Hölder continuity of the solution to the 3-dimensional stochastic wave equation

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Consider the wave equation on $\mathbb{R}^3$

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + b(t, x, u) + \sigma(t, x, u) \dot{W}(t, x),$$  \tag{1}

with zero initial conditions, and $t \in [0, T]$.

- $\dot{W}$ is a centered Gaussian noise with covariance

$$E[\dot{W}(t, x)\dot{W}(s, y)] = \delta(t - s)f(x - y),$$

where $f$ is a non-negative and non-negative definite, locally integrable function.
Gaussian noise

- $W = \{ W(\varphi), \varphi \in C_0^\infty([0, T] \times \mathbb{R}^3) \}$ is a zero mean Gaussian family with covariance

$$E(W(\varphi)W(\psi)) = \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(t, x)f(x - y)\psi(t, x)dx dy dt.$$ 

- $W$ can be extended to $L^2([0, T]; U)$, where $U$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the inner product

$$\langle h, g \rangle_U = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} h(x)g(y)f(x - y)dx dy.$$ 

Then $W_t(h) = W(1_{[0,t]}h)$, $h \in U$, defines a cylindrical Wiener process in $U$. 

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3D stochastic wave equation
We can define the stochastic integral

$$\int_0^T \int_{\mathbb{R}^3} g(t, x) W(dt, dx)$$

of any predictable process $g \in L^2(\Omega \times [0, T]; U)$ and we have

$$E \left( \int_0^T \int_{\mathbb{R}^3} g(t, x) W(dt, dx) \right)^2 = E \int_0^T \|g(t)\|^2_U dt.$$

- See Da Prato-Zabczyk '92, Dalang-Quer Sardanyons '11.
Let \( G_t = \frac{1}{4\pi t} \sigma_t \) be the fundamental solution to the 3-D wave equation, where \( \sigma_t \) is the uniform measure on the sphere or radius \( t \).

**Lemma**

Suppose

\[
\int_{|x| \leq 1} \frac{f(x)}{|x|} \, dx < \infty.
\]  

Then \( G_t \in U \) and

\[
\| G_t \|_U^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x - y) G_t(dx) G_t(dy) = \frac{1}{8\pi} \int_{|x| \leq 2t} \frac{f(x)}{|x|} \, dx
\]
Fundamental solution

Let $G_t = \frac{1}{4\pi t} \sigma_t$ be the fundamental solution to the 3-D wave equation, where $\sigma_t$ is the uniform measure on the sphere or radius $t$.

Lemma

Suppose

$$\int_{|x| \leq 1} \frac{f(x)}{|x|} dx < \infty. \quad (2)$$

Then $G_t \in U$ and

$$\|G_t\|_U^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x - y) G_t(dx) G_t(dy) = \frac{1}{8\pi} \int_{|x| \leq 2t} \frac{f(x)}{|x|} dx$$
This follows from the property

$$(G_t \ast G_t)(x) = \frac{1}{8\pi|x|}1_{[0,2]}(|x|).$$

**Probabilistic interpretation:** If $X$ and $Y$ are two independent three dimensional random variables uniformly distributed in the sphere of radius 1, then $X + Y$ has the density

$$\frac{1}{8\pi|x|}1_{[0,2]}(|x|).$$

Condition (2) implies that the stochastic convolution

$$\int_0^t G_{t-s}(dy) W(ds, dy)$$

is the solution to the 3D stochastic wave equation with additive noise.
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Fix $t > 0$. Let $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$, such that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\varphi(x)\varphi(y)| G(t, dx) G(t, dy) f(x - y) < \infty.$$ 

Then, $\varphi G(t)$ belongs to $U$ and

$$\|\varphi G(t)\|^2_U = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi(x)\varphi(y) G(t, dx) G(t, dy) f(x - y).$$
A predictable process \( u(t, x) \) is a mild solution of (1) if for all \((t, x)\)

\[
u(t, x) = \int_0^t \int_{\mathbb{R}^3} G_{t-s}(x - dy)\sigma(s, y, u(s, y))W(ds, dy) + \int_0^t [G_{t-s} * b(s, \cdot, u(s, \cdot))](x)ds,
\]

in the sense that the measure-valued process

\[
(s, y) \mapsto G_{t-s}(x - dy)\sigma(s, y, u(s, y))1_{[0,t]}(s)
\]

