Hölder continuity of the solution to the 3-dimensional stochastic wave equation

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Stochastic wave equation

Consider the wave equation on $\mathbb{R}^3$

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + b(t,x,u) + \sigma(t,x,u)\dot{W}(t,x), \qquad (1)$$

with zero initial conditions, and $t \in [0,T]$.

- $\dot{W}$ is a centered Gaussian noise with covariance

  $$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)f(x-y),$$

  where f is a non-negative and non-negative definite, locally integrable function.

Gaussian noise

- $W = \{W(\varphi), \varphi \in C_0^\infty([0,T] \times \mathbb{R}^3)\}$ is a zero mean Gaussian family with covariance

$$E(W(\varphi)W(\psi)) = \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(t,x) f(x-y) \psi(t,x) dxdydt.$$

- W can be extended to $L^2([0,T];U)$, where U is the completion of $C_0^\infty(\mathbb{R}^3)$ under the inner product

$$\langle h, g \rangle_U = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} h(x) g(y) f(x-y) dxdy.$$

Then $W_t(h) = W(\mathbf{1}_{[0,t]} h)$, $h \in U$, defines a cylindrical Wiener process in U.

Stochastic integral

We can define the stochastic integral

$$\int_0^T \int_{\mathbb{R}^3} g(t,x) W(dt,dx)$$

of any predictable process $g \in L^2(\Omega \times [0,T]; U)$ and we have

$$E\left(\int_0^T \int_{\mathbb{R}^3} g(t,x) W(dt,dx)\right)^2 = E \int_0^T \|g(t)\|_U^2 dt.$$

- See Da Prato-Zabczyk '92, Dalang-Quer Sardanyons '11.

Fundamental solution

Let $G_t = \frac{1}{4\pi t}\sigma_t$ be the fundamental solution to the 3-D wave equation, where σ_t is the uniform measure on the sphere or radius t.

Lemma

Suppose

$$\boxed{\int_{|x|\leq 1} \frac{f(x)}{|x|} dx < \infty.} \tag{2}$$

Then $G_t \in U$ and

$$\|G_t\|_U^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(x-y)G_t(dx)G_t(dy) = \frac{1}{8\pi}\int_{|x|\leq 2t} \frac{f(x)}{|x|} dx$$

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This follows from the property

$$(G_t * G_t)(x) = \frac{1}{8\pi|x|}\mathbf{1}_{[0,2t]}(|x|).$$

Probabilistic interpretation: If X and Y are two independent three dimensional random variables uniformly distributed in the sphere of radius 1, then $X + Y$ has the density

$$\frac{1}{8\pi|x|}\mathbf{1}_{[0,2]}(|x|)$$

- Condition (2) implies that the stochastic convolution $\int_0^t G_{t-s}(dy)W(ds, dy)$ is the solution to the 3D stochastic wave equation with additive noise.

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Fix $t > 0$. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$, such that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\varphi(x)\varphi(y)| G(t, dx) G(t, dy) f(x-y) < \infty.$$

Then, $\varphi G(t)$ belongs to U and

$$\|\varphi G(t)\|_U^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(x)\varphi(y) G(t, dx) G(t, dy) f(x-y).$$

Mild solution

A predictable process $u(t,x)$ is a mild solution of (1) if for all (t,x)

$$\begin{aligned} u(t,x) &= \int_0^t \int_{\mathbb{R}^3} G_{t-s}(x-dy)\sigma(s,y,u(s,y))W(ds,dy) \\ &+ \int_0^t [G_{t-s} * b(s,\cdot,u(s,\cdot))](x)ds, \end{aligned}$$

in the sense that the measure-valued process

$$(s,y) \mapsto G_{t-s}(x-dy)\sigma(s,y,u(s,y))\mathbf{1}_{[0,t]}(s)$$

is a predictable process in $L^2(\Omega \times [0,T]; U)$.

