) \\
\geq \inf |u_0|^2 + c \int_0^t I(s) \ ds \int_{-\infty}^{\infty} [p_{t-s}(y-x)]^2 \\
= \inf |u_0|^2 + c' \int_0^t \frac{I(s)}{\sqrt{t-s}} \ ds,
\]

where \( I(s) := \inf_{y \in \mathbb{R}} \mathbb{E}(\|u_s(y)\|^2) \Rightarrow I(t) \geq c_1 + c_2 \int_0^t \frac{I(s)}{\sqrt{t-s}} \ ds. \)
\[ I(s) := \inf_{y \in \mathbb{R}} \mathbb{E}(|u_s(y)|^2) \Rightarrow I(t) \geq c_1 + c_2 \int_0^t \frac{I(s)}{\sqrt{t-s}} \, ds. \]
\[ I(s) := \inf_{y \in \mathbb{R}} E(|u_s(y)|^2) \Rightarrow I(t) \geq c_1 + c_2 \int_0^t \frac{I(s)}{\sqrt{t-s}} \, ds. \]
Intermittency (Lecture 7)
Back to SPDEs

\[ I(s) := \inf_{y \in \mathbb{R}} E(|u_s(y)|^2) \implies I(t) \geq c_1 + c_2 \int_0^t \frac{I(s)}{\sqrt{t-s}} \, ds. \]


\[ I(t) \geq f(t), \text{ where } f \text{ solves the renewal equation} \]

\[ f(t) = c_1 + c_2 \int_0^t \frac{f(s)}{\sqrt{t-s}} \, ds. \]
Intermittency (Lecture 7)
Back to SPDEs

\[ I(s) := \inf_{y \in \mathbb{R}} \mathbb{E}(|u_s(y)|^2) \Rightarrow I(t) \geq c_1 + c_2 \int_0^t \frac{I(s)}{\sqrt{t-s}} \, ds. \]


\[ I(t) \geq f(t), \text{ where } f \text{ solves the renewal equation} \]

\[ f(t) = c_1 + c_2 \int_0^t \frac{f(s)}{\sqrt{t-s}} \, ds. \]

**Apply the “key renewal theorem” to see that**

\[ I(t) \geq c_3 \exp\{c_4 t\}. \]
We just showed that if $\inf |u_0| > 0$ then a cone condition such as “$L_\sigma := \inf_z |\sigma(z)/z| > 0$” automatically ensures weak intermittency $[\gamma_2 > 0]$. 

What if $\inf |u_0| = 0$, say $u_0$ has compact support?

From now, consider a non-random initial function $u_0: \mathbb{R} \to \mathbb{R}$ that is measurable and bounded [as before], has compact support, and is strictly positive on an open subinterval of $(0, \infty)$.

We examine the drift-free $[b \equiv 0]$ stochastic heat equation,

$$\partial_t u = \left( \frac{\nu}{2} \right) \partial_x^2 u + \sigma(u) \xi.$$

$\sigma: \mathbb{R} \to \mathbb{R}$ Lipschitz and $\sigma(0) = 0$.

Synopsis of behavior. A kind of weak intermittency occurs. Roughly, tall peaks arise as $t \to \infty$, but the farthest peaks move roughly linearly with time away from the origin.
We just showed that if \( \inf |u_0| > 0 \) then a cone conditions such as “\( L_\sigma := \inf_z |\sigma(z)/z| > 0 \)” automatically ensures weak intermittency \([\gamma_2 > 0]\).

What if \( \inf |u_0| = 0 \), say \( u_0 \) has compact support?
We just showed that if $\inf |u_0| > 0$ then a cone conditions such as “$L_\sigma := \inf_z |\sigma(z)/z| > 0$” automatically ensures weak intermittency [$\gamma_2 > 0$].

What if $\inf |u_0| = 0$, say $u_0$ has compact support?

From now, consider a non-random initial function $u_0 : \mathbb{R} \to \mathbb{R}$ that is measurable and bounded [as before], has compact support, and is strictly positive on an open subinterval of $(0, \infty)$. 

Synopsis of behavior. A kind of weak intermittency occurs. Roughly, tall peaks arise as $t \to \infty$, but the farthest peaks move roughly linearly with time away from the origin.
We just showed that if $\inf |u_0| > 0$ then a cone conditions such as “$L_\sigma := \inf_z |\sigma(z)/z| > 0$” automatically ensures weak intermittency [$\gamma_2 > 0$].

What if $\inf |u_0| = 0$, say $u_0$ has compact support?

From now, consider a non-random initial function $u_0 : \mathbb{R} \to \mathbb{R}$ that is measurable and bounded [as before], has compact support, and is strictly positive on an open subinterval of $(0, \infty)$.

We examine the drift-free [$b \equiv 0$] stochastic heat equation, $\partial_t u = (\nu/2)\partial^2_x u + \sigma(u)\xi$. 

$\sigma : \mathbb{R} \to \mathbb{R}$ Lipschitz and $\sigma(0) = 0$. 

Synopsis of behavior. A kind of weak intermittency occurs. Roughly, tall peaks arise as $t \to \infty$, but the farthest peaks move roughly linearly with time away from the origin.
We just showed that if \( \inf |u_0| > 0 \) then a cone conditions such as “\( L_\sigma := \inf_z |\sigma(z)/z| > 0 \)” automatically ensures weak intermittency \( [\gamma_2 > 0] \).

What if \( \inf |u_0| = 0 \), say \( u_0 \) has compact support?

From now, consider a non-random initial function \( u_0 : \mathbb{R} \to \mathbb{R} \) that is measurable and bounded [as before], has compact support, and is strictly positive on an open subinterval of \( (0, \infty) \).

We examine the drift-free \( [b \equiv 0] \) stochastic heat equation,
\[
\partial_t u = (\nu/2)\partial_x^2 u + \sigma(u)\xi.
\]

\( \sigma : \mathbb{R} \to \mathbb{R} \) Lipschitz and \( \sigma(0) = 0 \).
We just showed that if \( \inf |u_0| > 0 \) then a cone conditions such as “\( L_\sigma := \inf_z |\sigma(z)/z| > 0 \)” automatically ensures weak intermittency \([\gamma_2 > 0]\).

What if \( \inf |u_0| = 0 \), say \( u_0 \) has compact support?

From now, consider a non-random initial function \( u_0 : \mathbb{R} \to \mathbb{R} \) that is measurable and bounded [as before], has compact support, and is strictly positive on an open subinterval of \((0, \infty)\).

We examine the drift-free \([b \equiv 0]\) stochastic heat equation, 
\[
\partial_t u = (\nu/2) \partial_x^2 u + \sigma(u) \xi.
\]

\( \sigma : \mathbb{R} \to \mathbb{R} \) Lipschitz and \( \sigma(0) = 0 \).

Synopsis of behavior. A kind of weak intermittency occurs. Roughly, tall peaks arise as \( t \to \infty \), but the farthest peaks move roughly linearly with time away from the origin.

Intermittency Fronts (Lecture 8)
Define, for all $\alpha > 0$, 

$$\mathcal{I}(\alpha) := \limsup_{t \to \infty} \frac{1}{t} \sup_{|x| > \alpha t} \log E \left( |u_t(x)|^2 \right).$$
Define, for all $\alpha > 0$,

$$
\mathcal{I}(\alpha) := \limsup_{t \to \infty} \frac{1}{t} \sup_{|x| > \alpha t} \log E \left( |u_t(x)|^2 \right).
$$

Think of $\alpha_L$ as an **intermittency lower front** when $\mathcal{I}(\alpha) < 0$ for all $\alpha > \alpha_L$. 
Define, for all $\alpha > 0$,

$$\mathcal{I}(\alpha) := \limsup_{t \to \infty} \frac{1}{t} \sup_{|x| > \alpha t} \log \mathbb{E} \left( |u_t(x)|^2 \right).$$

Think of $\alpha_L$ as an **intermittency lower front** when $\mathcal{I}(\alpha) < 0$ for all $\alpha > \alpha_L$.

And $\alpha_U$ as an **intermittency upper front** when $\mathcal{I}(\alpha) > 0$ for all $\alpha < \alpha_U$. 
Define, for all $\alpha > 0$,

$$\mathcal{I}(\alpha) := \limsup_{t \to \infty} \frac{1}{t} \sup_{|x| > \alpha t} \log E \left( |u_t(x)|^2 \right).$$

Think of $\alpha_L$ as an \underline{intermittency lower front} when $\mathcal{I}(\alpha) < 0$ for all $\alpha > \alpha_L$.

And $\alpha_U$ as an \underline{intermittency upper front} when $\mathcal{I}(\alpha) > 0$ for all $\alpha < \alpha_U$.

If there exists $\alpha_*$ that is both a lower front and an upper front then $\alpha_*$ is \underline{the intermittency front}. phase transition
Theorem (Conus–K, 2012)

Under the present conditions, the SHE has a nontrivial intermittency lower front. In fact, \( \text{S}(\alpha) < 0 \) if \( \alpha > \frac{1}{2} \text{Lip}\). If, in addition, \( \text{L}_{\text{\sigma}} := \inf_{z \neq 0} |\text{\sigma}(z)/z| > 0 \), then there exists \( \alpha_0 > 0 \) such that \( \text{S}(\alpha) > 0 \) if \( \alpha \in (0, \alpha_0) \).

Conus–K showed in fact that \( \alpha_0 \geq \frac{\text{L}_{\text{\sigma}}}{2\pi} \).

When \( \text{\sigma}(x) = Cx \text{PAM Lip} = L_{\text{\sigma}} = C \). In that case, the work of Conus–K implies that, if there were an intermittency front, then it would lie between \( \frac{C}{2} \) and \( \frac{C}{2\pi} \).

The existence of an intermittency front has been proved recently by Le Chen–Dalang; in fact, they proved that the intermittency front is at \( \frac{C}{2} \).

A closely-related result: Because \( \text{\sigma}(0) = 0 \) and \( u_0 \in L^2(\mathbb{R}) \), \( u_t \in L^2(\mathbb{R}) \) a.s. for all \( t > 0 \) [Dalang–Mueller, 2003].
Theorem (Conus–K, 2012)

Under the present conditions, the SHE has a nontrivial intermittency lower front. In fact, $\mathcal{I}(\alpha) < 0$ if $\alpha > \frac{1}{2} \text{Lip}^2$.

If, in addition, $L_\sigma := \inf_{z \neq 0} |\sigma(z)/z| > 0$, then there exists $\alpha_0 > 0$ such that $\mathcal{I}(\alpha) > 0$ if $\alpha \in (0, \alpha_0)$. 
Theorem (Conus–K, 2012)

Under the present conditions, the SHE has a nontrivial intermittency lower front. In fact, $\mathcal{J}(\alpha) < 0$ if $\alpha > \frac{1}{2} \text{Lip}_\sigma^2$. If, in addition, $L_\sigma := \inf_{z \neq 0} |\sigma(z)/z| > 0$, then there exists $\alpha_0 > 0$ such that $\mathcal{J}(\alpha) > 0$ if $\alpha \in (0, \alpha_0)$.

Conus–K showed in fact that $\alpha_0 \geq \frac{L_\sigma^2}{2\pi}$.
Theorem (Conus–K, 2012)

Under the present conditions, the SHE has a nontrivial intermittency lower front. In fact, $\mathcal{I}(\alpha) < 0$ if $\alpha > \frac{1}{2} \text{Lip}_\sigma^2$. If, in addition, $L_\sigma := \inf_{z \neq 0} |\sigma(z)/z| > 0$, then there exists $\alpha_0 > 0$ such that $\mathcal{I}(\alpha) > 0$ if $\alpha \in (0, \alpha_0)$.

Conus–K showed in fact that $\alpha_0 \geq L_\sigma^2/(2\pi)$.

When $\sigma(x) = C x$ PAM $\text{Lip}_\sigma = L_\sigma = C$. In that case, the work of Conus–K implies that, if there were an intermittency front, then it would lie between $C^2/2\pi$ and $C^2/2$. 
Theorem (Conus–K, 2012)

Under the present conditions, the SHE has a nontrivial intermittency lower front. In fact, \( \mathcal{I}(\alpha) < 0 \) if \( \alpha > \frac{1}{2} \text{Lip}_\sigma^2 \). If, in addition, \( L_\sigma := \inf_{z \neq 0} |\sigma(z)/z| > 0 \), then there exists \( \alpha_0 > 0 \) such that \( \mathcal{I}(\alpha) > 0 \) if \( \alpha \in (0, \alpha_0) \).

- Conus–K showed in fact that \( \alpha_0 \geq \frac{L_\sigma^2}{2\pi} \).
- When \( \sigma(x) = Cx \), PAM \( \text{Lip}_\sigma = L_\sigma = C \). In that case, the work of Conus–K implies that, if there were an intermittency front, then it would lie between \( C^2 / 2\pi \) and \( C^2 / 2 \).
- The existence of an intermittency front has been proved recently by Le Chen–Dalang; in fact, they proved that the intermittency front is at \( C^2 / 2 \).
Theorem (Conus–K, 2012)

Under the present conditions, the SHE has a nontrivial intermittency lower front. In fact, $\mathcal{I}(\alpha) < 0$ if $\alpha > \frac{1}{2} \text{Lip}_\sigma^2$. If, in addition, $L_\sigma := \inf_{z \neq 0} |\sigma(z)/z| > 0$, then there exists $\alpha_0 > 0$ such that $\mathcal{I}(\alpha) > 0$ if $\alpha \in (0, \alpha_0)$.

- Conus–K showed in fact that $\alpha_0 \geq L_\sigma^2/(2\pi)$.
- When $\sigma(x) = Cx$ PAM $\text{Lip}_\sigma = L_\sigma = C$. In that case, the work of Conus–K implies that, if there were an intermittency front, then it would lie between $C^2/2\pi$ and $C^2/2$.
- The existence of an intermittency front has been proved recently by Le Chen–Dalang; in fact, they proved that the intermittency front is at $C^2/2$.
- A closely-related result: Because $\sigma(0) = 0$ and $u_0 \in L^2(\mathbb{R})$, $u_t \in L^2(\mathbb{R})$ a.s. for all $t > 0$ [Dalang–Mueller, 2003].
Intermittency Fronts (Lecture 8)

Sketch of proof

\[ N_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} \mathbb{E}(|\Phi_t(x)|^2) \right]^{1/2}. \]
Intermittency Fronts (Lecture 8)

Sketch of proof

\[ \mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} \mathbb{E} (|\Phi_t(x)|^2) \right]^{1/2}. \]

\[ \text{Warning.} \quad \mathcal{N}_{\beta,0}(\Phi) \text{ is what we used to write as } \mathcal{N}_{\beta/2,2}(\Phi). \]
Intermittency Fronts (Lecture 8)

Sketch of proof

\[ \mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} \mathbb{E} \left( |\Phi_t(x)|^2 \right) \right]^{1/2}. \]

- Warning. \( \mathcal{N}_{\beta,0}(\Phi) \) is what we used to write as \( \mathcal{N}_{\beta/2,2}(\Phi) \).
- The following uses similar ideas as the stochastic Young inequality of yore.
Sketch of proof

\[ \mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} \mathbb{E} \left( |\Phi_t(x)|^2 \right) \right]^{1/2}. \]

- Warning. \( \mathcal{N}_{\beta,0}(\Phi) \) is what we used to write as \( \mathcal{N}_{\beta/2,2}(\Phi) \).
- The following uses similar ideas as the stochastic Young inequality of yore.
Intermittency Fronts (Lecture 8)

Sketch of proof

- $\mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + c x} \mathbb{E} \left( |\Phi_t(x)|^2 \right) \right]^{1/2}$.

- Warning. $\mathcal{N}_{\beta,0}(\Phi)$ is what we used to write as $\mathcal{N}_{\beta/2,2}(\Phi)$.

- The following uses similar ideas as the stochastic Young inequality of yore.

