CTRW Limits: Governing Equations and Fractal Dimensions

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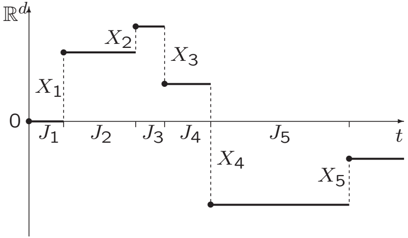
Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

Waiting time process

J_n's are nonnegative iid.
$T_n = J_1 + J_2 + \cdots + J_n$ is the time of the nth jump.
$N(t) = \max\{n \geq 0 : \ T_n \leq t\}$ is the number of jumps by time $t > 0$.
Suppose $P(J_n > t) \approx Ct^{-\beta}$ for $0 < \beta < 1$.
Scaling limit

$$c^{-1/\beta} T_{[ct]} \implies D(t)$$

is a β-stable subordinator.
Since $\{T_n \leq t\} = \{N(t) \geq n\}$

$$c^{-\beta} N(ct) \implies E(t) = \inf\{u > 0 : \quad D(u) > t\}.$$

The self-similar limit $E(ct) \stackrel{d}{=} c^{\beta} E(t)$ is non-Markovian.

Continuous time random walks (CTRW)

$S(n) = X_1 + \cdots + X_n$ particle location after n jumps
has scaling limit $c^{-1/2}S([ct]) \implies B(t)$ a Brownian motion.
Number of jumps has scaling limit $c^{-\beta}N(ct) \implies E(t)$.
CTRW is a random walk time-changed by (a renewal process) $N(t)$

$$S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}.$$

$S(N(t))$ is particle location at time $t > 0$.
CTRW scaling limit is a time-changed process:

$$\begin{aligned} c^{-\beta/2}S(N(ct)) &= (c^\beta)^{-1/2}S(c^\beta \cdot c^{-\beta}N(ct)) \\ &\approx (c^\beta)^{-1/2}S(c^\beta E(t)) \implies B(E(t)). \end{aligned}$$

The self-similar limit $B(E(ct)) \stackrel{d}{=} c^{\beta/2}B(E(t))$ is non-Markovian.

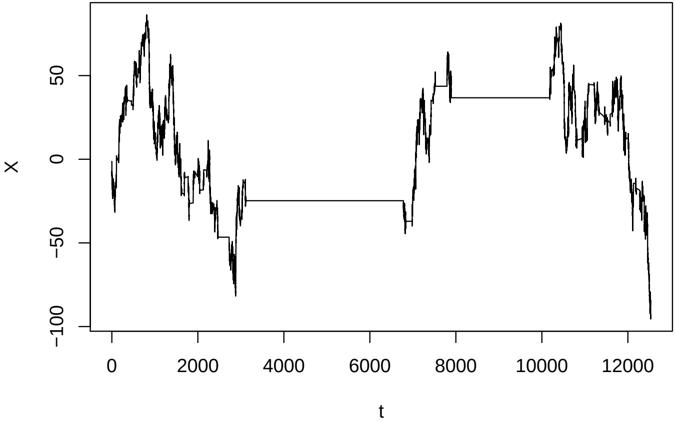


Figure : Typical sample path of the time-changed process $B(E(t))$. Here $B(t)$ is a Brownian motion and $E(t)$ is the inverse of a $\beta = 0.8$-stable subordinator. Graph has dimension $1 + \beta/2 = 1 + 0.4$. The limit process retains long resting times.

CTRW with serial dependence

Particle jumps $X_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$ with Z_n IID.

Short range dependence: $\sum_{n=1}^{\infty} |\mathbb{E}(X_n X_0)| < \infty \Longrightarrow$ the usual limit and PDE.

Long range dependence:
If Z_n has light tails: **time-changed fractional Brownian motion** limit $B_H(E(t))$. Hahn, Kobayashi and Umarov (2010) established a governing equation.

For heavy tails: **time-changed linear fractional stable motion** $L_{\alpha,H}(E(t))$.

Open problems: Governing equations, dependent waiting times.

Meerschaert, Nane and Xiao (2009).