Burkholder's inequality

For any predictable process $Z = \{Z(t,x), t \in [0,T], x \in \mathbb{R}^3\}$ and $p \geq 2$

$$E\left|\int_0^t \int_{\mathbb{R}^3} Z_{s,y} G_s(x - dy) W(ds, dy)\right|^p$$
$$\leq c_p E \left|\int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} Z_{s,x-y} Z_{s,x-z} G_s(dy) G_s(dz) f(y-z) ds\right|^{\frac{p}{2}}$$
$$\leq c_p' \int_0^t \left(\sup_{x \in \mathbb{R}^3} E|Z(s,x)|^p\right) \left(\int_{|x| \leq 2s} \frac{f(x)}{|x|}\right)^{\frac{p}{2}} ds.$$

Existence and uniqueness of solutions

Theorem (Dalang '99)

Fix $T > 0$. Assume condition (2). Suppose σ and b and Lipschitz functions with linear growth. Then, there is a unique mild solution such that for all $p \geq 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} E|u(t,x)|^p < \infty.$$

- The proof is done using the Fourier analysis. Suppose that f is the Fourier transform of a tempered measure μ. Then, for any $h \in C_0^\infty(\mathbb{R}^3)$,

$$\|h\|_U^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} h(x)h(y)f(x-y)dxdy = \int_{\mathbb{R}^3} |\mathcal{F}(h)(\xi)|^2\mu(d\xi).$$

- Condition (2) is equivalent to $\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$.

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Hölder continuity

Problem: We need to estimate the p-norm the increment

$$E|u(t,x_1)-u(t,x_2)|^p.$$

Using Burkholder's inequality we need to control terms like

$$\begin{aligned} E\Big| \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} & \sigma(s,y,u(s,y))f(y-z)\sigma(s,z,u(s,z)) \\ & \times(G_{t-s}(x_1-dy)-G_{t-s}(x_2-dy)) \\ & \times(G_{t-s}(x_1-dz)-G_{t-s}(x_2-dz))ds\Big|^{\frac{p}{2}}. \end{aligned}$$

- *Main technique*: Transfer the increments from G to σ and f.

- Dalang and Sanz-Solé '09 established the Hölder continuity when f is the Riesz kernel, $f(x) = |x|^{-\beta}$, where $\beta \in (0, 2)$, in the space and time variables, of order $1 - \frac{\beta}{2}$.
- They used Fourier analysis, the Sobolev embedding theorem, and fractional derivatives to express the increments of the Riesz kernel.

Theorem

Suppose that for some $\gamma \in (0,1]$ and $\gamma' \in (0,2]$ we have, for all $w \in \mathbb{R}^3$

(i) $\int_{|z|\leq 2T} \frac{|f(z+w)-f(z)|}{|z|} dz \leq C|w|^{\gamma}$

(ii) $\int_{|z|\leq 2T} \frac{|f(z+w)+f(z-w)-2f(z)|}{|z|} dz \leq C|w|^{\gamma'}$.

Set $\kappa_1 = \min(\gamma, \frac{\gamma'}{2})$. Then for any bounded rectangle $I \subset \mathbb{R}^3$ and $p \geq 2$, there exists a constant C such that

$$E|u(t,x) - u(t,y)|^p \leq C|x-y|^{\kappa_1 p}$$

for all $t \in [0,T]$ and $x, y \in I$.

Hölder continuity in time

Main ingredient:

The scaling property $G(t, dx) = t^{-2}G(1, t^{-1}dx)$ allows us to transform an increment like $G(t-s, dx)$ into an increment in the space variable.

Assumptions:

(iii) For some $\nu \in (0,1]$, $\int_{|z|\leq h} \frac{f(z)}{|z|} dz \leq Ch^{\nu}$.

(iv) If σ is the uniform measure in S^2, for some $\rho_1 \in (0,1]$

$$\int_0^T \int_{S^2} \int_{S^2} |f(s(\xi+\eta) + h(\xi+\eta)) - f(s(\xi+\eta) + h\eta)|$$
$$\times s\sigma(d\xi)\sigma(d\eta)ds \leq Ch^{\rho_1}.$$

(v) For some $\rho_2 \in (0,2]$

$$\int_0^T \int_{S^2} \int_{S^2} |f(s(\xi+\eta) + h(\xi+\eta)) - f(s(\xi+\eta) + h\xi)$$
$$-f\,(s(\xi+\eta) + h\eta) + f(s(\xi+\eta))|s^2\sigma(d\xi)\sigma(d\eta)ds \leq Ch^{\rho_2}.$$