- **Proposition**

  For all $c \in \mathbb{R}$, $\beta > c^2 \nu / 4$, and $\Phi \in \mathcal{L}^{\beta,2}$,

  $$\mathcal{N}_{\beta,c}(\rho \otimes \Phi) \leq \frac{\mathcal{N}_{\beta,c}(\Phi)}{(\nu(4\beta - c^2 \nu))^{1/4}}.$$
Intermittency Fronts (Lecture 8)

Sketch of proof $\mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} \mathbb{E} \left( |\Phi_t(x)|^2 \right) \right]^{1/2}$

$\mathcal{N}_{\beta,c}(p \ast \Phi) \leq \frac{\mathcal{N}_{\beta,c}(\Phi)}{(\nu(4\beta - c^2\nu))^{1/4}}$. 
Sketch of proof $N_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} E \left( |\Phi_t(x)|^2 \right) \right]^{1/2}$

$\triangleright \ N_{\beta,c}(p \ast \Phi) \leq \frac{N_{\beta,c}(\Phi)}{(\nu(4\beta - c^2\nu))^{1/4}}.$

$\triangleright$ Recall. $u_t^{(n+1)}(x) = (p_t * u_0)(x) + (p \ast \sigma(u^{(n)}))_t(x).$
Intermittency Fronts (Lecture 8)

Sketch of proof $\mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + c x} \mathbb{E} (|\Phi_t(x)|^2) \right]^{1/2}$

$\mathcal{N}_{\beta,c}(p \otimes \Phi) \leq \frac{\mathcal{N}_{\beta,c}(\Phi)}{(\nu(4\beta - c^2\nu))^{1/4}}$.

Recall. $u^{(n+1)}_t(x) = (p_t * u_0)(x) + (p \otimes \sigma(u^{(n)}))(x)$.

Because $\sigma(0) = 0$, $|\sigma(u^{(n)}_t(x))| \leq \text{Lip}_\sigma |u^{(n)}_t(x)| \Rightarrow$

$\mathcal{N}_{\beta,c}(u^{(n+1)}) \leq \mathcal{N}_{\beta,c}(p \otimes u_0) + \frac{\text{Lip}_\sigma}{(\nu(4\beta - c^2\nu))^{1/4}} \mathcal{N}_{\beta,c}(u^{(n)})$. 
Intermittency Fronts (Lecture 8)

Sketch of proof $N_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} E \left( |\Phi_t(x)|^2 \right) \right]^{1/2}$

- $N_{\beta,c}(p \otimes \Phi) \leq \frac{N_{\beta,c}(\Phi)}{(\nu(4\beta - c^2\nu))^{1/4}}$.

- Recall. $u_t^{(n+1)}(x) = (p_t * u_0)(x) + (p \otimes \sigma(u^{(n)}))_t(x)$.

- Because $\sigma(0) = 0$, $|\sigma(u_t^{(n)}(x))| \leq \text{Lip}_\sigma |u_t^{(n)}(x)| \Rightarrow$

  $N_{\beta,c} \left( u^{(n+1)} \right) \leq N_{\beta,c} \left( p \otimes u_0 \right) + \frac{\text{Lip}_\sigma}{(\nu(4\beta - c^2\nu))^{1/4}} N_{\beta,c} \left( u^{(n)} \right)$.

- Also,

  $e^{-\beta t + cx} (p_t * |u_0|)(x) = e^{-\beta t} \int_{-\infty}^{\infty} p_t(y - x) e^{-c(y-x)} e^{cy} |u_0(y)| \, dy$.
Intermittency Fronts (Lecture 8)

Sketch of proof $N_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} \mathbb{E} \left( |\Phi_t(x)|^2 \right) \right]^{1/2}$

- $N_{\beta,c}(p \otimes \Phi) \leq \frac{N_{\beta,c}(\Phi)}{\left( \nu(4\beta - c^2\nu) \right)^{1/4}}$.

- Recall. $u^{(n+1)}_t(x) = (p_t \ast u_0)(x) + (p \otimes \sigma(u^{(n)}))_t(x)$.

- Because $\sigma(0) = 0$, $|\sigma(u^{(n)}_t(x))| \leq \text{Lip}_\sigma |u^{(n)}_t(x)| \Rightarrow$

  $N_{\beta,c} \left( u^{(n+1)} \right) \leq N_{\beta,c}(p \ast u_0) + \frac{\text{Lip}_\sigma}{\left( \nu(4\beta - c^2\nu) \right)^{1/4}} N_{\beta,c} \left( u^{(n)} \right)$.

- Also,

  $$e^{-\beta t + cx}(p_t \ast |u_0|)(x) = e^{-\beta t} \int_{-\infty}^{\infty} p_t(y - x) e^{-c(y-x)} e^{cy} |u_0(y)| \, dy$$
Intermittency Fronts (Lecture 8)

Sketch of proof $\mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} \mathbb{E} \left( |\Phi_t(x)|^2 \right) \right]^{1/2}$

1. $\mathcal{N}_{\beta,c}(p \circ \Phi) \leq \frac{\mathcal{N}_{\beta,c}(\Phi)}{(\nu(4\beta - c^2 \nu))^{1/4}}$.

2. Recall. $u_{t}^{(n+1)}(x) = (p_t \ast u_0)(x) + (p \circ \sigma(u^{(n)}))_t(x)$.

3. Because $\sigma(0) = 0$, $|\sigma(u_{t}^{(n)}(x))| \leq \text{Lip}_\sigma |u_{t}^{(n)}(x)|$ \Rightarrow

$$\mathcal{N}_{\beta,c} \left( u^{(n+1)} \right) \leq \mathcal{N}_{\beta,c} \left( p \ast u_0 \right) + \frac{\text{Lip}_\sigma}{(\nu(4\beta - c^2 \nu))^{1/4}} \mathcal{N}_{\beta,c} \left( u^{(n)} \right).$$

4. Also,

$$e^{-\beta t + cx} (p_t \ast |u_0|)(x) = e^{-\beta t} \int_{-\infty}^{\infty} p_t(y - x)e^{-c(y-x)}e^{cy}|u_0(y)| \, dy$$

$$\leq \mathcal{N}_{0,c}(u_0) \exp \left( -t \left[ \beta - \frac{c^2 \nu}{2} \right] \right).$$

5. Apply this with $\beta := c^2 \nu / 2$ to see that

$$\mathcal{N}_{c^2 \nu / 2,c} \left( u^{(n+1)} \right) \leq \mathcal{N}_{0,c}(u_0) + \frac{\text{Lip}_\sigma}{\sqrt{|c| \nu}} \mathcal{N}_{c^2 \nu / 2,c} \left( u^{(n)} \right).$$
Intermittency Fronts (Lecture 8)

Sketch of proof \( \mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} [e^{-\beta t + cx} \mathbb{E}(|\Phi_t(x)|^2)]^{1/2} \)

\[\nabla \quad \mathcal{N}_{c^{2\nu/2},c} \left( u^{(n+1)} \right) \leq \mathcal{N}_{0,c}(u_0) + \frac{\text{Lip}_\sigma}{\sqrt{|c| \nu}} \mathcal{N}_{c^{2\nu/2},c} \left( u^{(n)} \right). \]
Sketch of proof $\mathcal{N}_{\beta, c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} \mathbb{E} \left( |\Phi_t(x)|^2 \right) \right]^{1/2}$

- $\mathcal{N}_{c^{2\nu/2}, c} \left( u^{(n+1)} \right) \leq \mathcal{N}_{0, c}(u_0) + \frac{\text{Lip}_\sigma}{\sqrt{|c|\nu}} \mathcal{N}_{c^{2\nu/2}, c} \left( u^{(n)} \right)$.

- $u_0$ has compact support
  $\Rightarrow \mathcal{N}_{0, c}(u_0) + \mathcal{N}_{0, -c}(u_0) < \infty \ \forall c \in \mathbb{R}$. 

Therefore, $\sup_{|x| > \alpha t} \mathbb{E} \left( |u_t(x)|^2 \right) \leq \text{const} \cdot \exp \left( -c |x| + c^{2\nu} t^2 \right)$.

The exponent is $< 0$ iff $\alpha > c\nu^{1/2}$ for any $c > \text{Lip}_\sigma^{2\nu}/\nu$.

$\therefore S(\alpha) < 0$ if $\alpha > \text{Lip}_\sigma^{2\nu}/\nu$. 1/2 of the thm
Intermittency Fronts (Lecture 8)

Sketch of proof $\mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} \mathbb{E} \left( |\Phi_t(x)|^2 \right) \right]^{1/2}$

- $\mathcal{N}_{c^{2\nu}/2,c} \left( u^{(n+1)} \right) \leq \mathcal{N}_{0,c}(u_0) + \frac{\text{Lip}_\sigma}{\sqrt{|c|\nu}} \mathcal{N}_{c^{2\nu}/2,c} \left( u^{(n)} \right)$.

- $u_0$ has compact support
  $\Rightarrow \mathcal{N}_{0,c}(u_0) + \mathcal{N}_{0,-c}(u_0) < \infty \ \forall c \in \mathbb{R}$.

- Iteration yields $\mathcal{N}_{c^{2\nu}/2,c}(u) < \infty$ when $\text{Lip}_\sigma/\sqrt{|c|\nu} < 1$. 
Intermittency Fronts (Lecture 8)

Sketch of proof \( \mathcal{N}_{\beta, c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} \mathbb{E} \left( |\Phi_t(x)|^2 \right) \right]^{1/2} \)

- \( \mathcal{N}_{c^{2\nu/2}, c} \left( u^{(n+1)} \right) \leq \mathcal{N}_{0, c}(u_0) + \frac{\text{Lip}_\sigma}{\sqrt{|c|\nu}} \mathcal{N}_{c^{2\nu/2}, c} \left( u^{(n)} \right) \).
- \( u_0 \) has compact support
  \[ \Rightarrow \mathcal{N}_{0, c}(u_0) + \mathcal{N}_{0, -c}(u_0) < \infty \quad \forall c \in \mathbb{R}. \]
- Iteration yields \( \mathcal{N}_{c^{2\nu/2}, c}(u) < \infty \) when \( \text{Lip}_\sigma / \sqrt{|c|\nu} < 1. \)
- Equivalently, if \( c > \text{Lip}_\sigma^2 / \nu \), then
  \[ \mathbb{E} \left( |u_t(x)|^2 \right) \leq \text{const} \cdot \exp \left( -c|x| + \frac{c^2 \nu t}{2} \right). \]
Intermittency Fronts (Lecture 8)

Sketch of proof \( \mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + cx} E \left( |\Phi_t(x)|^2 \right) \right]^{1/2} \)

- \( \mathcal{N}_{c^{2\nu/2},c} \left( u^{(n+1)} \right) \leq \mathcal{N}_{0,c}(u_0) + \frac{\text{Lip}_\sigma}{\sqrt{|c|\nu}} \mathcal{N}_{c^{2\nu/2},c} \left( u^{(n)} \right) \).

- \( u_0 \) has compact support
  \[ \Rightarrow \mathcal{N}_{0,c}(u_0) + \mathcal{N}_{0,-c}(u_0) < \infty \ \forall c \in \mathbb{R} . \]

- Iteration yields \( \mathcal{N}_{c^{2\nu/2},c}(u) < \infty \) when \( \text{Lip}_\sigma / \sqrt{|c|\nu} < 1 \).

- Equivalently, if \( c > \text{Lip}_\sigma^2 / \nu \), then
  \[ E \left( |u_t(x)|^2 \right) \leq \text{const} \cdot \exp \left( -c|x| + \frac{c^2 \nu t}{2} \right) . \]

- Therefore, \( \sup_{|x| > \alpha t} E \left( |u_t(x)|^2 \right) \leq \text{const} \cdot \exp \left( -c\alpha t + \frac{c^2 \nu t}{2} \right) . \)
Intermittency Fronts (Lecture 8)

Sketch of proof $\mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + c x} E \left( |\Phi_t(x)|^2 \right) \right]^{1/2}$

- $\mathcal{N}_{c^{2\nu/2},c} \left( u^{(n+1)} \right) \leq \mathcal{N}_{0,c} \left( u_0 \right) + \frac{\text{Lip}_\sigma}{\sqrt{|c|\nu}} \mathcal{N}_{c^{2\nu/2},c} \left( u^{(n)} \right)$.

- $u_0$ has compact support
  $\Rightarrow \mathcal{N}_{0,c} \left( u_0 \right) + \mathcal{N}_{0,-c} \left( u_0 \right) < \infty \ \forall \ c \in \mathbb{R}$.

- Iteration yields $\mathcal{N}_{c^{2\nu/2},c} \left( u \right) < \infty$ when $\text{Lip}_\sigma / \sqrt{|c|\nu} < 1$.

- Equivalently, if $c > \text{Lip}_\sigma^2 / \nu$, then

  $E \left( |u_t(x)|^2 \right) \leq \text{const} \cdot \exp \left( -c|x| + \frac{c^2\nu t}{2} \right)$.

- Therefore, $\sup_{|x| > \alpha t} E \left( |u_t(x)|^2 \right) \leq \text{const} \cdot \exp \left( -c\alpha t + \frac{c^2\nu t}{2} \right)$.

- The exponent is $< 0$ iff $\alpha > c\nu / 2$ for any $c > \text{Lip}_\sigma^2 / \nu$. 
Intermittency Fronts (Lecture 8)

Sketch of proof \( \mathcal{N}_{\beta,c}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}} \left[ e^{-\beta t + c x} E \left( |\Phi_t(x)|^2 \right) \right]^{1/2} \)

\[ \begin{align*}
&\quad \mathcal{N}_{c^{2}\nu/2,c} \left( u^{(n+1)} \right) \leq \mathcal{N}_{0,c}(u_0) + \frac{\text{Lip}_\sigma}{\sqrt{|c|\nu}} \mathcal{N}_{c^{2}\nu/2,c} \left( u^{(n)} \right). \\
&\quad u_0 \text{ has compact support} \\
&\quad \Rightarrow \mathcal{N}_{0,c}(u_0) + \mathcal{N}_{0,-c}(u_0) < \infty \quad \forall c \in \mathbb{R}. \\
&\quad \text{Iteration yields} \mathcal{N}_{c^{2}\nu/2,c}(u) < \infty \quad \text{when} \quad \text{Lip}_\sigma/\sqrt{|c|\nu} < 1. \\
&\quad \text{Equivalently, if} \quad c > \text{Lip}_\sigma^2/\nu, \text{ then} \\
&\quad \quad E \left( |u_t(x)|^2 \right) \leq \text{const} \cdot \exp \left( -c |x| + \frac{c^2 \nu t}{2} \right). \\
&\quad \quad \text{Therefore,} \quad \sup_{|x| > \alpha t} E \left( |u_t(x)|^2 \right) \leq \text{const} \cdot \exp \left( -c\alpha t + \frac{c^2 \nu t}{2} \right). \\
&\quad \quad \text{The exponent is} < 0 \quad \text{iff} \quad \alpha > c\nu/2 \quad \text{for any} \quad c > \text{Lip}_\sigma^2/\nu. \\
&\quad \quad \therefore \mathcal{I}(\alpha) < 0 \quad \text{if} \quad \alpha > \text{Lip}_\sigma^2/2. \quad 1/2 \text{ of the thm} \quad \square
\]
Recall. \( u_t(x) = (p_t * u_0)(x) + (p \circ \sigma(u))_t(x) \).
Recall. \( u_t(x) = (p_t * u_0)(x) + (p \otimes \sigma(u))_t(x) \).

Therefore, by the Walsh isometry,

\[
E(|u_t(x)|^2) = |(p_t * u_0)(x)|^2 + \int_0^t ds \int_{-\infty}^{\infty} dy \ [p_{t-s}(y-x)]^2 E[\sigma^2(u_s(y))] \]
Recall. \( u_t(x) = (p_t \ast u_0)(x) + (p \otimes \sigma(u))_t(x) \).

Therefore, by the Walsh isometry,

\[
E \left( |u_t(x)|^2 \right) = |(p_t \ast u_0)(x)|^2 + \int_0^t ds \int_{-\infty}^\infty dy \ [p_{t-s}(y-x)]^2 E \left[ \sigma^2(u_s(y)) \right]
\]
Recall. \( u_t(x) = (p_t * u_0)(x) + (p \otimes \sigma(u))_t(x) \).

Therefore, by the Walsh isometry,

\[
\begin{align*}
\mathbb{E}(|u_t(x)|^2) & = |(p_t * u_0)(x)|^2 + \int_0^t ds \int_{-\infty}^\infty dy \ [p_{t-s}(y-x)]^2 \mathbb{E}[\sigma^2(u_s(y))] \\
& \geq |(p_t * u_0)(x)|^2 + L_\sigma^2 \int_0^t ds \int_{-\infty}^\infty dy \ [p_{t-s}(y-x)]^2 \mathbb{E}(|u_s(y)|^2).
\end{align*}
\]

\[
\begin{align*}
1_{[\alpha t, \infty)}(x) & \geq 1_{[\alpha(t-s), \infty)}(y-x) \cdot 1_{[\alpha s, \infty)}(y).
\end{align*}
\]
Recall. \( u_t(x) = (p_t \ast u_0)(x) + (p \ast \sigma(u))_t(x) \).