Hausdorff dimension

For any $\alpha > 0$, the α-dimensional Hausdorff measure of $F \subseteq \mathbb{R}^d$ is defined by

$$s^\alpha\text{-}m(F) = \lim_{\epsilon \to 0} \inf \left\{ \sum_i (2r_i)^\alpha : F \subseteq \bigcup_{i=1}^\infty B(x_i, r_i),\ r_i < \epsilon \right\}, \quad (1)$$

where $B(x, r)$ denotes the open ball of radius r centered at x.
It is well-known that s^α-m is a metric outer measure and every Borel set in $\mathbb{R}^d$ is s^α-m measurable.
The Hausdorff dimension of F is defined by

$$\dim_{\mathrm{H}} F = \inf \left\{ \alpha > 0 :\ s^\alpha\text{-}m(F) = 0 \right\} = \sup \left\{ \alpha > 0 :\ s^\alpha\text{-}m(F) = \infty \right\}.$$

Hausdorff dimension of image

Let $Z = \{Z(t) = Y(E(t)), t \geq 0\}$ be the time-changed process with values in $\mathbb{R}^d$ where the processes Y and E independent and $E(t)$ is a nondecreasing continuous process. If $E(1) > 0$ a.s. and there exist a constant c_1 such that for all constants $0 < a < \infty$

$$\dim_{\mathrm{H}} Y([0, a]) = c_1, \ a.s. \tag{2}$$

then almost surely

$$\dim_{\mathrm{H}} Z([0, 1]) = c_1. \tag{3}$$

Applying this result to the space-time process $x \mapsto (x, Y(x))$: If there exist constants c_2 such that for all constants $0 < a < \infty$, $\dim_{\mathrm{H}} \mathrm{Gr}\, Y([0, a]) = c_2 \quad a.s.$, then

$$\dim_{\mathrm{H}} \{(E(t), Y(E(t))) : t \in [0, 1]\} = \dim_{\mathrm{H}} \mathrm{Gr}\, Y([0, 1]), \quad a.s.$$

Hausdorff dimension of graph

Let

$$A(x) = (D(x), Y(x)), \qquad \forall x \geq 0, \tag{4}$$

where $D = \{D(x), x \geq 0\}$ is defined by

$$D(x) = \inf \left\{ t > 0 : E(t) > x \right\}. \tag{5}$$

If $E(1) > 0$ a.s. and there exist a constant c_3 such that for all constants $0 < a < \infty$

$$\dim_{\mathrm{H}} A([0, a]) = c_3 \tag{6}$$

then

$$\begin{aligned} \dim_{\mathrm{H}} \mathrm{Gr} Z([0,1]) &= \dim_{\mathrm{H}} \{(t, Z(t)) : t \in [0,1]\} \\ &= \max \left\{1, \dim_{\mathrm{H}} A([0,1])\right\}, \qquad \text{a.s.} \end{aligned} \tag{7}$$

Meerschaert, Nane and Xiao (2013).

Examples

Let $Z = \{Y(E(t)), t \geq 0\}$, where $Y = \{Y(x) : \ x \geq 0\}$ is a **stable Lévy motion** of index $\alpha \in (0, 2]$ with values in $\mathbb{R}^d$ and $E(t)$ is the **inverse of a stable subordinator** of index $0 < \beta < 1$, independent of Y. Then

$$\dim_{\mathrm{H}} Z([0,1]) = \dim_{\mathrm{H}} Y([0,1]) = \min\{d, \alpha\}, \qquad \text{a.s.} \tag{8}$$

$$\dim_{\mathrm{H}} \mathrm{Gr} Z([0,1]) = \begin{cases} \max\{1, \alpha\} & \text{if } \alpha \leq d, \\ 1 + \beta(1 - \frac{1}{\alpha}) & \text{if } \alpha > d = 1, \end{cases} \qquad \text{a.s.} \tag{9}$$

compare to

$$\dim_{\mathrm{H}} \mathrm{Gr} Y([0,1]) = \dim_{\mathrm{H}} \big\{ (E(t), Y(E(t))) : t \in [0,1] \big\}$$
$$= \begin{cases} \max\{1, \alpha\} & \text{if } \alpha \leq d, \\ 2 - \frac{1}{\alpha} = 1 + 1 - \frac{1}{\alpha} & \text{if } \alpha > d = 1. \end{cases} \qquad \text{a.s.} \tag{10}$$

Let $Z = \{Y(E(t)), t \geq 0\}$, where Y is a **fractional Brownian motion** with values in $\mathbb{R}^d$ of index $H \in (0,1)$ and $E(t)$ is the the **inverse of a β-stable subordinator** D which is independent of Y. Then

$$\dim_{\mathrm{H}} Z([0,1]) = \dim_{\mathrm{H}} Y([0,1]) = \min\left\{d, \frac{1}{H}\right\}, \qquad \text{a.s.} \qquad (11)$$

$$\dim_{\mathrm{H}} \mathrm{Gr} Z([0,1]) = \begin{cases} \frac{1}{H} & \text{if } 1 \leq Hd, \\ \beta + (1 - H\beta)d = d + \beta(1 - Hd) & \text{if } 1 > Hd, \end{cases} \qquad (12)$$

compare to

$$\dim_{\mathrm{H}} \{(E(t), Y(E(t))) : t \in [0,1]\} = \dim_{\mathrm{H}} \mathrm{Gr} Y([0,1]) = \begin{cases} \frac{1}{H} & \text{if } 1 \leq Hd, \\ 1 + (1 - H)d = d + (1 - Hd) & \text{if } 1 > Hd, \end{cases} \quad \text{a.s.} \qquad (13)$$