is a predictable process in \( L^2(\Omega \times [0, T]; U) \).
For any predictable process $Z = \{Z(t, x), t \in [0, T], x \in \mathbb{R}^3\}$ and $p \geq 2$

$$E \left| \int_0^t \int_{\mathbb{R}^3} Z_{s, y} G_s(x - dy) W(ds, dy) \right|^p \leq c_p E \left| \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} Z_{s, x-y} Z_{s, x-z} G_s(dy) G_s(dz) f(y - z) ds \right|^{p/2}$$

$$\leq c'_p \int_0^t \left( \sup_{x \in \mathbb{R}^3} E|Z(s, x)|^p \right) \left( \int_{|x| \leq 2s} \frac{f(x)}{|x|} \right)^{p/2} ds.$$
Existence and uniqueness of solutions

Theorem (Dalang ’99)

Fix $T > 0$. Assume condition (2). Suppose $\sigma$ and $b$ and Lipschitz functions with linear growth. Then, there is a unique mild solution such that for all $p \geq 1$,

$$
\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} E|u(t, x)|^p < \infty.
$$

The proof is done using the Fourier analysis. Suppose that $f$ is the Fourier transform of a tempered measure $\mu$. Then, for any $h \in C_0^\infty(\mathbb{R}^3)$,

$$
\|h\|_U^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} h(x)h(y)f(x-y)dx dy = \int_{\mathbb{R}^3} |\mathcal{F}(h)(\xi)|^2 \mu(d\xi).
$$

Condition (2) is equivalent to

$$
\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty.
$$
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3D stochastic wave equation
Hölder continuity

**Problem:** We need to estimate the $p$-norm the increment

$$E|u(t, x_1) - u(t, x_2)|^p.$$ 

Using Burkholder’s inequality we need to control terms like

$$E\left| \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma(s, y, u(s, y)) f(y - z) \sigma(s, z, u(s, z)) \times (G_{t-s}(x_1 - dy) - G_{t-s}(x_2 - dy)) \times (G_{t-s}(x_1 - dz) - G_{t-s}(x_2 - dz)) ds \right|^\frac{p}{2}.$$ 

**Main technique:** Transfer the increments from $G$ to $\sigma$ and $f$. 

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3D stochastic wave equation
Dalang and Sanz-Solé ’09 established the Hölder continuity when $f$ is the Riesz kernel, $f(x) = |x|^{-\beta}$, where $\beta \in (0, 2)$, in the space and time variables, of order $1 - \frac{\beta}{2}$.

They used Fourier analysis, the Sobolev embedding theorem, and fractional derivatives to express the increments of the Riesz kernel.
Hölder continuity in space

Theorem

Suppose that for some $\gamma \in (0, 1]$ and $\gamma' \in (0, 2]$ we have, for all $w \in \mathbb{R}^3$

(i) $\int_{|z| \leq 2T} \frac{|f(z+w)-f(z)|}{|z|} dz \leq C|w|^\gamma$

(ii) $\int_{|z| \leq 2T} \frac{|f(z+w)+f(z-w)-2f(z)|}{|z|} dz \leq C|w|^{\gamma'}$.

Set $\kappa_1 = \min(\gamma, \frac{\gamma'}{2})$. Then for any bounded rectangle $I \subset \mathbb{R}^3$ and $p \geq 2$, there exists a constant $C$ such that

$$E|u(t, x) - u(t, y)|^p \leq C|x - y|^\kappa_1p$$

for all $t \in [0, T]$ and $x, y \in I$. 
Main ingredient:

The scaling property $G(t, dx) = t^{-2}G(1, t^{-1}dx)$ allows us to transform an increment like $G(t - s, dx)$ into an increment in the space variable.
Assumptions:

(iii) For some \( \nu \in (0, 1] \), \( \int_{|z| \leq h} \frac{f(z)}{|z|} \, dz \leq Ch^\nu \).