Therefore, by the Walsh isometry,

\[
E\left( |u_t(x)|^2 \right) = |(p_t \ast u_0)(x)|^2 + \int_0^t ds \int_{-\infty}^\infty dy \ [p_{t-s}(y-x)]^2 E \left[ \sigma^2(u_s(y)) \right]
\]

\[
\geq |(p_t \ast u_0)(x)|^2 + L_\sigma^2 \int_0^t ds \int_{-\infty}^\infty dy \ [p_{t-s}(y-x)]^2 E \left( |u_s(y)|^2 \right).
\]

\[
1_{[\alpha t, \infty)}(x) \geq 1_{[\alpha(t-s), \infty)}(y-x) \cdot 1_{[\alpha s, \infty)}(y).
\]

\[
\therefore \int_\infty^t dx \int_0^t ds \int_{-\infty}^\infty dy \ [p_{t-s}(y-x)]^2 E \left( |u_s(y)|^2 \right)
\]

\[
\geq \int_0^t ds \left( \int_\infty^{\alpha(t-s)} [p_{-s}(z)]^2 dz \right) \left( \int_{\alpha s}^\infty E \left( |u_s(y)|^2 \right) dy \right)
\]
Intermittency Fronts (Lecture 8)
Sketch of proof

\[ \therefore M_+(t) := \int_{\alpha t}^{\infty} E \left( |u_t(x)|^2 \right) \, dx \text{ solves} \]

\[ M_+(t) \geq \int_{\alpha t}^{\infty} \left| (p_t \ast u_0)(x) \right|^2 \, dx + L_2^2(T \ast M_+)(t), \]

where \( T(t) := \int_{\alpha t}^{\infty} [p_t(z)]^2 \, dz. \)
Intermittency Fronts (Lecture 8)

Sketch of proof

\[ M_+(t) := \int_{\alpha t}^{\infty} \mathbb{E} \left( |u_t(x)|^2 \right) \, dx \text{ solves} \]

\[ M_+(t) \geq \int_{\alpha t}^{\infty} |(p_t \ast u_0)(x)|^2 \, dx + \mathbb{L}^2_\sigma (T \ast M_+)(t), \]

where \( T(t) := \int_{\alpha t}^{\infty} [p_t(z)]^2 \, dz. \)

\[ \therefore M_-(t) := \int_{-\alpha t}^{-\infty} \mathbb{E} \left( |u_t(x)|^2 \right) \, dx \text{ solves} \]

\[ M_-(t) \geq \int_{-\infty}^{-\alpha t} |(p_t \ast u_0)(x)|^2 \, dx + \mathbb{L}^2_\sigma (T \ast M_-)(t), \]

Laplace transform: \( (L \phi)(\beta) := \int_{0}^{\infty} e^{-\beta t} \phi(t) \, dt. \)
Intermittency Fronts (Lecture 8)

Sketch of proof

\[ \therefore M_+(t) := \int_{\alpha t}^{\infty} E \left( |u_t(x)|^2 \right) \, dx \text{ solves} \]

\[ M_+(t) \geq \int_{\alpha t}^{\infty} |(p_t * u_0)(x)|^2 \, dx + L_\sigma^2(T * M_+)(t), \]

where \( T(t) := \int_{\alpha t}^{\infty} [p_t(z)]^2 \, dz \).

\[ \therefore M_-(t) := \int_{-\infty}^{-\alpha t} E \left( |u_t(x)|^2 \right) \, dx \text{ solves} \]

\[ M_-(t) \geq \int_{-\infty}^{-\alpha t} |(p_t * u_0)(x)|^2 \, dx + L_\sigma^2(T * M_-)(t), \]

\[ \therefore M(t) := \int_{|x|>\alpha t} E \left( |u_t(x)|^2 \right) \, dx \text{ solves} \]

\[ M(t) \geq \int_{|x|>\alpha t} |(p_t * u_0)(x)|^2 \, dx + L_\sigma^2(T * M)(t). \]
Intermittency Fronts (Lecture 8)

Sketch of proof

\[ M_+(t) := \int_{\alpha t}^{\infty} E \left( |u_t(x)|^2 \right) \, dx \] solves

\[ M_+(t) \geq \int_{\alpha t}^{\infty} |(p_t * u_0)(x)|^2 \, dx + L^2_\sigma (T * M_+)(t), \]

where \( T(t) := \int_{\alpha t}^{\infty} [p_t(z)]^2 \, dz. \)

\[ M_-(t) := \int_{-\infty}^{-\alpha t} E \left( |u_t(x)|^2 \right) \, dx \] solves

\[ M_-(t) \geq \int_{-\infty}^{-\alpha t} |(p_t * u_0)(x)|^2 \, dx + L^2_\sigma (T * M_-)(t), \]

\[ M(t) := \int_{|x| > \alpha t} E \left( |u_t(x)|^2 \right) \, dx \] solves

\[ M(t) \geq \int_{|x| > \alpha t} |(p_t * u_0)(x)|^2 \, dx + L^2_\sigma (T * M)(t). \]

\[ \text{Laplace transform: } (\mathcal{L} \phi)(\beta) := \int_0^\infty e^{-\beta t} \phi(t) \, dt. \]
Sketch of proof

- $M(t) := \int_{|x| > \alpha t} \mathbb{E}(|u_t(x)|^2) \, dx$, $T(t) = \int_{\alpha t}^{\infty} [p_t(z)]^2 \, dz$. 

Therefore, $\exists \alpha, \beta > 0$ such that $(LM)(\beta) = \infty$.
Intermittency Fronts (Lecture 8)

Sketch of proof

\[ M(t) := \int_{|x| > \alpha t} E(|u_t(x)|^2) \, dx, \quad T(t) = \int_{\alpha t}^{\infty} [p_t(z)]^2 \, dz. \]

\[(\mathcal{L} M)(\beta) \geq \int_{0}^{\infty} e^{-\beta t} \, dt \int_{|x| > \alpha t} dx \, |(p_t \ast u_0)(x)|^2 + L_{\sigma}^2(\mathcal{L} T)(\beta)(\mathcal{L} M)(\beta).\]
Intermittency Fronts (Lecture 8)

Sketch of proof

\[ M(t) := \int_{|x| > \alpha t} E(|u_t(x)|^2) \, dx, \quad T(t) = \int_{\alpha t}^{\infty} [p_t(z)]^2 \, dz. \]

\[ (\mathcal{L}M)(\beta) \geq \int_0^\infty e^{-\beta t} dt \int_{|x| > \alpha t} dx \, |(p_t * u_0)(x)|^2 + L_\sigma^2(\mathcal{L}T)(\beta)(\mathcal{L}M)(\beta). \]

Direct computation:

\[ (\mathcal{L}T)(0) = \frac{1}{2\nu\pi} \int_0^\infty \frac{dt}{t} \int_{\alpha t}^{\infty} dz \, e^{-z^2/(\nu t)} \rightarrow \infty \text{ as } \alpha \downarrow 0. \]
Intermittency Fronts (Lecture 8)

Sketch of proof

\[ M(t) := \int_{|x| > \alpha t} E(|u_t(x)|^2) \, dx, \quad T(t) = \int_{\alpha t}^\infty [p_t(z)]^2 \, dz. \]

\[ (\mathcal{LM})(\beta) \geq \int_0^\infty e^{-\beta t} \, dt \int_{|x| > \alpha t} dx \, |(p_t * u_0)(x)|^2 + L_\sigma^2 (\mathcal{LT})(\beta) (\mathcal{LM})(\beta). \]

Direct computation:

\[ (\mathcal{LT})(0) = \frac{1}{2 \nu \pi} \int_0^\infty \frac{dt}{t} \int_{\alpha t}^\infty dz \, e^{-z^2/(\nu t)} \to \infty \text{ as } \alpha \downarrow 0. \]

Therefore, there exist \( \alpha, \beta > 0 \) such that \( (\mathcal{LM})(\beta) = \infty. \)
Intermittency Fronts (Lecture 8)
Sketch of proof

$M(t) := \int_{|x| > \alpha t} E(|u_t(x)|^2) \, dx$, $T(t) = \int_{\alpha t}^{\infty} [p_t(z)]^2 \, dz$.

$(\mathcal{L} M)(\beta) \geq \int_{0}^{\infty} e^{-\beta t} dt \int_{|x| > \alpha t} dx \ |(p_t \ast u_0)(x)|^2 + L^2_{\sigma}(\mathcal{L}T)(\beta)(\mathcal{L}M)(\beta)$.

Direct computation:
$(\mathcal{L}T)(0) = \frac{1}{2 \nu \pi} \int_{0}^{\infty} \frac{dt}{t} \int_{\alpha t}^{\infty} \, dz \ e^{-z^2/(\nu t)} \to \infty \text{ as } \alpha \downarrow 0$.

Therefore, $\exists \alpha, \beta > 0$ such that $(\mathcal{L}M)(\beta) = \infty$.

Argue by contradiction to see that for this choice of $\alpha, \beta$, $\mathcal{S}(\alpha) = \limsup_{t \to \infty} t^{-1} \sup_{|x| > \alpha t} \log E(|u_t(x)|^2) \geq \beta > 0$. □
Aside: Regularity Theory (Appendix C)
Kolmogorov’s Continuity Theorem

\[
\{X_t\}_{t \in T} \text{ a stochastic process, where } T \subset \mathbb{R}^m \text{ is meas. and bdd.}
\]
Aside: Regularity Theory (Appendix C)
Kolmogorov’s Continuity Theorem

- \{X_t\}_{t \in T} a stochastic process, where \( T \subset \mathbb{R}^m \) is meas. and bdd.
- If \( 0 < \alpha_1, \ldots, \alpha_m < 1 \), then \( \rho \) is a ”distance” on \( \mathbb{R}^m \)—compatible with the topology of \( \mathbb{R}^m \)—where

\[
\rho(w) := \sum_{j=1}^{m} |w_j|^\alpha_j \quad (w \in \mathbb{R}^m).
\]
Aside: Regularity Theory (Appendix C)
Kolmogorov’s Continuity Theorem

- \( \{X_t\}_{t \in T} \) a stochastic process, where \( T \subset \mathbb{R}^m \) is meas. and bdd.
- If \( 0 < \alpha_1, \ldots, \alpha_m < 1 \), then \( \varrho \) is a ”distance” on \( \mathbb{R}^m \)—compatible with the topology of \( \mathbb{R}^m \)—where

\[
\varrho(w) := \sum_{j=1}^{m} |w_j|^{\alpha_j} \quad (w \in \mathbb{R}^m).
\]
Theorem (KCT)

Suppose \( \exists \) finite \( C > 0 \) and \( k > H := \sum_{j=1}^{m} \alpha_j^{-1} \) so that

\[
\|X_t - X_s\|_k \leq C \varrho(t - s) \quad \forall s, t \in T.
\]

Then \( X \) has a continuous modification \( \tilde{X} \) that is Hölder continuous. In fact, \( \forall q \in (0, 1 - (H/k)) \),

\[
E \left( \sup_{s, t \in T: s \neq t \atop s, t \in T} \left| \frac{\tilde{X}_t - \tilde{X}_s}{\varrho(t - s)^q} \right|^k \right) < \infty.
\]
Aside: Regularity Theory (Appendix C)

- “Garsia’s theorem” (1970/1971) nowadays refers to a class of theorems that show that if \( f : \mathbb{R}^m \to \mathbb{R} \) has good integrability properties, then \( f \) is in fact smooth.
“Garsia’s theorem” (1970/1971) nowadays refers to a class of theorems that show that if $f : \mathbb{R}^m \to \mathbb{R}$ has good integrability properties, then $f$ is in fact smooth.

Let $\varrho$ := a “norm” on $\mathbb{R}^m$ that is compatible with the topology of $\mathbb{R}^m$. $\varrho$ := continuous; 0 at 0; subadditive.
“Garsia’s theorem” (1970/1971) nowadays refers to a class of theorems that show that if \( f : \mathbb{R}^m \rightarrow \mathbb{R} \) has good integrability properties, then \( f \) is in fact smooth.

Let \( \varrho \) := a “norm” on \( \mathbb{R}^m \) that is compatible with the topology of \( \mathbb{R}^m \). \( \varrho \) := continuous; 0 at 0; subadditive

\[ B_\varrho(x; r) := \{ y \in \mathbb{R}^m : \varrho(y - x) \leq r \} . \]
Aside: Regularity Theory (Appendix C)

▶ “Garsia’s theorem” (1970/1971) nowadays refers to a class of theorems that show that if \( f : \mathbb{R}^m \to \mathbb{R} \) has good integrability properties, then \( f \) is in fact smooth.

▶ Let \( \varrho := \) a “norm” on \( \mathbb{R}^m \) that is compatible with the topology of \( \mathbb{R}^m \). \( \varrho := \) continuous; 0 at 0; subadditive

▶ \( B_\varrho(x ; r) := \{ y \in \mathbb{R}^m : \varrho(y - x) \leq r \} \).

▶ \( f : \mathbb{R}^m \to \mathbb{R} \) locally integrable \( f \in L^1_{loc}(\mathbb{R}^m) \).
“Garsia’s theorem” (1970/1971) nowadays refers to a class of theorems that show that if $f : \mathbb{R}^m \to \mathbb{R}$ has good integrability properties, then $f$ is in fact smooth.

Let $\varrho :=$ a “norm” on $\mathbb{R}^m$ that is compatible with the topology of $\mathbb{R}^m$. $\varrho :=$ continuous; 0 at 0; subadditive

$B_\varrho(x ; r) := \{ y \in \mathbb{R}^m : \varrho(y - x) \leq r \}$.

$f : \mathbb{R}^m \to \mathbb{R}$ locally integrable $f \in L^1_{loc}(\mathbb{R}^m)$

$\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, strictly increasing function with $\mu(0) = 0$. 
“Garsia’s theorem” (1970/1971) nowadays refers to a class of theorems that show that if $f : \mathbb{R}^m \rightarrow \mathbb{R}$ has good integrability properties, then $f$ is in fact smooth.

Let $\varrho :=$ a “norm” on $\mathbb{R}^m$ that is compatible with the topology of $\mathbb{R}^m$. $\varrho :=$ continuous; 0 at 0; subadditive

$B_{\varrho}(x ; r) := \{ y \in \mathbb{R}^m : \varrho(y - x) \leq r \}.$

$f : \mathbb{R}^m \rightarrow \mathbb{R}$ locally integrable $f \in L^1_{loc}(\mathbb{R}^m)$

$\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous, strictly increasing function with $\mu(0) = 0$.