Theorem

Assume conditions (iii) to (v). Suppose also that for some $\kappa_1 \in [0,1]$*, for all* $p \geq 2$ *and for all* $x, y \in I$*, I bounded rectangle of* $\mathbb{R}^3$*,*

$$E|u(t,x) - u(t,y)|^p \leq C|x-y|^{p\kappa_1}.$$

Set $\kappa_2 = \min(\frac{\nu+1}{2}, \frac{\rho_1+\kappa_1}{2}, \frac{\rho_2}{2})$*.*
Then for any bounded rectangle $I \subset \mathbb{R}^3$ *and* $p \geq 2$*, there exists a constant C such that*

$$E|u(t,x) - u(s,x)|^p \leq C|t-s|^{\kappa_2 p}$$

for all $t \in [0,T]$ *and* $x, y \in I$*.*

Hölder continuity in space using Fourier transform

Theorem

Assume that for some $\gamma \in (0, 1]$

(i') *The Fourier transform of the measure* $|\zeta|^{2\gamma}\mu(d\zeta)$ *is a nonnegative locally integrable function*

(ii')
$$\int_{\mathbb{R}^3} \frac{\mu(d\zeta)}{1 + |\zeta|^{2-2\gamma}} < \infty. \tag{3}$$

Then for any bounded rectangle $I \subset \mathbb{R}^3$ *and* $p \geq 2$*, there exists a constant* C *such that*

$$E|u(t,y) - u(t,x)|^p \leq C|x-y|^{p\gamma}$$

for all $t \in [0, T]$ *and* $x, y \in I$.

Main ingredients of the proof:

- Consider the term

$$Q = \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t-s, d\xi) G(t-s, d\eta) \\ \times (f(\eta - \xi + w) - f(\eta - \xi)) \Sigma_x^k(s,\xi) \Sigma_{x,y}^k(s,\eta) ds;$$

where $w = x_1 - x_2$ and

$$\begin{aligned} \Sigma_x(s,\xi) &= \sigma(s, x-\xi, u(s, x-\xi)) \\ \Sigma_{x,y}(s,\xi) &= \sigma(s, x-\xi, u(s, x-\xi)) - \sigma(s, y-\xi, u(s, y-\xi)). \end{aligned}$$

- Using the Fourier transform

$$Q = \int_0^t ds \int_{\mathbb{R}^3} \overline{\mathcal{F}\left(\Sigma_x^k(s,\cdot)G(t-s)\right)(\zeta)}$$
$$\times \mathcal{F}\left(\Sigma_{x,y}^k(s,\cdot)G(t-s)\right)(\zeta)(e^{-iw\cdot\zeta}-1)\mu(d\zeta).$$

- By the estimate $|e^{-iw\cdot\zeta}-1| \leq |w|^{\gamma}|\zeta|^{\gamma}$ for every $0<\gamma\leq 1$, and Cauchy-Schwartz inequality, we need to control

$$Q_1 = |w|^{2\gamma} \int_0^t ds \int_{\mathbb{R}^3} \left|\mathcal{F}\left(\Sigma_x^k(s,\cdot)G(t-s)\right)(\zeta)\right|^2 |\zeta|^{2\gamma}\mu(d\zeta)$$
$$= w|^{2\gamma} \int_0^t ds \int_{\mathbb{R}^3} d\eta g(\eta)$$
$$\times \left(\Sigma_x^k(s,\cdot)G(t-s)\right) * \left(\widetilde{\Sigma_x^k(s,\cdot)G(t-s)}\right)(\eta),$$

where g is the Fourier transform of $\zeta|^{2\gamma}\mu(d\zeta)$.