Fractional time derivative: Two approaches

- Riemann-Liouville fractional derivative of order $0 < \beta < 1$;

$$\mathbb{D}_t^{\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \left[\int_0^t g(s) \frac{ds}{(t-s)^{\beta}} \right]$$

  with Laplace transform $s^{\beta}\tilde{g}(s)$, $\tilde{g}(s) = \int_0^{\infty} e^{-st} g(t) dt$ denotes the usual Laplace transform of g.

- Caputo fractional derivative of order $0 < \beta < 1$;

$$D_t^{\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t-s)^{\beta}} \tag{14}$$

  was invented to properly handle initial values (Caputo 1967). Laplace transform of $D_t^{\beta} g(t)$ is $s^{\beta}\tilde{g}(s) - s^{\beta-1} g(0)$ incorporates the initial value in the same way as the first derivative.

examples

- $$D_t^\beta(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$
- $$D_t^\beta(e^{\lambda t}) = \lambda^\beta e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)}?$$
- $$D_t^\beta(\sin t) = \sin(t + \frac{\pi\beta}{2})$$

Diffusion

Let L_x be the generator of some continuous Markov process $X(t)$. Then $p(t,x) = \mathbb{E}_x[f(X(t))]$ is the unique solution of the heat-type Cauchy problem

$$\partial_t p(t,x) = L_x p(t,x),\ \ t > 0, x \in \mathbb{R}^d;\ \ \ p(0,x) = f(x),\ \ x \in \mathbb{R}^d$$

Examples:
X: Brownian motion then $L_x = \Delta_x$, BM is a stochastic solution of the heat equation
X: Symmetric α-stable process then $L_x = -(-\Delta)^{\alpha/2}$

Time-fractional model for Anamolous sub-diffusion

Let $0 < \beta < 1$. Nigmatullin (1986) gave a physical derivation of fractional diffusion

$$\partial_t^\beta u(t,x) = L_x u(t,x); \quad u(0,x) = f(x) \tag{15}$$

Zaslavsky (1994) used this to model Hamiltonian chaos.
(**??**) has the unique solution

$$u(t,x) = \mathbb{E}_x[f(X(E(t)))] = \int_0^\infty p(s,x) g_{E(t)}(s) ds$$

where $p(t,x) = \mathbb{E}_x[f(X(t))]$ and $E(t) = \inf\{\tau > 0 : \; D_\tau > t\}$, $D(t)$ is a stable subordinator with index β and $\mathbb{E}(e^{-sD(t)}) = e^{-ts^\beta}$ (Baeumer and Meerschaert, 2002).
$\mathbb{E}_x(B(E(t)))^2 = \mathbb{E}(E(t)) \approx t^\beta$.

Heat equation in bounded domains

Denote the eigenvalues and the eigenfunctions of Δ on a bounded domain D with Dirichlet boundary conditions by $\{\mu_n, \phi_n\}_{n=1}^{\infty}$;

$$\Delta\phi_n(x) = -\mu_n\phi_n(x), \quad x \in D; \phi_n(x) = 0, \quad x \in \partial D.$$

$\tau_D(X) = \inf\{t \geq 0 : \; X(t) \notin D\}$ is the first exit time of a process X, and let $\bar{f}(n) = \int_D f(x)\phi_n(x)dx$. The semigroup given by

$$T_D(t)f(x) \;\; = \;\; \mathbb{E}_x[f(B(t))I(\tau_D(B) > t)] = \sum_{n=1}^{\infty} e^{-\mu_n t}\phi_n(x)\bar{f}(n)$$

solves the heat equation in D with Dirichlet boundary conditions:

$$\begin{aligned} \partial_t u(t,x) &= \Delta_x u(t,x), \quad x \in D, \; t > 0, \\ u(t,x) &= 0, \quad x \in \partial D; \quad u(0,x) = f(x), \quad x \in D. \end{aligned}$$

Fractional diffusion in bounded domains

$$\begin{aligned} \partial_t^\beta u(t,x) &= \Delta_x u(t,x); \quad x \in D, \ t > 0 \\ u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D. \end{aligned} \tag{16}$$

has the unique (classical) solution

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)M_\beta(-\mu_n t^\beta) = \int_0^\infty T_D(l)f(x)g_{E(t)}(l)dl \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))] \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B(E)) > t)] \end{aligned}$$

Meerschaert, Nane and Vellaisamy (2009).