Garsia’s integral[s]:

$$\mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x - y))} \right|^k \quad \forall k \geq 1.$$
Theorem (Garsia’s theorem)

If \( \mathcal{I}_k < \infty \) for some \( k \in [1, \infty) \) and \( \int_{0}^{r_0} |B_\varrho(r)|^{-2/k} \, d\mu(r) < \infty \), then \( f = \bar{f} \) a.e., where \( \bar{f} : \mathbb{R}^m \to \mathbb{R} \) satisfies

\[
|\bar{f}(s) - \bar{f}(t)| \leq 12\mathcal{I}_k^{1/k} \cdot \int_{0}^{\varrho(s-t)} |B_\varrho(r)|^{-2/k} \, d\mu(r),
\]

for all \( s, t \in \mathbb{R}^m \) that satisfy \( \varrho(s - t) \leq r_0 \).
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

\[ \forall \text{ meas. } Q \subset \mathbb{R}^m \text{ with } |Q| > 0, \text{ define} \]

\[ \bar{f}_Q(x) := \frac{1}{|Q|} \int_Q f(x + z) \, dz \quad (x \in \mathbb{R}^m). \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- \( \forall \) meas. \( Q \subset \mathbb{R}^m \) with \( |Q| > 0 \), define

\[ \tilde{f}_Q(x) := \frac{1}{|Q|} \int_Q f(x + z) \, dz \quad (x \in \mathbb{R}^m). \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

∀ meas. \( Q \subset \mathbb{R}^m \) with \(|Q| > 0\), define

\[ \bar{f}_Q(x) := \frac{1}{|Q|} \int_Q f(x + z) \, dz \quad (x \in \mathbb{R}^m). \]

Lemma (Garsia’s lemma)

∀ \( k \geq 1 \) and bounded and meas. \( Q \subset Q' \subset \mathbb{R}^m \) with \(|Q| > 0\),

\[ \sup_{x \in \mathbb{R}^m} \left| \bar{f}_Q(z) - \bar{f}_{Q'}(z) \right| \leq \sup_{a \in Q, b \in Q'} \mu(\varrho(a - b)) \cdot \left( \frac{I_k}{|Q|^2} \right)^{1/k}. \]
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(x-y)} \right|^k \]

- Jensen’s inequality:

\[ \left| \bar{f}_Q(z) - \bar{f}_{Q'}(z) \right|^k = \left| \frac{1}{|Q| \cdot |Q'|} \int_Q dx \int_{Q'} dy \left( f(x + z) - f(y + z) \right) \right|^k \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- Jensen’s inequality:

\[ \left| \bar{f}_Q(z) - \bar{f}_Q'(z) \right|^k = \left| \frac{1}{|Q| \cdot |Q'|} \int_Q dx \int_{Q'} dy \ (f(x + z) - f(y + z)) \right|^k \]
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- Jensen’s inequality:

\[
| \bar{f}_Q(z) - \bar{f}_{Q'}(z) |^k = \left| \frac{1}{|Q| \cdot |Q'|} \int_{Q} dx \int_{Q'} dy \ (f(x + z) - f(y + z)) \right|^k \\
\leq \frac{1}{|Q|^2} \int_{Q} dx \int_{Q'} dy |f(x + z) - f(y + z)|^k.
\]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\rho(x-y))} \right|^k \]

- Jensen’s inequality:

\[
\left| \bar{f}_Q(z) - \bar{f}_{Q'}(z) \right|^k = \left| \frac{1}{|Q| \cdot |Q'|} \int_Q dx \int_{Q'} dy \left( f(x + z) - f(y + z) \right) \right|^k \\
\leq \frac{1}{|Q|^2} \int_Q dx \int_{Q'} dy \left| f(x + z) - f(y + z) \right|^k.
\]

- Choose and fix \( \alpha > \sup_{a \in Q} \sup_{b \in Q'} \mu(\rho(a - b)) \Rightarrow \)
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- Jensen’s inequality:

\[
\left| \bar{f}_Q(z) - \bar{f}_{Q'}(z) \right|^k \leq \frac{1}{|Q| \cdot |Q'|} \int_Q dx \int_{Q'} dy \left( f(x+z) - f(y+z) \right) \left| \left| \int_{Q} dx \int_{Q'} dy \left( f(x+z) - f(y+z) \right) \right|^k \]

\[
\leq \frac{1}{|Q|^2} \int_Q dx \int_{Q'} dy \left| f(x+z) - f(y+z) \right|^k.
\]

- Choose and fix \( \alpha > \sup_{a \in Q} \sup_{b \in Q'} \mu(\varrho(a-b)) \Rightarrow \)
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- Jensen’s inequality:

\[ \left| \bar{f}_Q(z) - \bar{f}_{Q'}(z) \right|^k = \left| \frac{1}{|Q| \cdot |Q'|} \int_{Q} dx \int_{Q'} dy \left( f(x + z) - f(y + z) \right) \right|^k \]

\[ \leq \frac{1}{|Q|^2} \int_{Q} dx \int_{Q'} dy \left| f(x + z) - f(y + z) \right|^k. \]

- Choose and fix \( \alpha > \sup_{a \in Q} \sup_{b \in Q'} \mu(\varrho(a - b)) \) \( \Rightarrow \)

\[ \left| \bar{f}_Q(z) - \bar{f}_{Q'}(z) \right| \leq \alpha \left( \frac{\mathcal{I}_k}{|Q|^2} \right)^{1/k}. \]
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- Jensen’s inequality:

\[ \left| \bar{f}_Q(z) - \bar{f}_{Q'}(z) \right|^k = \left| \frac{1}{|Q| \cdot |Q'|} \int_Q dx \int_{Q'} dy \left( f(x + z) - f(y + z) \right) \right|^k \leq \frac{1}{|Q|^2} \int_Q dx \int_{Q'} dy \left| f(x + z) - f(y + z) \right|^k. \]

- Choose and fix \( \alpha > \sup_{a \in Q} \sup_{b \in Q'} \mu(\varrho(a - b)) \) \( \Rightarrow \)

\[ \left| \bar{f}_Q(z) - \bar{f}_{Q'}(z) \right| \leq \alpha \left( \frac{\mathcal{I}_k}{|Q|^2} \right)^{1/k}. \]

- Let \( \alpha \downarrow \sup_{a \in Q} \sup_{b \in Q'} \mu(\varrho(a - b)) \) to finish. \( \square \)
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

Define \( r_n \downarrow 0 \) via: \( r_0 > 0 \) fixed; \( \mu(2r_n) = 2^{-n} \mu(2r_0) \); equivalently,

\[ r_{n+1} := \sup \left\{ r > 0 : \mu(2r) = \frac{1}{2} \mu(2r_n) \right\} . \]
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- Define \( r_n \downarrow 0 \) via: \( r_0 > 0 \) fixed; \( \mu(2r_n) = 2^{-n} \mu(2r_0) \); equivalently,

\[ r_{n+1} := \sup \left\{ r > 0 : \mu(2r) = \frac{1}{2} \mu(2r_n) \right\} . \]

- Define

\[ \bar{f}_n(z) := \bar{f}_{B_\varrho(r_n)}(z) \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- Define \( r_n \downarrow 0 \) via: \( r_0 > 0 \) fixed; \( \mu(2r_n) = 2^{-n}\mu(2r_0) \); equivalently,

\[ r_{n+1} := \sup \left\{ r > 0 : \mu(2r) = \frac{1}{2}\mu(2r_n) \right\} \]

- Define

\[ \bar{f}_n(z) := \bar{f}_{B_\varrho(r_n)}(z) \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x - y))} \right|^k \]

- Define \( r_n \downarrow 0 \) via: \( r_0 > 0 \) fixed; \( \mu(2r_n) = 2^{-n} \mu(2r_0) \); equivalently,

\[
 r_{n+1} := \sup \left\{ r > 0 : \mu(2r) = \frac{1}{2} \mu(2r_n) \right\}.
\]

- Define

\[
 \bar{f}_n(z) := \bar{f}_{\varrho(r_n)}(z) = \frac{1}{|B_{\varrho(r_n)}|} \int_{B_{\varrho(r_n)}} f(x + z) \, dx.
\]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

Lemma

Suppose \( \exists k \in [1, \infty) \) so that:

\[ I_k < \infty \] and

\[ \int_{\mathbb{R}^m} \left| B_{\varrho(r)} \right| - \frac{2}{k} d\mu(r) < \infty. \]

Then, \( \bar{f} := \lim_{n \to \infty} \bar{f}_n \) exists, and

\[ \sup_{z \in \mathbb{R}^m} \left| \bar{f}(z) - \bar{f}_\ell(z) \right| \leq \frac{4I_1}{k} \]

\( \cdot \int_{\mathbb{R}^{\ell+1}} \left| B_{\varrho(r)} \right| - \frac{2}{k} d\mu(r) \) for all \( \ell \in \{1, 2, \ldots\} \).

Consequently, \( f = \bar{f} \) a.e.
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

Lemma

Suppose \( \exists k \in [1, \infty) \) so that:

\[ \sup_{z \in \mathbb{R}^m} |\bar{f}(z) - \bar{f}_\ell(z)| \leq \frac{4I_1}{k} \cdot \int_{\mathbb{R}^{\ell+1}} |B(\varrho(r))|^{-2/k} d\mu(r) \quad \forall \ell \in \{1, 2, \ldots\} \]

Consequently, \( f = \bar{f} \text{ a.e.} \)
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

**Lemma**

Suppose \( \exists k \in [1, \infty) \) so that:

- \( I_k < \infty \): and
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x)-f(y)}{\mu(\varrho(x-y))} \right|^k \]

**Lemma**

Suppose \( \exists k \in [1, \infty) \) so that:

- \( I_k < \infty \): and
Aside: Regularity Theory (Appendix C)

\( I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \)

▶ **Lemma**

Suppose \( \exists k \in [1, \infty) \) so that:

▶ \( I_k < \infty \): and

▶ \( \int_0^{r_0} |B_{\varrho}(r)|^{-2/k} d\mu(r) < \infty \).
Aside: Regularity Theory (Appendix C)

$I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\rho(x-y))} \right|^k$

**Lemma**

Suppose $\exists k \in [1, \infty)$ so that:

- $I_k < \infty$: and
- $\int_0^{r^0} |B_\rho(r)|^{-2/k} d\mu(r) < \infty$. 

Lemma

Suppose \( \exists k \in [1, \infty) \) so that:

\( I_k < \infty \): and

\( \int_{r_0}^{\infty} |B_{\rho}(r)|^{-2/k} \, d\mu(r) < \infty. \)

Then, \( \bar{f} := \lim_{n \to \infty} \bar{f}_n \) exists, and

\[
\sup_{z \in \mathbb{R}^m} |\bar{f}(z) - \bar{f}_\ell(z)| \leq 4I_k^{1/k} \cdot \int_{0}^{r_{\ell+1}} |B_{\rho}(r)|^{-2/k} \, d\mu(r) \quad \forall \ell \in \{1, 2, \ldots\}.
\]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

**Lemma**

Suppose \( \exists k \in [1, \infty) \) so that:

- \( I_k < \infty \): and
- \( \int_0^{r_0} |B_\varrho(r)|^{-2/k} d\mu(r) < \infty. \)

Then, \( \bar{f} := \lim_{n \to \infty} \bar{f}_n \) exists, and

\[
\sup_{z \in \mathbb{R}^m} |\bar{f}(z) - \bar{f}_\ell(z)| \leq 4I_k^{1/k} \cdot \int_0^{r_{\ell+1}} |B_\varrho(r)|^{-2/k} d\mu(r) \quad \forall \ell \in \{1, 2, \ldots\}.
\]

Consequently, \( f = \bar{f} \) a.e.
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

Proof. If \( a \in B_\varrho(r_n) \) and \( b \in B_\varrho(r_{n+1}) \), then

\[ \varrho(a - b) \leq \varrho(a) + \varrho(b) \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x - y))} \right|^k \]

Proof. If \( a \in B_\varrho(r_n) \) and \( b \in B_\varrho(r_{n+1}) \), then

\[ \varrho(a - b) \leq \varrho(a) + \varrho(b) \]
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- Proof. If \( a \in B_{\varrho}(r_n) \) and \( b \in B_{\varrho}(r_{n+1}) \), then

\[ \varrho(a - b) \leq \varrho(a) + \varrho(b) \leq r_n + r_{n+1} \leq 2r_n. \]
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- **Proof.** If \( a \in B_\varrho(r_n) \) and \( b \in B_\varrho(r_{n+1}) \), then

\[ \varrho(a - b) \leq \varrho(a) + \varrho(b) \leq r_n + r_{n+1} \leq 2r_n. \]

- **Garsia’s lemma ⇒**

\[ \left| \bar{f}_{\ell+L}(z) - \bar{f}_\ell(z) \right| \leq \sum_{n=\ell}^{\ell+L-1} \left| \bar{f}_{n+1}(z) - \bar{f}_n(z) \right| \]
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\rho(x-y))} \right|^k \]

- **Proof.** If \( a \in B_\rho(r_n) \) and \( b \in B_\rho(r_{n+1}) \), then
  \[
  \rho(a - b) \leq \rho(a) + \rho(b) \leq r_n + r_{n+1} \leq 2r_n.
  \]

- **Garsia’s lemma ⇒**
  \[
  \left| \bar{f}_{\ell+L}(z) - \bar{f}_\ell(z) \right| \leq \sum_{n=\ell}^{\ell+L-1} \left| \bar{f}_{n+1}(z) - \bar{f}_n(z) \right|
  \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x - y))} \right|^k \]

- Proof. If \( a \in B_\varrho(r_n) \) and \( b \in B_\varrho(r_{n+1}) \), then
  \[ \varrho(a - b) \leq \varrho(a) + \varrho(b) \leq r_n + r_{n+1} \leq 2r_n. \]

- Garsia’s lemma \( \Rightarrow \)

\[ \left| \bar{f}_{\ell+L}(z) - \bar{f}_\ell(z) \right| \leq \sum_{n=\ell}^{\ell+L-1} \left| \bar{f}_{n+1}(z) - \bar{f}_n(z) \right| \]

\[ \leq I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_n)}{|B_\varrho(r_{n+1})|^{2/k}} \quad \forall \ell, L \geq 0. \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x - y))} \right|^k \]

- **Proof.** If \( a \in B_\varrho(r_n) \) and \( b \in B_\varrho(r_{n+1}) \), then

\[ \varrho(a - b) \leq \varrho(a) + \varrho(b) \leq r_n + r_{n+1} \leq 2r_n. \]

- **Garsia’s lemma \( \Rightarrow \)**

\[
\left| \bar{f}_{\ell+L}(z) - \bar{f}_\ell(z) \right| \leq \sum_{n=\ell}^{\ell+L-1} \left| \bar{f}_{n+1}(z) - \bar{f}_n(z) \right|
\]

\[
\leq I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_n)}{|B_\varrho(r_{n+1})|^{2/k}} \quad \forall \ell, L \geq 0.
\]

- **Since** \( \mu(2r_n) = 2\mu(2r_{n+1}) \), we can write

\[ \mu(2r_n) = 4\{\mu(2r_{n+1}) - \mu(2r_{n+2})\}. \]
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\rho(x-y))} \right|^k \]

\[ \therefore \left| \bar{f}_{\ell+L}(z) - \bar{f}_{\ell}(z) \right| \leq 4 \mathcal{I}_k^{1/k} \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{\left| B_\rho(r_{n+1}) \right|^{2/k}} \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(g(x-y))} \right|^k \]

\[
\therefore \left| \bar{f}_\ell + L(z) - \bar{f}_\ell(z) \right| \leq 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|B_{\varrho}(r_{n+1})|^{2/k}}
\]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

\[ \therefore \quad |\bar{f}_{\ell+L}(z) - \bar{f}_{\ell}(z)| \leq 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|B_\varrho(r_{n+1})|^{2/k}} \]

\[ = 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_{n+2}}^{r_{n+1}} \frac{d\mu(r)}{|B_\varrho(r_{n+1})|^{2/k}} \]

\[ \therefore \quad f = \bar{f} \text{ a.e.} \]
Aside: Regularity Theory (Appendix C)

$$I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k$$

\[ \therefore \quad \left| \bar{f}_{\ell + L}(z) - \bar{f}_{\ell}(z) \right| \leq 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|B_\varrho(r_{n+1})|^{2/k}} \]

\[ = 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_{n+2}}^{r_{n+1}} \frac{d\mu(r)}{|B_\varrho(r_{n+1})|^{2/k}} \]

\[ \leq 4I_k^{1/k} \cdot \int_{0}^{r_{\ell + 1}} \frac{d\mu(r)}{|B_\varrho(r)|^{2/k}}. \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

\[ \therefore \quad \left| \bar{f}_{\ell + L}(z) - \bar{f}_\ell(z) \right| \leq 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|B_\varrho(r_{n+1})|^{2/k}} \]

\[ = 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_{n+2}}^{r_{n+1}} \frac{d\mu(r)}{|B_\varrho(r_{n+1})|^{2/k}} \]

\[ \leq 4I_k^{1/k} \cdot \int_0^{r_{\ell+1}} \frac{d\mu(r)}{|B_\varrho(r)|^{2/k}}. \]

Therefore, \( \ell \mapsto \bar{f}_\ell \) is uniformly Cauchy and \( \bar{f} := \lim_{\ell \to \infty} \bar{f}_\ell \exists \)
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\rho(x - y))} \right|^k \]

\[ \therefore \quad \left| \bar{f}_{\ell + L}(z) - \bar{f}_\ell(z) \right| \leq 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|B_\rho(r_{n+1})|^{2/k}} \]

\[ = 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_{n+2}}^{r_{n+1}} \frac{d\mu(r)}{|B_\rho(r_{n+1})|^{2/k}} \]

\[ \leq 4I_k^{1/k} \cdot \int_{0}^{r_{\ell+1}} \frac{d\mu(r)}{|B_\rho(r)|^{2/k}}. \]