(iv) If \( \sigma \) is the uniform measure in \( S^2 \), for some \( \rho_1 \in (0, 1] \)

\[
\int_0^T \int_{S^2} \int_{S^2} |f(s(\xi + \eta) + h(\xi + \eta)) - f(s(\xi + \eta) + h\eta)| \\
\times s\sigma(d\xi)\sigma(d\eta) \, ds \leq Ch^{\rho_1}.
\]

(v) For some \( \rho_2 \in (0, 2] \)

\[
\int_0^T \int_{S^2} \int_{S^2} \left| f(s(\xi + \eta) + h(\xi + \eta)) - f(s(\xi + \eta) + h\xi) \\
- f(s(\xi + \eta) + h\eta) + f(s(\xi + \eta)) \right| s^2\sigma(d\xi)\sigma(d\eta) \, ds \leq Ch^{\rho_2}.
\]
Assume conditions (iii) to (v). Suppose also that for some \( \kappa_1 \in [0, 1] \), for all \( p \geq 2 \) and for all \( x, y \in I \), \( I \) bounded rectangle of \( \mathbb{R}^3 \),

\[
E|u(t, x) - u(t, y)|^p \leq C|x - y|^p \kappa_1.
\]

Set \( \kappa_2 = \min\left(\frac{\nu + 1}{2}, \frac{\rho_1 + \kappa_1}{2}, \frac{\rho_2}{2}\right) \).

Then for any bounded rectangle \( I \subset \mathbb{R}^3 \) and \( p \geq 2 \), there exists a constant \( C \) such that

\[
E|u(t, x) - u(s, x)|^p \leq C|t - s|^{\kappa_2 p}
\]

for all \( t \in [0, T] \) and \( x, y \in I \).
Hölder continuity in space using Fourier transform

**Theorem**

Assume that for some \( \gamma \in (0, 1] \)

(i’) The Fourier transform of the measure \( |\zeta|^{2\gamma} \mu(d\zeta) \) is a nonnegative locally integrable function

(ii’) \[
\int_{\mathbb{R}^3} \frac{\mu(d\zeta)}{1 + |\zeta|^{2-2\gamma}} < \infty.
\] (3)

Then for any bounded rectangle \( I \subset \mathbb{R}^3 \) and \( p \geq 2 \), there exists a constant \( C \) such that

\[ E|u(t, y) - u(t, x)|^p \leq C|x - y|^{p\gamma} \]

for all \( t \in [0, T] \) and \( x, y \in I \).
Main ingredients of the proof:

Consider the term

\[ Q = \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t-s, d\xi)G(t-s, d\eta) \times (f(\eta - \xi + w) - f(\eta - \xi)) \Sigma_x^k(s, \xi) \Sigma_x^k s_x, y(s, \eta) ds; \]

where \( w = x_1 - x_2 \) and

\[ \Sigma_x(s, \xi) = \sigma(s, x - \xi, u(s, x - \xi)) \]
\[ \Sigma_x, y(s, \xi) = \sigma(s, x - \xi, u(s, x - \xi)) - \sigma(s, y - \xi, u(s, y - \xi)). \]
Using the Fourier transform

\[ Q = \int_0^t ds \int_{\mathbb{R}^3} \mathcal{F} \left( \Sigma^k_x(s, \cdot) G(t - s) \right) (\zeta) \times \mathcal{F} \left( \Sigma^k_{x,y}(s, \cdot) G(t - s) \right) (\zeta) (e^{-iw \cdot \zeta} - 1) \mu(d\zeta). \]

By the estimate \(|e^{-iw \cdot \zeta} - 1| \leq |w| \gamma |\zeta| \gamma\) for every \(0 < \gamma \leq 1\), and Cauchy-Schwartz inequality, we need to control

\[ Q_1 = |w|^{2\gamma} \int_0^t ds \int_{\mathbb{R}^3} \left| \mathcal{F} \left( \Sigma^k_x(s, \cdot) G(t - s) \right) (\zeta) \right|^2 |\zeta|^{2\gamma} \mu(d\zeta) \]

\[ = w^{2\gamma} \int_0^t ds \int_{\mathbb{R}^3} d\eta g(\eta) \times \left( \Sigma^k_x(s, \cdot) G(t - s) \right) \ast \left( \Sigma^k_x(s, \cdot) \widehat{G}(t - s) \right)(\eta), \]

where \(g\) is the Fourier transform of \(|\zeta|^{2\gamma} \mu(d\zeta)\).
Because $g \geq 0$ and using again the Fourier transform and the integrability condition (ii’) yields

$$E|Q_1|^{\frac{q}{2}} \leq C_1 |w|^{2\gamma} \times \int_0^t ds \left( \int_{\mathbb{R}^3} d\eta g(\eta) G(t-s) \ast G(t-s)(\eta) \right)^{\frac{q}{2}} \leq C_2 |w|^{2\gamma}.$$
Examples