- Because $g \geq 0$ and using again the Fourier transform and the integrability condition (ii') yields

$$
\begin{aligned}
E|Q_1|^{\frac{q}{2}} \leq C_1 |w|^{2\gamma} \\
&\times \int_0^t ds \left(\int_{\mathbb{R}^3} d\eta g(\eta) G(t-s) * G(t-s)(\eta) \right)^{\frac{q}{2}} \\
&\leq C_2 |w|^{2\gamma}.
\end{aligned}
$$

Examples

Riesz kernel:

$$f(x) = |x|^{-\beta}, \quad 0 < \beta.$$

- Condtion (2) holds is $0 < \beta < 2$.
- Then $\mu(d\xi) = C|\xi|^{-3+\beta} d\xi$ satisfies (3) for any $\gamma \in (0, \frac{2-\beta}{2})$.
- We have

$$\mathcal{F}\big(|\xi|^{2\gamma}\mu(d\xi)\big)(x) = \mathcal{F}(|\xi|^{-3+\beta+2\gamma})(x) = C|x|^{-(\beta+2\gamma)}$$

  which is a nonnegative function for $0 < \gamma < \frac{2-\beta}{2}$.
- f satisfies conditions (iii), (iv) and (v) with $\nu = 2 - \beta$, $0 < \rho_1 < \min(2 - \beta, 1)$ and $0 < \rho_2 < 2 - \beta$.

As a consequence the solution if locally Hölder continuous in space and time of order $\frac{2-\beta}{2}$.

Bessel kernel:

$$f(x) = \int_0^\infty w^{\frac{\alpha-5}{2}} e^{-w} e^{-\frac{|x|^2}{4w}} dw, \quad \alpha > 0.$$

- f satisfies (2) if $\alpha > 1$.
- f satisfies (i), (ii), (iii), (iv) and (v), for any $\gamma, \rho_1, \nu < \min(\alpha - 1, 1)$ and $\gamma', \rho_2 < \min(\alpha - 1, 2)$.

As a consequence the solution if locally Hölder continuous in space and time of order $\frac{\alpha-1}{2} \wedge 1$.

Fractional kernel:

- Let $\dot{W}(t,x)$ be a fractional Brownian noise with Hurst parameters $H_1, H_2, H_3 \in (1/2, 1)$.
- $\dot{W}(t,x)$ is the formal partial derivative $\frac{\partial^4 W}{\partial t \partial x_1 \partial x_2 \partial x_3}(t,x)$, where $W(t,x)$ is a centered Gaussian field with covariance

$$E[W(s,x)W(t,y)] = (s \wedge t) \prod_{i=1}^{3} R_i(x_i, y_i),$$

where

$$R_i(u,v) = \frac{1}{2}\left(|u|^{2H_i} + |v|^{2H_i} - |u-v|^{2H_i}\right).$$

- This example corresponds to the covariance function

$$f(x) = C_H |x_1|^{2H_1-2} |x_2|^{2H_2-2} |x_3|^{2H_3-2}.$$

- f satisfies (2) if $H_1 + H_2 + H_3 > 2$.

Set

$$\kappa = H_1 + H_2 + H_3 - 2 > 0.$$

Then, for any $\kappa_i < \min(H_i - \frac{1}{2}, \kappa)$, $i = 1, 2, 3$, if $\kappa_0 = \kappa_1 \wedge \kappa_2 \wedge \kappa_3$, we obtain for $t, s \in [0, T]$ and $|x|, |y| \leq M$,

$$|u(t,x) - u(s,y)| \leq K_M(|x_1 - y_1|^{\kappa_1} + |x_2 - y_2|^{\kappa_2} + |x_3 - y_3|^{\kappa_3} + |s - t|^{\kappa_0}),$$

for some random variable K_M depending on the κ_i's.

- In the case of an additive noise, we have

$$c_1|x-y|^{2\kappa} \leq E|u(t,x)-u(t,y)|^2 \leq c_2|x-y|^{2\kappa}$$

and

$$d_1|s-t|^{2\kappa} \leq E|u(t,x)-u(s,x)|^2 \leq c_2|s-t|^{2\kappa}$$

for any $s,t \in [0,T]$ and x,y in a bounded rectangle.
- Then we can take $\kappa_i < \kappa$ for all $i = 0,1,2,3$ and the exponent κ is optimal.
- *Question:* Are the additional conditions $\kappa_i < H_i - \frac{1}{2}$ due to the nonlinearity or to the limitation of the method?