Therefore, \( \ell \mapsto \bar{f}_\ell \) is uniformly Cauchy and \( \bar{f} := \lim_{\ell \to \infty} \bar{f}_\ell \exists \)

\[ \text{Remains: } f = \bar{f} \text{ a.e.: } \forall \text{ nice } \phi, \]

\[ \int \phi(x) \bar{f}(x) \, dx = \lim_{\ell \to \infty} \int \phi(x) \bar{f}_\ell(x) \, dx \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

\[ \therefore \left| \bar{f}_{\ell+L}(z) - \bar{f}_\ell(z) \right| \leq 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|B_\varrho(r_{n+1})|^{2/k}} \]

\[ = 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_{n+2}}^{r_{n+1}} \frac{d\mu(r)}{|B_\varrho(r_{n+1})|^{2/k}} \]

\[ \leq 4I_k^{1/k} \cdot \int_{0}^{r_{\ell+1}} \frac{d\mu(r)}{|B_\varrho(r)|^{2/k}}. \]

Therefore, \( \ell \mapsto \bar{f}_\ell \) is uniformly Cauchy and \( \bar{f} := \lim_{\ell \to \infty} \bar{f}_\ell \exists \)

\[ \Rightarrow \text{Remains: } f = \bar{f} \text{ a.e.: } \forall \text{ nice } \phi, \]

\[ \int \phi(x) \bar{f}(x) \, dx = \lim_{\ell \to \infty} \int \phi(x) \bar{f}_\ell(x) \, dx \]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

\[ \therefore \ |\bar{f}_{\ell+1}(z) - \bar{f}_\ell(z)| \leq 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|B_\varrho(r_{n+1})|^{2/k}} \]

\[ = 4I_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_{n+2}}^{r_{n+1}} \frac{d\mu(r)}{|B_\varrho(r_{n+1})|^{2/k}} \]

\[ \leq 4I_k^{1/k} \cdot \int_{0}^{r_{\ell+1}} \frac{d\mu(r)}{|B_\varrho(r)|^{2/k}}. \]

Therefore, \( \ell \mapsto \bar{f}_\ell \) is uniformly Cauchy and \( \bar{f} := \lim_{\ell \to \infty} \bar{f}_\ell \exists \)

Remains: \( f = \bar{f} \) a.e.: \( \forall \) nice \( \phi \),

\[ \int \phi(x) \bar{f}(x) \, dx = \lim_{\ell \to \infty} \int \phi(x) \bar{f}_\ell(x) \, dx \]

\[ \int \phi(x) \bar{f}_\ell(x) \, dx = \int \bar{\phi}_\ell(x) f(x) \, dx \]
Aside: Regularity Theory (Appendix C)

\[ \mathcal{I}_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\rho(x-y))} \right|^k \]

\[ \therefore \left| \tilde{f}_{\ell+L}(z) - \tilde{f}_\ell(z) \right| \leq 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \frac{\mu(2r_{n+1}) - \mu(2r_{n+2})}{|B_\rho(r_{n+1})|^{2/k}} \]

\[ = 4\mathcal{I}_k^{1/k} \cdot \sum_{n=\ell}^{\infty} \int_{r_n+2}^{r_{n+1}} \frac{d\mu(r)}{|B_\rho(r_{n+1})|^{2/k}} \]

\[ \leq 4\mathcal{I}_k^{1/k} \cdot \int_0^{r_{\ell+1}} \frac{d\mu(r)}{|B_\rho(r)|^{2/k}}. \]

Therefore, \( \ell \mapsto \tilde{f}_\ell \) is uniformly Cauchy and \( \tilde{f} := \lim_{\ell \to \infty} \tilde{f}_\ell \exists \)

- Remains: \( f = \tilde{f} \) a.e.: \( \forall \) nice \( \phi \),

\[ \int \phi(x) \tilde{f}(x) \, dx = \lim_{\ell \to \infty} \int \phi(x) \tilde{f}_\ell(x) \, dx \]

\[ \int \phi(x) \tilde{f}_\ell(x) \, dx = \int \phi_\ell(x) f(x) \, dx \to \int \phi(x) f(x) \, dx. \]

- Therefore \( f = \tilde{f} \) a.e.

\( \square \)
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- Similar ideas yield the following [see lecture notes]:
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x-y))} \right|^k \]

- Similar ideas yield the following [see lecture notes]:

**Lemma**

> Under the preceding integrability conditions, \( \bar{f} \) is continuous. In fact, if \( s, t \in \mathbb{R}^m \) satisfy \( \varrho(s-t) \leq r_0 \), then

\[
|\bar{f}(s) - \bar{f}(t)| \leq 12I_k^{1/k} \cdot \int_0^{\varrho(s-t)} |B_\varrho(r)|^{-2/k} \, d\mu(r).
\]
Aside: Regularity Theory (Appendix C)

\[ I_k := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} dy \left| \frac{f(x) - f(y)}{\mu(\varrho(x - y))} \right|^k \]

- Similar ideas yield the following [see lecture notes]:

**Lemma**

*Under the preceding integrability conditions, \( \bar{f} \) is continuous. In fact, if \( s, t \in \mathbb{R}^m \) satisfy \( \varrho(s - t) \leq r_0 \), then*

\[ |\bar{f}(s) - \bar{f}(t)| \leq 12I_k^{1/k} \cdot \int_0^{\varrho(s-t)} |B_{\varrho}(r)|^{-2/k} \, d\mu(r). \]

- This concludes the proof of Garsia’s theorem. □
Back to PAM:
Back to PAM:

THERE WILL BE LUNCH VOUCHERS TOMORROW
Back to PAM:

THERE WILL BE LUNCH VOUCHERS TOMORROW

DON’T WORRY!
Back to PAM:
THERE WILL BE LUNCH VOUCHERS TOMORROW
DON’T WORRY!
Back to PAM: $\partial_t u = (\nu/2) \partial_x^2 u + u \xi$, s.t. $u_0 \equiv 1$. 
Intermittency Islands (Lecture 9)

- Back to PAM: $\partial_t u = (\nu/2)\partial_x^2 u + u\xi$, s.t. $u_0 \equiv 1$.
- The solution is $u_t(x) = 1 + (p \otimes u)_t(x)$. 

Theorem (Conus–Joseph–K, 2013)

$0 < \limsup_{|x| \to \infty} \left| \log |u_t(x)| \right| \left( \log |x| \right)^{2/3} < \infty$ a.s. $\forall t > 0$.

- The biggest peaks grow as $\exp\{c (\log |x|)^2/3\}$ with $x$.
- Mueller's comparison principle: $u_t(x) > 0$ a.s. $\Rightarrow \log |u_t(x)| = \log u_t(x)$.
- $E u_t(x) = 1 = \|u_t(x)\|_1$. Therefore, by Fatou's lemma, $\liminf_{|x| \to \infty} u_t(x) < \infty$. So the $\limsup$ is not a $\lim$. 
Intermittency Islands (Lecture 9)

- Back to PAM: $\partial_t u = (\nu/2)\partial_x^2 u + u\xi$, s.t. $u_0 \equiv 1$.
- The solution is $u_t(x) = 1 + (p \otimes u)_t(x)$. 

Theorem (Conus–Joseph–K, 2013)

$0 < \limsup_{|x| \to \infty} \frac{\log |u_t(x)|}{\log |x|^{2/3}} < \infty$ a.s. for all $t > 0$.

- The biggest peaks grow as $\exp\left\{c \left(\log |x|\right)^{2/3}\right\}$ with $x$.
- Mueller's comparison principle: $u_t(x) > 0$ a.s. $\Rightarrow \log |u_t(x)| = \log u_t(x)$.

$\mathbb{E} u_t(x) = 1 = \|u_t(x)\|_1$. Therefore, by Fatou's lemma, $\liminf_{|x| \to \infty} u_t(x) < \infty$. So the lim sup is not a lim.
Intermittency Islands (Lecture 9)

- Back to PAM: \( \partial_t u = (\nu/2)\partial_x^2 u + u\xi \), s.t. \( u_0 \equiv 1 \).
- The solution is \( u_t(x) = 1 + (p \otimes u)_t(x) \).

- **Theorem (Conus–Joseph–K, 2013)**

\[
0 < \limsup_{|x| \to \infty} \frac{\log |u_t(x)|}{(\log |x|)^{2/3}} < \infty \text{ a.s. } \forall t > 0.
\]
► Back to PAM: $\partial_t u = (\nu/2)\partial_x^2 u + u\xi$, s.t. $u_0 \equiv 1$.

► The solution is $u_t(x) = 1 + (p \otimes u)_t(x)$.

► **Theorem (Conus–Joseph–K, 2013)**

$$0 < \limsup_{|x| \to \infty} \frac{\log |u_t(x)|}{(\log |x|)^{2/3}} < \infty \quad a.s. \quad \forall t > 0.$$

► The biggest peaks grow as $\exp\{c(\log |x|)^{2/3}\}$ with $x$. 

Back to PAM: \( \partial_t u = (\nu/2)\partial_x^2 u + u\xi \), s.t. \( u_0 \equiv 1 \).

The solution is \( u_t(x) = 1 + (p \otimes u)_t(x) \).

**Theorem (Conus–Joseph–K, 2013)**

\[
0 < \limsup_{|x| \to \infty} \frac{\log |u_t(x)|}{(\log |x|)^{2/3}} < \infty \text{ a.s. } \forall t > 0.
\]

The biggest peaks grow as \( \exp\{c(\log |x|)^{2/3}\} \) with \( x \).

Mueller’s comparison principle: \( u_t(x) > 0 \text{ a.s.} \)

\( \Rightarrow \log |u_t(x)| = \log u_t(x) \).
Back to PAM: \( \partial_t u = (\nu/2)\partial_x^2 u + u\xi \), s.t. \( u_0 \equiv 1 \).

The solution is \( u_t(x) = 1 + (p \otimes u)_t(x) \).

**Theorem (Conus–Joseph–K, 2013)**

\[
0 < \limsup_{|x| \to \infty} \frac{\log |u_t(x)|}{(\log |x|)^{2/3}} < \infty \quad \text{a.s. \( \forall t > 0 \).}
\]

The biggest peaks grow as \( \exp\{c(\log |x|)^{2/3}\} \) with \( x \).

Mueller’s comparison principle: \( u_t(x) > 0 \) a.s.

\[ \Rightarrow \log |u_t(x)| = \log u_t(x). \]

\( E u_t(x) = 1 = \|u_t(x)\|_1 \). Therefore, by Fatou’s lemma,

\[ \liminf_{|x| \to \infty} u_t(x) < \infty. \] So the \( \limsup \) is not a \( \lim \).
\( h_t(x) = \log u_t(x) \) “Cole–Hopf solution to KPZ”:

\[
0 < \limsup_{|x| \to \infty} \frac{h_t(x)}{(\log |x|)^{2/3}} < \infty. \quad \liminf \leq 0
\]
Intermittency Islands (Lecture 9)

- \( h_t(x) = \log u_t(x) \) “Cole–Hopf solution to KPZ”:

\[
0 < \limsup_{|x| \to \infty} \frac{h_t(x)}{(\log |x|)^{2/3}} < \infty. \quad \liminf \leq 0
\]

- The KPZ equation is:

\[
\frac{\partial}{\partial t} h = \nu \frac{\partial^2}{\partial x^2} h - \left[ \frac{\partial}{\partial x} h \right]^2 + \xi - \infty.
\]
Let us say that $F(z) \asymp G(z)$ for all $z > 1$ when $\exists c \in (1, \infty)$ such that

$$c^{-1}G(z) \leq F(z) \leq cG(z) \quad \forall z > 1.$$
Let us say that $F(z) \preceq G(z)$ for all $z > 1$ when $\exists c \in (1, \infty)$ such that

$$c^{-1} G(z) \leq F(z) \leq c G(z) \quad \forall z > 1.$$ 

The upper bound follows from a careful application of Chebyshev inequality, sub sequencing, and the upper bound of the following tail probability bound:

$$\log P\{h_t(x) > z\} \approx -\frac{3}{2} \log z.$$ 

Well-known conjecture [based on RMT]:

$$\log P\{h_t(x) < -z\} \leq -cz^3/2 \quad 0 < z \ll 1.$$ 


$$\log P\{h_t(x) < -z\} \leq -c \left|\log\left(\frac{1}{z}\right)\right|^3/2 \quad 0 < z \ll 1.$$
Let us say that $F(z) \asymp G(z)$ for all $z > 1$ when $\exists c \in (1, \infty)$ such that
\[ c^{-1}G(z) \leq F(z) \leq cG(z) \quad \forall z > 1. \]

The upper bound follows from a careful application of Chebyshev inequality, sub sequencing, and the upper bound of the following tail probability bound:

**Theorem**

$\forall t > 0$ and $z > 1$, \(\log P \{u_t(x) > z\} \asymp -(\log z)^{3/2}\).
Intermittency Islands (Lecture 9)

- Let us say that $F(z) \asymp G(z)$ for all $z > 1$ when $\exists c \in (1, \infty)$ such that

$$c^{-1}G(z) \leq F(z) \leq cG(z) \quad \forall z > 1.$$

- The upper bound follows from a careful application of Chebyshev inequality, sub sequencing, and the upper bound of the following tail probability bound:

**Theorem**

$\forall t > 0$ and $z > 1$, $\log \mathbb{P} \{u_t(x) > z\} \asymp -(\log z)^{3/2}$.

- In terms of KPZ, $h := \log u$ we have

  $\log \mathbb{P} \{h_t(x) > z\} \asymp z^{3/2}$. \quad z \gg 1
Let us say that $F(z) \asymp G(z)$ for all $z > 1$ when $\exists c \in (1, \infty)$ such that
\[
c^{-1}G(z) \leq F(z) \leq cG(z) \quad \forall z > 1.
\]

The upper bound follows from a careful application of Chebyshev inequality, sub sequencing, and the upper bound of the following tail probability bound:

**Theorem**
\[
\forall t > 0 \text{ and } z > 1, \log P \{ u_t(x) > z \} \asymp -(\log z)^{3/2}.
\]

In terms of KPZ, $h := \log u$ we have
\[
\log P \{ h_t(x) > z \} \asymp z^{3/2}. \quad z \gg 1
\]

Well-known conjecture [based on RMT]:
\[
\log P \{ h_t(x) < -z \} \leq -cz^3? \quad 0 < z \ll 1
\]
Let us say that $F(z) \asymp G(z)$ for all $z > 1$ when $\exists c \in (1, \infty)$ such that

$$c^{-1} G(z) \leq F(z) \leq c G(z) \quad \forall z > 1.$$ 

The upper bound follows from a careful application of Chebyshev inequality, sub sequencing, and the upper bound of the following tail probability bound:

**Theorem**

$\forall t > 0$ and $z > 1$, $\log P \{ u_t(x) > z \} \asymp -(\log z)^{3/2}$.

In terms of KPZ, $h := \log u$ we have

$\log P \{ h_t(x) > z \} \asymp z^{3/2}$. $z \gg 1$

Well-known conjecture [based on RMT]:

$\log P \{ h_t(x) < -z \} \leq -cz^3? \quad 0 < z \ll 1$


$$\log P \{ h_t(x) < -z \} \leq -c \left| \frac{\log(1/z)}{z} \right|^{3/2} \quad 0 < z \ll 1$$
The following will yield the upper bound.

Lemma
Suppose \( X \) is non negative and \( \exists a, C > 0 \) and \( b > 1 \) such that
\[
E(X^k) \leq C k^{\exp\{ak^{b}\}} \quad \forall k \in [1, \infty).
\]
Then,
\[
E \exp(\alpha (\log + X) b / (b - 1)) < \infty \quad \forall \alpha \in (0, 1 - \frac{b - 1}{ab^{1/(b - 1)}}).
\]

In particular, Chebyshev inequality
\[
P\{X > z\} \leq c_1 \exp(-c_2 [\log + z] b / (b - 1)) \quad \forall z > 1.
\]

The probability bound follows from the expectation bound and the Chebyshev inequality.
The following will yield the upper bound.