Corollary

Mittag-Leffler function is defined by

$$M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

$$||u(t,\cdot)||_{L^2} \sim C M_\beta(-\mu_1 t^\beta) \sim \frac{C}{\mu_1 t^\beta}$$

In the Heat-equation case, since $\beta = 1$ we have $M_\beta(-\mu_1 t^\beta) = e^{-\mu_1 t}$ so

$$||u(t,\cdot)||_{L^2} \sim C M_1(-\mu_1 t) = C e^{-\mu_1 t}$$

New time operators

Laplace symbol: $\psi(s)$	inverse subordinator	time operator
$\int_0^\infty (1 - e^{-sy})\nu(dy)$	$E_\psi(t)$	$\psi(\partial_t) - \delta(0)\nu(t,\infty)$
s^β	$E(t)$	∂_t^β, Caputo
$\int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta)$	$E_\mu(t)$	$\int_0^1 \partial_t^\beta \Gamma(1-\beta)\mu(d\beta)$
$(s+\lambda)^\beta - \lambda^\beta$	$E_\lambda(t)$	$\partial_t^{\beta,\lambda}$ in (17)

$$
\begin{aligned}
\partial_t^{\beta,\lambda} g(t) &= \psi_\lambda(\partial_t) g(t) - g(0)\phi_\lambda(t,\infty) \\
&= e^{-\lambda t}\frac{1}{\Gamma(1-\beta)} d_t\left[\int_0^t \frac{e^{\lambda s} g(s)\, ds}{(t-s)^\beta}\right] - \lambda^\beta g(t) \\
&\quad - \frac{g(0)}{\Gamma(1-\beta)}\int_t^\infty e^{-\lambda r}\beta r^{-\beta-1}\, dr.
\end{aligned} \tag{17}
$$

Subdiffusion: $0 < \beta < 1$, $\mathbb{E}_x(B(E(t)))^2 = \mathbb{E}(E(t)) \approx t^\beta$.
Ultraslow diffusion: For special $\mu \in RV_0(\theta - 1)$ for some $\theta > 0$:
$\mathbb{E}_x(B(E_\mu(t)))^2 = \mathbb{E}(E_\mu(t)) \approx (\log t)^\theta$ for some $\theta > 0$.
Intermediate between subdiffusion and diffusion: Tempered fractional diffusion

$$\mathbb{E}_x(B(E_\lambda(t)))^2 \approx \begin{cases} t^\beta/\Gamma(1+\beta), & t << 1 \\ t/\beta, & t >> 1. \end{cases}$$

$B(E_\lambda(t))$ occupies an intermediate place between subdiffusion and diffusion (Stanislavsky et al., 2008)

New space operators

Laplace exp.: $\psi(s)$	subord.	process	Generator
$\int_0^\infty (1 - e^{-sy})\nu(dy)$	$D_\psi(t)$	$B(D_\psi(t))$	$-\psi(-\Delta)$
s^β	$D(t)$	$B(D(t))$	$-(-\Delta)^\beta$
$(s + m^{1/\beta})^\beta - m$	$T_\beta(t, m)$	$B(T_\beta(t, m))$	$-[(-\Delta + m^{1/\beta})^\beta - m]$
$\log(1 + s^\beta)$	$D_{\log}(t)$	$B(D_{\log}(t))$	$-\log(1 + (-\Delta)^\beta)$

$B(t)$, Brownian motion
$B(D(t))$, symmetric stable process
$B(T_\beta(t, m))$, relativistic stable process
$B(D_{\log}(t))$, geometric stable process

Space-time fractional diffusion in bounded domains

$$\begin{aligned}
[\psi_1(\partial_t) - \delta(0)\nu(t,\infty)]u(t,x) &= -\psi_2(-\Delta_x)u(t,x); \quad x \in D,\ t > 0 \\
u(t,x) &= 0,\ x \in \partial D \ (\text{or } x \in D^C),\ t > 0; \\
u(0,x) &= f(x),\ x \in D.
\end{aligned}$$

has the unique (classical) solution

$$\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)h_{\psi_1}(t,\lambda_n) \\
&= \mathbb{E}_x[f(B(D_{\psi_2}(E_{\psi_1}(t))))I(\tau_D(B(D_{\psi_2}(E_{\psi_1}))) > t)]
\end{aligned}$$