Riesz kernel:

\[ f(x) = |x|^{-\beta}, \quad 0 < \beta. \]

- Condition (2) holds is \( 0 < \beta < 2 \).
- Then \( \mu(d\xi) = C|\xi|^{-3+\beta}d\xi \) satisfies (3) for any \( \gamma \in (0, \frac{2-\beta}{2}) \).
- We have

\[
\mathcal{F}(|\xi|^{2\gamma}\mu(d\xi))(x) = \mathcal{F}(|\xi|^{-\beta+2\gamma})(x) = C|x|^{-(\beta+2\gamma)}
\]

which is a nonnegative function for \( 0 < \gamma < \frac{2-\beta}{2} \).

- \( f \) satisfies conditions (iii), (iv) and (v) with \( \nu = 2 - \beta \),
  \( 0 < \rho_1 < \min(2 - \beta, 1) \) and \( 0 < \rho_2 < 2 - \beta \).

As a consequence the solution is locally Hölder continuous in space and time of order \( \frac{2-\beta}{2} \).
Bessel kernel:

\[ f(x) = \int_0^\infty w^{\frac{\alpha-5}{2}} e^{-w} e^{-\frac{|x|^2}{4w}} \, dw, \quad \alpha > 0. \]

- \( f \) satisfies (2) if \( \alpha > 1 \).
- \( f \) satisfies (i), (ii), (iii), (iv) and (v), for any \( \gamma, \rho_1, \nu < \min(\alpha - 1, 1) \) and \( \gamma', \rho_2 < \min(\alpha - 1, 2) \).

As a consequence the solution if locally Hölder continuous in space and time of order \( \frac{\alpha-1}{2} \wedge 1 \).
Fractional kernel:

- Let $\dot{W}(t, x)$ be a fractional Brownian noise with Hurst parameters $H_1, H_2, H_3 \in (1/2, 1)$.
- $\dot{W}(t, x)$ is the formal partial derivative $\frac{\partial^4 W}{\partial t \partial x_1 \partial x_2 \partial x_3}(t, x)$, where $W(t, x)$ is a centered Gaussian field with covariance

$$E[W(s, x)W(t, y)] = (s \wedge t) \prod_{i=1}^{3} R_i(x_i, y_i),$$

where

$$R_i(u, v) = \frac{1}{2} \left( |u|^{2H_i} + |v|^{2H_i} - |u - v|^{2H_i} \right).$$

- This example corresponds to the covariance function

$$f(x) = C_H |x_1|^{2H_1-2} |x_2|^{2H_2-2} |x_3|^{2H_3-2}.$$
• $f$ satisfies (2) if $H_1 + H_2 + H_3 > 2$.

Set

$$
\kappa = H_1 + H_2 + H_3 - 2 > 0.
$$

Then, for any $\kappa_i < \min(H_i - \frac{1}{2}, \kappa)$, $i = 1, 2, 3$, if $\kappa_0 = \kappa_1 \wedge \kappa_2 \wedge \kappa_3$, we obtain for $t, s \in [0, T]$ and $|x|, |y| \leq M$,

$$
|u(t, x) - u(s, y)| \leq K_M(|x_1 - y_1|^\kappa_1 + |x_2 - y_2|^\kappa_2 + |x_3 - y_3|^\kappa_3 + |s - t|^\kappa_0),
$$

for some random variable $K_M$ depending on the $\kappa_i$'s.
In the case of an additive noise, we have

\[ c_1 |x - y|^{2\kappa} \leq E |u(t, x) - u(t, y)|^2 \leq c_2 |x - y|^{2\kappa} \]

and

\[ d_1 |s - t|^{2\kappa} \leq E |u(t, x) - u(s, x)|^2 \leq c_2 |s - t|^{2\kappa} \]

for any \( s, t \in [0, T] \) and \( x, y \) in a bounded rectangle.

Then we can take \( \kappa_i < \kappa \) for all \( i = 0, 1, 2, 3 \) and the exponent \( \kappa \) is optimal.

**Question:** Are the additional conditions \( \kappa_i < H_i - \frac{1}{2} \) due to the nonlinearity or to the limitation of the method?