Lemma
Suppose $X$ is non negative and $\exists a, C > 0$ and $b > 1$ such that $E(X^k) \leq C k \exp\{ak b\} \forall k \in [1, \infty)$. Then, $E \exp(\alpha (\log + X) b/(b - 1)) < \infty \forall \alpha \in (0, 1-b-1/ab 1/(b-1))$.

In particular, Chebyshev inequality $P\{X > z\} \leq c_1 \exp(-c_2 \log z b/(b - 1)) \forall z > 1$.

The probability bound follows from the expectation bound and the Chebyshev inequality.
The following will yield the upper bound.

**Lemma**

Suppose $X$ is non negative and $\exists a, C > 0$ and $b > 1$ such that $E(X^k) \leq C^k \exp\{ak^b\}$ $\forall k \in [1, \infty)$. Then,

$$E\exp\left(\alpha (\log_+ X)^{b/(b-1)}\right) < \infty \quad \forall \alpha \in \left(0, \frac{1 - b^{-1}}{(ab)^{1/(b-1)}}\right).$$

In particular, Chebyshev inequality

$$P\{X > z\} \leq c_1 \exp\left(-c_2[\log_+ z]^{b/(b-1)}\right) \quad \forall z > 1.$$
The following will yield the upper bound.

**Lemma**

*Suppose $X$ is non negative and $\exists a, C > 0$ and $b > 1$ such that $E(X^k) \leq C^k \exp\{ak^b\} \quad \forall k \in [1, \infty)$. Then,*

$$E \exp\left(\alpha (\log_+ X)^{b/(b-1)}\right) < \infty \quad \forall \alpha \in \left(0, \frac{1 - b^{-1}}{(ab)^{1/(b-1)}}\right).$$

*In particular, Chebyshev inequality*

$$P\{X > z\} \leq c_1 \exp \left(-c_2 [\log_+ z]^{b/(b-1)}\right) \quad \forall z > 1.$$

*The probability bound follows from the expectation bound and the Chebyshev inequality.*
Proof of the expectation bound

- WLOG I will assume that $E(X^k) \leq C^k \exp\{ak^b\}$. $C := 1$
Intermittency Islands (Lecture 9)
Proof of the expectation bound

- WLOG I will assume that $E(X^k) \leq C^k \exp\{ak^b\}$. $C := 1$
- If $z > e$ and $k \geq 1$, then by Chebyshev’s inequality,

$$P \left\{ e^{\alpha (\log_+ X)^b/(b-1)} > z \right\} = P \left\{ X > \exp \left[ (\log(z)/\alpha)^{(b-1)/b} \right] \right\}$$
Intermittency Islands (Lecture 9)
Proof of the expectation bound

- WLOG I will assume that $E(X^k) \leq C^k \exp\{ak^b\}$.  $C := 1$
- If $z > e$ and $k \geq 1$, then by Chebyshev’s inequality,

$$P \left\{ e^{\alpha (\log_+ X)^{b/(b-1)}} > z \right\} = P \left\{ X > \exp \left[ (\log(z)/\alpha)^{(b-1)/b} \right] \right\}$$
Intermittency Islands (Lecture 9)
Proof of the expectation bound

- WLOG I will assume that $E(X^k) \leq C^k \exp\{ak^b\}$.  \( C := 1 \)
- If $z > e$ and $k \geq 1$, then by Chebyshev’s inequality,
\[
P \left\{ e^{\alpha (\log + X)^b/(b-1)} > z \right\} = P \left\{ X > \exp \left[ (\log(z)/\alpha)^{(b-1)/b} \right] \right\} \leq E(X^k) \exp \left[-k(\log(z)/\alpha)^{(b-1)/b} \right]
\]
Intermittency Islands (Lecture 9)

Proof of the expectation bound

- WLOG I will assume that $E(X^k) \leq C^k \exp\{ak^b\}$. $C := 1$
- If $z > e$ and $k \geq 1$, then by Chebyshev’s inequality,

$$P\left\{e^{\alpha(\log X)^b/(b-1)} > z\right\} = P\left\{X > \exp\left[(\log(z)/\alpha)^{(b-1)/b}\right]\right\} \leq E(X^k) \exp\left[-k(\log(z)/\alpha)^{(b-1)/b}\right] \leq \exp\left(\frac{ak^b - k(\log(z)/\alpha)^{(b-1)/b}}{:= -g(k)}\right).$$

- $\therefore P\left\{e^{\alpha(\log X)^b/(b-1)} > z\right\} \leq e^{-\sup_{k \geq 1} g(k)}$
Intermittency Islands (Lecture 9)
Proof of the expectation bound

- WLOG I will assume that $E(X^k) \leq C^k \exp\{ak^b\}$.  \( C := 1 \)
- If \( z > e \) and \( k \geq 1 \), then by Chebyshev’s inequality,

\[
P \left\{ e^{\alpha (\log + X)^{b/(b-1)}} > z \right\} = P \left\{ X > \exp \left[ (\log(z)/\alpha)^{(b-1)/b} \right] \right\} 
\leq E(X^k) \exp \left[ -k (\log(z)/\alpha)^{(b-1)/b} \right]
\leq \exp \left( ak^b - k (\log(z)/\alpha)^{(b-1)/b} \right) \tag{:= -g(k)}
\]

- \( \therefore \ P \left\{ e^{\alpha (\log + X)^{b/(b-1)}} > z \right\} \leq e^{-\sup_{k \geq 1} g(k)} \)
Intermittency Islands (Lecture 9)
Proof of the expectation bound

▶ WLOG I will assume that $\mathbb{E}(X^k) \leq C^k \exp\{ak^b\}$.

▶ If $z > e$ and $k \geq 1$, then by Chebyshev’s inequality,

$$
\mathbb{P}\left\{ e^{\alpha (\log + X)^b/(b-1)} > z \right\} = \mathbb{P}\left\{ X > \exp\left[ \left( \log(z)/\alpha \right)^{(b-1)/b} \right] \right\}
$$

$$
\leq \mathbb{E}(X^k) \exp\left[ -k (\log(z)/\alpha)^{(b-1)/b} \right]
$$

$$
\leq \exp\left( ak^b - k \log(z)/\alpha)^{(b-1)/b} \right) \quad \text{:=} \quad -g(k).$$

▶ $\therefore \mathbb{P}\left\{ e^{\alpha (\log + X)^b/(b-1)} > z \right\} \leq e^{-\sup_{k \geq 1} g(k)} = e^{-c \log(z)} = z^{-c}$, where $c := \frac{1 - b^{-1}}{\alpha \cdot (ab)^{1/(b-1)}}$. 

Proof of the expectation bound

- WLOG I will assume that $E(X^k) \leq C^k \exp\{ak^b\}$.
- $C := 1$

- If $z > e$ and $k \geq 1$, then by Chebyshev’s inequality,

$$
P \left\{ e^{\alpha (\log_+ X)^b/(b-1)} > z \right\} = P \left\{ X > \exp \left[ (\log(z)/\alpha)^{(b-1)/b} \right] \right\}
\leq E(X^k) \exp \left[ -k(\log(z)/\alpha)^{(b-1)/b} \right]
\leq \exp \left( \frac{ak^b - k(\log(z)/\alpha)^{(b-1)/b}}{\alpha \cdot (ab)^{1/(b-1)}} \right).
\quad \overset{\text{:=} -g(k)}{=}$$

- Choose $\alpha$ small so that $c > 1$.

$$P \left\{ e^{\alpha (\log_+ X)^b/(b-1)} > z \right\} \leq e^{-\sup_{k \geq 1} g(k)} = e^{-c \log(z)} = z^{-c}, \quad \text{where } c := \frac{1 - b^{-1}}{\alpha \cdot (ab)^{1/(b-1)}}.$$
For the lower bound we need 2 easy lemmas.
For the lower bound we need 2 easy lemmas.

Lemma (Easy Lemma 1)
Suppose $X$ is a non negative r.v. with $\exp(\ell_k t) \leq E(X) \leq L_k \exp(L_k t)$. Then, $\exists Q > 0$ s.t.,

$$P\{X > z\} \geq \exp\left(-Q \frac{\log z}{3} \frac{1 + o(1)}{\sqrt{t \ell}}\right) (z \to \infty).$$

Proof. By the Paley–Zygmund inequality,

$$P\{X \geq \frac{1}{2} \|X\|_m\} \geq \left(1 - 2^{-2} 2^{-2}\|X\|_2^{2m}\right)^{\frac{1}{2m}} \geq \exp\left(-\frac{Q_m}{2} t\right)$$

with $Q_m := 8 \ell - 2L$. 

$\Rightarrow Q > 0 \Rightarrow solved$.
For the lower bound we need 2 easy lemmas.

Lemma (Easy Lemma 1)

Suppose $X$ is a non negative r.v. with
$\exp(\ell k^3 t) \leq E(X^k) \leq L^k \exp(Lk^3 t)$. Then, $\exists Q > 0$ s.t.,

$$P\{X > z\} \geq \exp\left(-Q \frac{|\log z|^{3/2}(1 + o(1))}{\sqrt{t\ell^3}}\right) \quad (z \to \infty).$$
Intermittency Islands (Lecture 9)

- For the lower bound we need 2 easy lemmas.

- **Lemma (Easy Lemma 1)**

  Suppose $X$ is a non negative r.v. with $\exp(\ell k^3 t) \leq \mathbb{E}(X^k) \leq L^k \exp(Lk^3 t)$. Then, $\exists Q > 0$ s.t.,

  $$\mathbb{P}\{X > z\} \geq \exp\left(-Q \frac{\log z^{3/2}(1 + o(1))}{\sqrt{t\ell^3}}\right) \quad (z \to \infty).$$

- **Proof.** By the Paley–Zygmund inequality,

  $$\mathbb{P}\left\{ X \geq \frac{1}{2} \|X\|_m \right\} \geq (1 - 2^{-m})^2 \frac{[\mathbb{E}(X^m)]^2}{\mathbb{E}(X^{2m})}$$
For the lower bound we need 2 easy lemmas.

**Lemma (Easy Lemma 1)**

Suppose $X$ is a non negative r.v. with $\exp(\ell k^3 t) \leq E(X^k) \leq L^k \exp(Lk^3 t)$. Then, $\exists Q > 0$ s.t.,

$$
P\{X > z\} \geq \exp\left( -Q \frac{|\log z|^{3/2}(1 + o(1))}{\sqrt{t\ell^3}} \right) \quad (z \rightarrow \infty).
$$

**Proof.** By the Paley–Zygmund inequality,

$$
P \left\{ X \geq \frac{1}{2} \|X\|_m \right\} \geq \left(1 - 2^{-m}\right)^2 \frac{[E(X^m)]^2}{E(X^{2m})}
$$
Intermittency Islands (Lecture 9)

- For the lower bound we need 2 easy lemmas.

- **Lemma (Easy Lemma 1)**

  Suppose $X$ is a non negative r.v. with
  
  $\exp(\ell k^3 t) \leq \mathbb{E}(X^k) \leq L^k \exp(L k^3 t)$. Then, $\exists Q > 0$ s.t.,

  \[
  \mathbb{P}\{X > z\} \geq \exp\left(-Q \frac{|\log z|^{3/2}(1 + o(1))}{\sqrt{t \ell^3}}\right) \quad (z \to \infty).
  \]

- Proof. By the Paley–Zygmund inequality,

  \[
  \mathbb{P}\left\{ X \geq \frac{1}{2} \|X\|_m \right\} \geq (1 - 2^{-m})^2 \frac{[\mathbb{E}(X^m)]^2}{\mathbb{E}(X^{2m})}
  \geq (1 - 2^{-2})^2 \frac{L^{2m} \exp(2Lm^3 t)}{\exp(8\ell m^3 t)}
  \]
For the lower bound we need 2 easy lemmas.

**Lemma (Easy Lemma 1)**

Suppose $X$ is a non negative r.v. with

$$\exp(\ell k^3 t) \leq \mathbb{E}(X^k) \leq L^k \exp(L k^3 t).$$

Then, $\exists Q > 0$ s.t.,

$$P\{X > z\} \geq \exp\left(-Q \frac{\log z^{3/2}(1 + o(1))}{\sqrt{t \ell^3}}\right) \quad (z \to \infty).$$

**Proof.** By the Paley–Zygmund inequality,

$$P \left\{ X \geq \frac{1}{2} \|X\|_m \right\} \geq \left(1 - 2^{-m}\right)^2 \frac{[\mathbb{E}(X^m)]^2}{\mathbb{E}(X^{2m})}$$

$$\geq \left(1 - 2^{-2}\right)^2 \frac{L^{2m} \exp(2Lm^3t)}{\exp(8\ell m^3t)}$$

$$\geq \frac{9}{16} L^{2m} \exp(-Qm^3 t) \quad Q := 8\ell - 2L.$$
For the lower bound we need 2 easy lemmas.

**Lemma (Easy Lemma 1)**

Suppose $X$ is a non negative r.v. with 
$\exp(\ell k^3 t) \leq E(X^k) \leq L^k \exp(Lk^3 t)$. Then, $\exists Q > 0$ s.t.,

$$P\{X > z\} \geq \exp\left(-Q\frac{\log z^{3/2}(1+o(1))}{\sqrt{t\ell^3}}\right) \quad (z \to \infty).$$

**Proof.** By the Paley–Zygmund inequality,

$$P\left\{ X \geq \frac{1}{2} \|X\|_m \right\} \geq \left(1 - 2^{-m}\right)^2 \frac{[E(X^m)]^2}{E(X^{2m})}$$

$$\geq \left(1 - 2^{-2}\right)^2 \frac{L^{2m} \exp(2Lm^3t)}{\exp(8\ell m^3 t)}$$

$$\geq \frac{9}{16} L^{2m} \exp\left(-Qm^3 t\right) \quad Q := 8\ell - 2L.$$

$\Rightarrow Q > 0 \Rightarrow$ solve.
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erd˝os, 1947, 1952; Erd˝os–R´enyi, 1959; Lamperti, 1963)

Let $E_1, E_2, \ldots$ be events such that:

- $\sum_n P(E_n) = \infty$
- $\exists \theta \in [1, \infty)$ s.t. $P(E_j | E_i) \leq \theta P(E_j) \forall j \geq i$.

Then $P\{E_n \text{ i.o.}\} \geq \theta - 1$.

Proof.

JN := $\sum_{n=1}^{N} 1_{E_n}$.

$E(JN) \to \infty$ as $N \to \infty$;

$E(JN^2) \leq \theta [E(JN)]^2$.

Apply the Paley–Zygmund inequality: $\forall \delta \in (0, 1)$,

$P\{JN \geq \delta E(JN)\} \geq (1 - \delta)^2 (E[JN])^2 E(JN^2) \geq (1 - \delta)^2 \theta$.
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erdős, 1947, 1952; Erdős–Rényi, 1959; Lamperti, 1963)

Let $E_1, E_2, \ldots$ be events such that:

$$
\sum_{n} P(E_n) = \infty;
$$

$$
\exists \theta \in [1, \infty) \text{ s.t. } P(E_j \mid E_i) \leq \theta P(E_j) \quad \forall j \geq 1.
$$

Then $P\{E_n \text{i.o.}\} \geq \theta^{-1}$. 

Proof.

$$
J_N := \sum_{n=1}^{N} 1_{E_n}.
$$

$$
E(J_N) \to \infty \text{ as } N \to \infty;
$$

$$
E(J_N^2) \leq \theta [E(J_N)]^2.
$$

Apply the Paley–Zygmund inequality:

$$
P\{J_N \geq \delta E(J_N)\} \geq (1 - \delta)^2 (E[J_N])^2 \geq (1 - \delta) \theta.
$$
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erdős, 1947, 1952; Erdős–Rényi, 1959; Lamperti, 1963)

Let $E_1, E_2, \ldots$ be events such that:

$\sum_n P(E_n) = \infty$;
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erdős, 1947, 1952; Erdős–Rényi, 1959; Lamperti, 1963)

Let $E_1, E_2, \ldots$ be events such that:

- $\sum_n P(E_n) = \infty$;
- $\exists \theta \in [1, \infty)$ s.t. $P(E_j \mid E_i) \leq \theta P(E_j) \ \forall j > i \geq 1$.

Proof.