$h_{\psi_1}(t,\lambda) = \mathbb{E}(e^{-\lambda E_{\psi_1}(t)})$ is the solution of

$$[\psi_1(\partial_t) - \delta(0)\nu(t,\infty)]h_{\psi_1}(t,\lambda) = -\lambda h_{\psi_1}(t,\lambda); \quad h_{\psi_1}(0,\lambda) = 1;$$

$$-\psi_2(-\Delta_x)\phi_n(x) = -\lambda_n\phi_n(x);\ \phi_n(x) = 0,\ x \in \partial D \ (\text{or } x \in D^C).$$

$\psi_2(s) = s$ with

- $\psi_1(s) = s^\beta$: subdiffusion
- $\psi_1(s) = \int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta)$: Ultraslow diffusion
- $\psi_1(s) = (s+\lambda)^\beta - \lambda^\beta$: tempered fractional diffusion.

$\psi_2(s) = s^{\beta_2}$ with the three ψ_1s: space-time fractional diffusion, $(-\Delta)^{\beta_2}$

Joint work with Meerschaert, Vellaisamy and Chen (2009,2010, 2012).

Time fractional models in Manifolds

Let $(\mathcal{M}, \mu)$ be a smooth connected Riemannian manifold of dimension $n \geq 1$ with Riemannian metric g. The associated Laplace-Beltrami operator $\triangle = \triangle_{\mathcal{M}}$ in $\mathcal{M}$ is an elliptic, second order, differential operator defined in the space $C_0^\infty(\mathcal{M})$. In local coordinates, this operator is written as

$$\triangle = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right) \tag{18}$$

where $\{g_{ij}\}$ is the matrix of the Riemannian metric, $\{g^{ij}\}$ and g are respectively the inverse and the determinant of $\{g_{ij}\}$.

Let $\mathcal{M}$ be a connected and compact manifold. The unique strong solution to the fractional Cauchy problem

$$\begin{cases} \partial_t^\beta u(m,t) = \triangle u(m,t), \quad m \in \mathcal{M}, t > 0 \\ u(m,0) = f(m), \quad m \in \mathcal{M}, \quad f \in H^s(\mathcal{M}) \end{cases} \tag{19}$$

is given by

$$u(m,t) = \mathbb{E} f(B^m_{E_t}) = \sum_{j=1}^{\infty} M_\beta(-t^\beta \lambda_j)\phi_j(m) \int_{\mathcal{M}} \phi_j(y) f(y) \mu(dy) \tag{20}$$

where B^m_t is a Brownian motion in $\mathcal{M}$ and $E_t = E^\beta_t$ is inverse to a stable subordinator with index $0 < \beta < 1$.

D'Ovidio and Nane (2013).

Fractional Diffusion in the sphere

In the sphere: $x \in \mathbb{S}^2_r$ can be represented as

$$x = (r \sin\vartheta \cos\varphi, r \sin\vartheta \sin\varphi, r \cos\vartheta).$$

The spherical Laplacian is defined by

$$\triangle_{\mathbb{S}^2_r} = \frac{1}{r^2}\left[\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)\right], \quad \vartheta \in [0,\pi],\ \varphi \in [0,2\pi].$$

Sometimes $\triangle_{\mathbb{S}^2_r}$ is called Laplace operator on the sphere. In this case the eigenfunctions of the spherical Laplacian are called Spherical harmonics obtained from Legendre polynomials.

We can use $B^m(E_t)$ as a process to introduce time dependence in a Gaussian random field T to model "Cosmic Microwave Background (CMB) radiation". CMB radiation is thermal radiation filling the observable universe almost uniformly and is well explained as radiation associated with an early stage in the development of the universe.

A further remarkable feature is that the new random field $T(B^m(E_t))$ has short-range dependence for $\beta \to 1$ and long-range dependence for $\beta \in (0, 1)$.

Further research

- Work in progress for other Subordinated Brownian motions, e.g. relativistic stable process as the outer process.... corresponding to the operator $(-\Delta + m^{1/\beta})^\beta - m$ for $0 < \alpha \le 2,\ m \ge 0$.
- Extension to Neumann boundary conditions...
- Extensions to anisotropic/nonsymmetric space operators..
- Fractal properties of $B(E(t))$ and other time-changed processes
- Applications-interdisciplinary research
- solutions to the spde with space time white noise ξ:

$$\partial_t^\beta u(t,x) \ = \ \partial_x^2 u(t,x) + \xi;\ x \in \mathbb{R},\ t > 0;\ u(0,x) = f(x),\ x \in \mathbb{R}.$$

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Thank You!