Apply the Paley–Zygmund inequality: for all $\delta \in (0, 1)$,

$$P\left\{J_N \geq \delta E(J_N)\right\} \geq (1 - \delta)^2 (E\left[J_N\right])^2 E(J_{2N}) \geq (1 - \delta)^2 \theta.$$
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erdős, 1947, 1952; Erdős–Rényi, 1959; Lamperti, 1963)

Let $E_1, E_2, \ldots$ be events such that:

- $\sum_n P(E_n) = \infty$;
- $\exists \theta \in [1, \infty)$ s.t. $P(E_j \mid E_i) \leq \theta P(E_j) \ \forall j > i \geq 1$.

Proof.

$J_N := \sum_{n=1}^{N} 1_{E_n}$.

$E(J_N) \to \infty$ as $N \to \infty$;

$E(J_N^2) \leq \theta E(J_N)$.

Apply the Paley–Zygmund inequality:

$\forall \delta \in (0, 1)$,

$P\{J_N \geq \delta E(J_N)\} \geq (1 - \delta)^2 \frac{E(J_N)}{E(J_N^2)} \geq (1 - \delta)^2 \theta$. 

$\square$
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erdős, 1947, 1952; Erdős–Rényi, 1959; Lamperti, 1963)

Let $E_1, E_2, \ldots$ be events such that:

- $\sum_n P(E_n) = \infty$;
- $\exists \theta \in [1, \infty)$ s.t. $P(E_j \mid E_i) \leq \theta P(E_j) \ \forall j > i \geq 1$.

Then $P\{E_n \text{ i.o.}\} \geq \theta^{-1}$.

Proof. $J_N := \sum_{n=1}^{N} 1_{E_n}$.
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erdős, 1947, 1952; Erdős–Rényi, 1959; Lamperti, 1963)

Let $E_1, E_2, \ldots$ be events such that:

- $\sum_n P(E_n) = \infty$;
- $\exists \theta \in [1, \infty)$ s.t. $P(E_j | E_i) \leq \theta P(E_j) \ \forall j > i \geq 1$.

Then $P\{E_n \ i.o.\} \geq \theta^{-1}$.

Proof. $J_N := \sum_{n=1}^{N} 1_{E_n}$.

- $E(J_N) \to \infty$ as $N \to \infty$;
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erdős, 1947, 1952; Erdős–Rényi, 1959; Lamperti, 1963)

Let \( E_1, E_2, \ldots \) be events such that:

1. \( \sum_n P(E_n) = \infty \); 
2. \( \exists \theta \in [1, \infty) \) s.t. \( P(E_j \mid E_i) \leq \theta P(E_j) \) \( \forall j > i \geq 1 \).

Then \( P\{E_n \text{ i.o.}\} \geq \theta^{-1} \).

Proof. \( J_N := \sum_{n=1}^N 1_{E_n} \).

1. \( E(J_N) \to \infty \) as \( N \to \infty \); 
2. \( E(J_N^2) \leq \theta [EJ_N]^2 \).
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erdős, 1947, 1952; Erdős–Rényi, 1959; Lamperti, 1963)

Let $E_1, E_2, \ldots$ be events such that:

- $\sum_n P(E_n) = \infty$;
- $\exists \theta \in [1, \infty)$ s.t. $P(E_j \mid E_i) \leq \theta P(E_j) \ \forall j > i \geq 1$.

Then $P\{E_n \text{ i.o.}\} \geq \theta^{-1}$.

Proof. $J_N := \sum_{n=1}^N 1_{E_n}$.

- $E(J_N) \to \infty$ as $N \to \infty$;
- $E(J_{N}^2) \leq \theta [EJ_N]^2$.

Apply the Paley–Zygmund inequality: $\forall \delta \in (0, 1),$

$$P\{J_N \geq \delta E(J_N)\} \geq (1 - \delta)^2 \frac{(E[J_N])^2}{E(J_N^2)}$$
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erdős, 1947, 1952; Erdős–Rényi, 1959; Lamperti, 1963)

Let $E_1, E_2, \ldots$ be events such that:

- $\sum_n P(E_n) = \infty$;
- $\exists \theta \in [1, \infty)$ s.t. $P(E_j | E_i) \leq \theta P(E_j) \ \forall j > i \geq 1$.

Then $P\{E_n \ i.o.\} \geq \theta^{-1}$.

Proof. $J_N := \sum_{n=1}^N 1_{E_n}$.

- $E(J_N) \to \infty$ as $N \to \infty$;
- $E(J_N^2) \leq \theta [EJ_N]^{-2}$.

Apply the Paley–Zygmund inequality: $\forall \delta \in (0, 1)$,

$$P\{J_N \geq \delta E(J_N)\} \geq (1 - \delta)^2 \frac{(E[J_N])^2}{E(J_N^2)}$$
Lemma (Easy lemma 2; Paley–Zygmund, 1932; Chung–Erdős, 1947, 1952; Erdős–Rényi, 1959; Lamperti, 1963)

Let $E_1, E_2, \ldots$ be events such that:

- $\sum_n P(E_n) = \infty$;
- $\exists \theta \in [1, \infty)$ s.t. $P(E_j \mid E_i) \leq \theta P(E_j) \ \forall j > i \geq 1$.

Then $P\{E_n \text{ i.o.}\} \geq \theta^{-1}$.

Proof. $J_N := \sum_{n=1}^{N} 1_{E_n}$.

- $E(J_N) \to \infty$ as $N \to \infty$;
- $E(J_N^2) \leq \theta [EJ_N]^2$.

Apply the Paley–Zygmund inequality: $\forall \delta \in (0, 1)$,

$$P \{ J_N \geq \delta E(J_N) \} \geq (1 - \delta) \frac{E[ J_N^2 ]}{E(J_N^2)} \geq \frac{(1 - \delta)^2}{\theta}.$$
Correlation length (Lecture 10)

- PAM: $\partial_t u = (\nu/2)\partial_x^2 u + u \xi; \quad u_0 \equiv 1.$
PAM: $\partial_t u = (\nu/2) \partial_x^2 u + u \xi; \ u_0 \equiv 1.$

Basic problems:
Correlation length (Lecture 10)

- PAM: $\partial_t u = (\nu/2)\partial_x^2 u + u\xi$; $u_0 \equiv 1$.
- Basic problems:
  - If $x \gg x'$ then $u_t(x)$ is “nearly independent” from $u_t(x')$. 

\[ u_t(x) = 1 + \int_{(0,t)} \times \mathcal{R}_{t-s}(y-x) u_s(y) \xi(ds)dy. \]

Roughly speaking, $\mathcal{R}_{t-s}(y-x) \approx 0$ when $|x-y| \gg 1$.

Therefore, $u_t(x)$ and $u_t(x')$ “use up” different parts of white noise.

This isn’t quite honest. Long-range interactions $\exists$.

If $u_t(x) \gg 1$ then $|x| \gg 1$; therefore, the collection of $y$ “near” $x$ with $u_t(y) \gg 1$ should be “relatively small.”
PAM: $\frac{\partial u}{\partial t} = \left(\nu/2\right)\frac{\partial^2 u}{\partial x^2} + u\xi; \ u_0 \equiv 1.$

Basic problems:

- If $x \gg x'$ then $u_t(x)$ is “nearly independent” from $u_t(x').$
  - $u_t(x) = 1 + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x)u_s(y)\xi(ds\,dy).$
Correlation length (Lecture 10)

- PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u\xi; \ u_0 \equiv 1. \)

- Basic problems:
  - If \( x \gg x' \) then \( u_t(x) \) is “nearly independent” from \( u_t(x') \).
  - \( u_t(x) = 1 + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x)u_s(y)\xi(ds\,dy). \)
  - Roughly speaking, \( p_{t-s}(y-x) \approx 0 \) when \( |x-y| \gg 1. \)
Correlation length (Lecture 10)

- PAM: \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi; \, u_0 \equiv 1. \)
- Basic problems:
  - If \( x \gg x' \) then \( u_t(x) \) is “nearly independent” from \( u_t(x') \).
    - \( u_t(x) = 1 + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) u_s(y) \xi(ds \, dy). \)
    - Roughly speaking, \( p_{t-s}(y-x) \approx 0 \) when \( |x - y| \gg 1. \)
  - Therefore, \( u_t(x) \) and \( u_t(x') \) “use up” different parts of white noise.
PAM: $\partial_t u = (\nu/2)\partial_x^2 u + u \xi; \ u_0 \equiv 1$.

Basic problems:

- If $x \gg x'$ then $u_t(x)$ is “nearly independent” from $u_t(x')$.
  - $u_t(x) = 1 + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x)u_s(y)\xi(ds \, dy)$.
  - Roughly speaking, $p_{t-s}(y-x) \approx 0$ when $|x-y| \gg 1$.
  - Therefore, $u_t(x)$ and $u_t(x')$ “use up” different parts of white noise.

- This isn’t quite honest. long-range interactions $\exists$
Correlation length (Lecture 10)

- PAM: $\partial_t u = (\nu/2) \partial_x^2 u + u \xi$; $u_0 \equiv 1$.
- Basic problems:
  - If $x \gg x'$ then $u_t(x)$ is “nearly independent” from $u_t(x')$.
    - $u_t(x) = 1 + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) u_s(y) \xi(ds\,dy)$.
    - Roughly speaking, $p_{t-s}(y-x) \approx 0$ when $|x - y| \gg 1$.
    - Therefore, $u_t(x)$ and $u_t(x')$ “use up” different parts of white noise.
  - This isn’t quite honest. long-range interactions $\exists$
  - If $u_t(x) \gg 1$ then $|x| \gg 1$; therefore, the collection of $y$ “near” $x$ with $u_t(y) \gg 1$ should be “relatively small.”

Anderson localization
Correlation length (Lecture 10)

\[ \partial_t u = (\nu/2) \partial_x^2 u + u \xi; \quad u_0 \equiv 1. \]
Correlation length (Lecture 10)

\[ \partial_t u = (\nu/2)\partial_x^2 u + u\xi; \quad u_0 \equiv 1. \]
Correlation length (Lecture 10)

$\partial_t u = (\nu/2) \partial_x^2 u + u\xi; \quad u_0 \equiv 1.$

**Definition**

$b > a > 0$ and $t > 0$ non random, fixed. We say that a random subinterval $[c, d]$ of $\mathbb{R}_+$ is an $(a, b)$-island for $u$ at time $t$ if:

- $u_t(c) = u_t(d) = a$;
- $u_t(x) > a$ for all $x \in (c, d)$; and
- $\sup_{x \in [c, d]} u_t(x) > b$.

$d - c :=$ the length of the $(a, b)$-island $[c, d]$.

$\sup_{x \in [c, d]} u_t(x) :=$ the height.

All of the preceding "peaks" are $(1, 2)$-islands [say].
Correlation length (Lecture 10)

- $\partial_t u = (\nu/2) \partial_x^2 u + u \xi; \ u_0 \equiv 1.$

- **Definition**
  
  $b > a > 0$ and $t > 0$ non random, fixed. We say that a random subinterval $[c, d]$ of $\mathbb{R}_+$ is an $(a, b)$-island for $u$ at time $t$ if:
  
  - $u_t(c) = u_t(d) = a;$
  
  - $d - c :=$ the length of the $(a, b)$-island.
  
  - $\sup_{x \in [c, d]} u_t(x) :=$ the height.
  
  All of the preceding "peaks" are $(1, 2)$-islands [say].

- **A priori fact.** $(1, 2)$-islands exist.
Correlation length (Lecture 10)

\[ \partial_t u = \left( \frac{\nu}{2} \right) \partial_x^2 u + u \xi; \quad u_0 \equiv 1. \]

**Definition**

\( b > a > 0 \) and \( t > 0 \) non random, fixed. We say that a random subinterval \([c, d]\) of \( \mathbb{R}_+ \) is an \((a, b)\)-island for \( u \) at time \( t \) if:

- \( u_t(c) = u_t(d) = a; \)
- \( u_t(x) > a \) for all \( x \in (c, d); \) and

\( d - c := \text{the length of the } (a, b)\text{-island } [c, d]. \)

\( \sup_{x \in [c, d]} u_t(x) := \text{the height} \ldots. \)

All of the preceding "peaks" are \((1, 2)\)-islands [say].

A priori fact. \((1, 2)\)-islands exist.
Correlation length (Lecture 10)

- $\partial_t u = (\nu/2)\partial_x^2 u + u\xi; \ u_0 \equiv 1$.

- **Definition**
  $b > a > 0$ and $t > 0$ non random, fixed. We say that a random subinterval $[c, d]$ of $\mathbb{R}_+$ is an $(a, b)$-island for $u$ at time $t$ if:
  - $u_t(c) = u_t(d) = a$;
  - $u_t(x) > a$ for all $x \in (c, d)$; and
  - $\sup_{x \in [c, d]} u_t(x) > b$. 

- $d - c :=$ the length of the $(a, b)$-island $[c, d]$.

- $\sup_{x \in [c, d]} u_t(x) :=$ the height of the $(a, b)$-island $[c, d]$ in the $u_t$ picture.

All of the preceding “peaks” are $(1, 2)$-islands [say].

A priori fact. $(1, 2)$-islands exist.
Correlation length (Lecture 10)

\[ \partial_t u = (\nu/2)\partial_x^2 u + u\xi; \quad u_0 \equiv 1. \]

Definition

\[ b > a > 0 \text{ and } t > 0 \text{ non random, fixed. We say that a random subinterval } [c, d] \text{ of } \mathbb{R}_+ \text{ is an } (a, b)-\text{island} \text{ for } u \text{ at time } t \text{ if:} \]

\[ \begin{align*}
&\quad u_t(c) = u_t(d) = a; \\
&\quad u_t(x) > a \text{ for all } x \in (c, d); \text{ and} \\
&\quad \sup_{x \in [c, d]} u_t(x) > b. 
\end{align*} \]

\[ d - c := \text{the length of the } (a, b)-\text{island } [c, d]. \]
Correlation length (Lecture 10)

\[ \partial_t u = (\nu/2) \partial_x^2 u + u \xi; \quad u_0 \equiv 1. \]

Definition $b > a > 0$ and $t > 0$ non random, fixed. We say that a random subinterval $[c, d]$ of $\mathbb{R}_+$ is an $(a, b)$-island for $u$ at time $t$ if:

- $u_t(c) = u_t(d) = a$;
- $u_t(x) > a$ for all $x \in (c, d)$; and
- $\sup_{x \in [c, d]} u_t(x) > b$.

\[ d - c := \text{the length of the } (a, b)\text{-island } [c, d]. \]
\[ \sup_{x \in [c, d]} u_t(x) := \text{the height } \ldots . \]
Correlation length (Lecture 10)

\[ \partial_t u = (\nu/2) \partial_x^2 u + u \xi; \quad u_0 \equiv 1. \]

- **Definition**
  
  $b > a > 0$ and $t > 0$ non random, fixed. We say that a random subinterval $[c, d]$ of $\mathbb{R}_+$ is an $(a, b)$-island for $u$ at time $t$ if:
  
  - $u_t(c) = u_t(d) = a$;
  - $u_t(x) > a$ for all $x \in (c, d)$; and
  - $\sup_{x \in [c, d]} u_t(x) > b$.

- $d - c :=$ the length of the $(a, b)$-island $[c, d]$.
- $\sup_{x \in [c, d]} u_t(x) :=$ the height . . . . .
- All of the preceding “peaks” are $(1, 2)$-islands [say].
Correlation length (Lecture 10)

\[ \partial_t u = (\nu/2) \partial_x^2 u + u \xi; \quad u_0 \equiv 1. \]

**Definition**

\( b > a > 0 \) and \( t > 0 \) non random, fixed. We say that a random subinterval \([c, d]\) of \( \mathbb{R}^+ \) is an \((a, b)\)-island for \( u \) at time \( t \) if:

- \( u_t(c) = u_t(d) = a; \)
- \( u_t(x) > a \) for all \( x \in (c, d); \) and
- \( \sup_{x \in [c, d]} u_t(x) > b. \)

\[ d - c := \text{the length of the } (a, b)\text{-island } [c, d]. \]
\[ \sup_{x \in [c, d]} u_t(x) := \text{the height } \ldots . \]

All of the preceding “peaks” are \((1, 2)\)-islands [say].

A priori fact. \((1, 2)\)-islands exist.
\[ \partial_t u = (\nu/2) \partial_x^2 u + u \xi; \quad u_0 \equiv 1. \]
Correlation length (Lecture 10)

\[ \partial_t u = (\nu/2)\partial_x^2 u + u\xi; \quad u_0 \equiv 1. \]
Correlation length (Lecture 10)

\[ \partial_t u = (\nu/2) \partial_x^2 u + u \xi; \quad u_0 \equiv 1. \]

**Definition**

\[ J_t(a, b; R) := \text{the length of the largest } (a, b)\text{-island that is contained entirely in } [0, R]. \quad \exists \Rightarrow J := R + 1 \]
Correlation length (Lecture 10)

\[ \partial_t u = (\nu/2)\partial_x^2 u + u\xi; \quad u_0 \equiv 1. \]

Definition
\[ J_t(a, b; R) := \text{the length of the largest } (a, b)\text{-island that is contained entirely in } [0, R]. \]
\[ \exists \Rightarrow J := R + 1 \]
Correlation length (Lecture 10)

- \[ \partial_t u = (\nu/2)\partial_x^2 u + u\xi; \quad u_0 \equiv 1. \]

- **Definition**
  \[ J_t(a, b; R) := \text{the length of the largest} \ (a, b)\text{-island that is} \]
  \[ \text{contained entirely in} \ [0, R]. \quad \exists J \Rightarrow J := R + 1 \]

- **Theorem (Conus–Joseph–K, 2013)**
  \[ \forall t > 0 \ \exists b > a > 1 \text{ such that} \ J_t(a, b; R) = O (|\log R|^2). \]
Correlation length (Lecture 10)

$\partial_t u = (\nu/2)\partial_x^2 u + u\xi; \ u_0 \equiv 1.$

**Definition**

$J_t(a, b; R) :=$ the length of the largest $(a, b)$-island that is contained entirely in $[0, R]$. \[\exists \Rightarrow J := R + 1\]

**Theorem (Conus–Joseph–K, 2013)**

$\forall t > 0 \ \exists \ b > a > 1$ such that $J_t(a, b; R) = O (|\log R|^2)$.

- By comparison: If $\sigma \approx 1$, then $u_t(x) \approx \sqrt{\log |x|};$ and
Correlation length (Lecture 10)

- \( \partial_t u = (\nu/2) \partial_x^2 u + u \xi; \ u_0 \equiv 1. \)

- **Definition**
  \( J_t(a, b; R) := \) the length of the largest \((a, b)\)-island that is contained entirely in \([0, R]\). \( \forall R \Rightarrow J := R + 1 \)

- **Theorem (Conus–Joseph–K, 2013)**
  \( \forall t > 0 \ \exists b > a > 1 \ such \ that \ J_t(a, b; R) = O(\log R^2). \)
  - By comparison: If \( \sigma \approx 1, \) then \( u_t(x) \approx \sqrt{\log |x|}; \) and
  - \( J_t(a, b; R) = O\left(\log R[\log \log R]^{3/2}\right). \)
\[ \frac{\partial}{\partial t} u = \left( \frac{\nu}{2} \right) \frac{\partial^2}{\partial x^2} u + u \xi; \quad u_0 \equiv 1. \]

**Definition**

\[ J_t(a, b; R) := \text{the length of the largest (a, b)-island that is contained entirely in } [0, R]. \]

\[ \exists \Rightarrow J := R + 1 \]

**Theorem (Conus–Joseph–K, 2013)**

\[ \forall t > 0 \ \exists \ b > a > 1 \text{ such that } J_t(a, b; R) = O \left( |\log R|^2 \right). \]

- By comparison: If \( \sigma \asymp 1 \), then \( u_t(x) \asymp \sqrt{\log |x|} \); and
- \[ J_t(a, b; R) = O \left( \log R [\log \log R]^{3/2} \right). \]
- I am not aware of any non-trivial lower bounds on \( J_t(a, b; R) \).
Correlation length (Lecture 10)

- \( \partial_t u = (\nu/2)\partial_x^2 u + u\xi; \ u_0 \equiv 1. \)

- **Definition**
  \( J_t(a, b; R) := \) the length of the largest \((a, b)\)-island that is contained entirely in \([0, R]\).
  \( \not\exists \Rightarrow J := R + 1 \)

- **Theorem (Conus–Joseph–K, 2013)**
  \( \forall t > 0 \ \exists b > a > 1 \) such that \( J_t(a, b; R) = O(| \log R |^2) \).
  - By comparison: If \( \sigma \asymp 1 \), then \( u_t(x) \asymp \sqrt{\log |x|} \); and
  - \( J_t(a, b; R) = O \left( \log R \log \log R \right)^{3/2} \).
  - I am not aware of any non-trivial lower bounds on \( J_t(a, b; R) \).
  - E.g., does \( J_t(a, b; R) \to \infty \) as \( R \to \infty \)?
Theorem

Fix $t > 0$ and $N \in \{0, 1, \ldots\}$. Then there exist $C = C(t)$ and $\{Y(x)\}_{x \in \mathbb{R}}$ such that:

$\mathbb{E}(|u_t(x) - Y(x)|^N) \leq C N e^{-CN^3}$;

and

if $x_1, x_2, \ldots \in \mathbb{R}$ are non random and $|x_i - x_j| > 2N^3/\sqrt{t}$ then $Y(x_1), Y(x_2), \ldots$ are totally independent.

Step 1. Solve for all $\beta > 0$, $U(\beta, t)(x) = 1 + \int_{(0, t)} \left[ x \pm \sqrt{\beta t} \right] p_{t-s}(y-x) U(\beta, s)(y) \xi(dsdy)$.

Existence, uniqueness, . . . , all ok.

$U(\beta, n+1)(t)(x) = 1 + \int_{(0, t)} \left[ x \pm \sqrt{\beta t} \right] p_{t-s}(y-x) U(\beta, n)(s)(y) \xi(dsdy)$.

If $\beta \gg 1$ then $U(\beta) \approx u_\beta$.

Step 2. $U(\beta, N)(t)(x) \approx U(\beta)(t)$. $N \propto k^2$. $Y(x) := U(\beta, N)(t)(x)$. 

Theorem

Fix $t > 0$ and $N \in \{0, 1, \cdots \}$. Then $\exists C = C(t)$ and 
$\{Y(x)\}_{x \in \mathbb{R}}$ such that:

- $E(|u_t(x) - Y(x)|^N) \leq C N e^{-CN^3}$.
- If $|x_1 - x_2| > 2N^3/\sqrt{t}$ then $Y(x_1), Y(x_2), \ldots$ are totally independent.

Step 1. Solve for all $\beta > 0$, $U(\beta) t(x) = 1 + \int_{(0,t)} \left[ x \pm \sqrt{\beta t} \right] p_t - s(y - x) U(\beta) s(y) \xi(ds dy)$.

Existence, uniqueness, \ldots, all ok.

Step 2. $U(\beta, N+1) t(x) = 1 + \int_{(0,t)} \left[ x \pm \sqrt{\beta t} \right] p_t - s(y - x) U(\beta, N) s(y) \xi(ds dy)$.

If $\beta \gg 1$ then $U(\beta) \approx u(\beta) \propto k^2$.
Correlation length (Lecture 10)
Remarks on how to prove asymptotic independence

Theorem

Fix \( t > 0 \) and \( N \in \{0, 1, \cdots\} \). Then \( \exists C = C(t) \) and \( \{Y(x)\}_{x \in \mathbb{R}} \) such that:

- \( \mathbb{E} \left( \left| u_t(x) - Y(x) \right|^N \right) \leq C^N e^{-CN^3} \); and

Step 1. Solve for all \( \beta > 0 \),

\[
U(\beta) t(x) = 1 + \int_{(0,t)} x \pm \sqrt{\beta t} p_{t-s}(y-x) U(\beta) s(y) \xi(ds)dy.
\]
Existence, uniqueness, . . . , all ok.

Step 2.

\[
U(\beta,N+1) t(x) = 1 + \int_{(0,t)} x \pm \sqrt{\beta t} p_{t-s}(y-x) U(\beta,N) s(y) \xi(ds)dy.
\]

If \( \beta \gg 1 \) then \( U(\beta) \approx u(\beta) \). \( \beta \propto k^2 \).
Correlation length (Lecture 10)
Remarks on how to prove asymptotic independence

▶ **Theorem**
Fix $t > 0$ and $N \in \{0, 1, \cdots \}$. Then $\exists C = C(t)$ and
\[ \{Y(x)\}_{x \in \mathbb{R}} \] such that:

▶ $E\left(|u_t(x) - Y(x)|^N\right) \leq C^N e^{-CN^3}$; and

▶ If $x_1, x_2, \ldots \in \mathbb{R}$ are non random and $|x_i - x_j| > 2N^{3/2} \sqrt{t}$
then $Y(x_1), Y(x_2), \ldots$ are totally independent.
Correlation length (Lecture 10)
Remarks on how to prove asymptotic independence

▶ **Theorem**

Fix $t > 0$ and $N \in \{0, 1, \cdots \}$. Then $\exists C = C(t)$ and
\[
\{Y(x)\}_{x \in \mathbb{R}} 
\] such that:

▶ $E\left(|u_t(x) - Y(x)|^N\right) \leq C^N e^{-CN^3}$; and

▶ If $x_1, x_2, \ldots \in \mathbb{R}$ are non random and $|x_i - x_j| > 2N^{3/2}\sqrt{t}$ then $Y(x_1), Y(x_2), \ldots$ are totally independent.

▶ **Step 1.** Solve for all $\beta > 0$,

\[
U_t^{(\beta)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{\beta t}]} p_{t-s}(y-x)U_s^{(\beta)}(y) \xi(ds \, dy).
\]
Correlation length (Lecture 10)
Remarks on how to prove asymptotic independence

▶ **Theorem**
Fix $t > 0$ and $N \in \{0, 1, \cdots \}$. Then $\exists C = C(t)$ and 
\{\(Y(x)\)\}_{x \in \mathbb{R}} such that:

- $E \left( |u_t(x) - Y(x)|^N \right) \leq C^N e^{-CN^3}$; and
- If $x_1, x_2, \ldots \in \mathbb{R}$ are non random and $|x_i - x_j| > 2N^{3/2} \sqrt{t}$
  then $Y(x_1), Y(x_2), \ldots$ are totally independent.

▶ **Step 1.** Solve for all $\beta > 0,$

\[ U_t^{(\beta)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{\beta t}]} p_{t-s}(y - x)U_s^{(\beta)}(y) \xi(ds \, dy). \]

- Existence, uniqueness, \ldots, all ok.
Theorem

Fix $t > 0$ and $N \in \{0, 1, \cdots \}$. Then $\exists C = C(t)$ and $
abla \{Y(x)\}_{x \in \mathbb{R}}$ such that:

- $\mathbb{E} \left( |u_t(x) - Y(x)|^N \right) \leq C^N e^{-CN^3}$; and
- If $x_1, x_2, \ldots \in \mathbb{R}$ are non random and $|x_i - x_j| > 2N^{3/2}\sqrt{t}$ then $Y(x_1), Y(x_2), \ldots$ are totally independent.

Step 1. Solve for all $\beta > 0$,

$$U_t^{(\beta)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{bt}]} p_{t-s}(y-x)U_s^{(\beta)}(y) \xi(ds \, dy).$$

- Existence, uniqueness, ..., all ok.

- $U_t^{(\beta,n+1)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{bt}]} p_{t-s}(y-x)U_s^{(\beta,n)}(y) \xi(ds \, dy)$. 
Correlation length (Lecture 10)
Remarks on how to prove asymptotic independence

- **Theorem**
  Fix $t > 0$ and $N \in \{0, 1, \cdots\}$. Then $\exists C = C(t)$ and \{\(Y(x)\)\}_{x \in \mathbb{R}} such that:
  - $E\left(|u_t(x) - Y(x)|^N\right) \leq C^N e^{-CN^3}$; and
  - If $x_1, x_2, \ldots \in \mathbb{R}$ are non random and $|x_i - x_j| > 2N^{3/2}\sqrt{t}$ then $Y(x_1), Y(x_2), \ldots$ are totally independent.

- **Step 1.** Solve for all $\beta > 0$,
  $$U_t^{(\beta)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{\beta t}]} p_{t-s}(y-x)U_s^{(\beta)}(y) \xi(ds \, dy).$$
  - Existence, uniqueness, \ldots, all ok.
  - $U_t^{(\beta, n+1)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{\beta t}]} p_{t-s}(y-x)U_s^{(\beta, n)}(y) \xi(ds \, dy)$.
  - If $\beta \gg 1$ then $U^{(\beta)} \approx u$. $\beta \propto k^2$
Correlation length (Lecture 10)
Remarks on how to prove asymptotic independence

- **Theorem**
  Fix $t > 0$ and $N \in \{0, 1, \cdots\}$. Then $\exists C = C(t)$ and $\{Y(x)\}_{x \in \mathbb{R}}$ such that:
  - $E \left( |u_t(x) - Y(x)|^N \right) \leq C^N e^{-CN^3}$; and
  - If $x_1, x_2, \ldots \in \mathbb{R}$ are non random and $|x_i - x_j| > 2N^{3/2} \sqrt{t}$ then $Y(x_1), Y(x_2), \ldots$ are totally independent.

- **Step 1.** Solve for all $\beta > 0$,
  
  $$U_t^{(\beta)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{\beta}t]} p_{t-s}(y-x)U_s^{(\beta)}(y) \xi(ds \, dy).$$

  - Existence, uniqueness, \ldots, all ok.
  - $U_t^{(\beta,n+1)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{\beta}t]} p_{t-s}(y-x)U_s^{(\beta,n)}(y) \xi(ds \, dy)$.
  - If $\beta \gg 1$ then $U^{(\beta)} \approx u. \quad \beta \propto k^2$

- **Step 2.** $U^{(\beta,N)} \approx U^{(\beta)}. \quad N \propto k^2$. 
Correlation length (Lecture 10)
Remarks on how to prove asymptotic independence

▶ **Theorem**
Fix $t > 0$ and $N \in \{0, 1, \ldots \}$. Then $\exists C = C(t)$ and 
\{\(Y(x)\)\}$_{x \in \mathbb{R}}$ such that:

- $E\left(|u_t(x) - Y(x)|^N\right) \leq C^N e^{-CN^3}$; and
- If $x_1, x_2, \ldots \in \mathbb{R}$ are non random and $|x_i - x_j| > 2N^{3/2}\sqrt{t}$
then $Y(x_1), Y(x_2), \ldots$ are totally independent.

▶ Step 1. Solve for all $\beta > 0$,
\[
U_t^{(\beta)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{\beta t}]} p_{t-s}(y-x)U_s^{(\beta)}(y) \xi(ds \, dy).
\]

- Existence, uniqueness, . . . , all ok.
- $U_t^{(\beta,n+1)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{\beta t}]} p_{t-s}(y-x)U_s^{(\beta,n)}(y) \xi(ds \, dy)$.
- If $\beta \gg 1$ then $U^{(\beta)} \approx u$. $\beta \propto k^2$

▶ Step 2. $U^{(\beta,N)} \approx U^{(\beta)}$. $N \propto k^2$.
▶ $Y(x) := U_t^{(\beta,N)}(x)$. 


Correlation length (Lecture 10)
Remarks on how to prove asymptotic independence

**Theorem**
Fix $t > 0$ and $N \in \{0, 1, \cdots\}$. Then $\exists \, C = C(t)$ and 
$\{Y(x)\}_{x \in \mathbb{R}}$ such that:

- $\mathbb{E}\left(|u_t(x) - Y(x)|^N\right) \leq C^N e^{-CN^3}$; and
- If $x_1, x_2, \ldots \in \mathbb{R}$ are non random and $|x_i - x_j| > 2N^{3/2}\sqrt{t}$ then $Y(x_1), Y(x_2), \ldots$ are totally independent.

**Step 1.** Solve for all $\beta > 0$,

$$U_t^{(\beta)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{\beta}t]} p_{t-s}(y-x)U_s^{(\beta)}(y) \xi(ds \, dy).$$

- Existence, uniqueness, \ldots, all ok.
- $U_t^{(\beta,n+1)}(x) = 1 + \int_{(0,t) \times [x \pm \sqrt{\beta}t]} p_{t-s}(y-x)U_s^{(\beta,n)}(y) \xi(ds \, dy)$.
- If $\beta \gg 1$ then $U^{(\beta)} \approx u$. $\beta \propto k^2$

**Step 2.** $U^{(\beta,N)} \approx U^{(\beta)}$. $N \propto k^2$.

$Y(x) := U_t^{(\beta,N)}(x)$. 