Hitting probabilities for general Gaussian processes

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NSF/CBMS Conference: Analysis of SPDEs

Michigan State University, August 2013

A general class of Gaussian processes

- Let $B = (B(t), t \in \mathbb{R}_+)$ be a centered continuous Gaussian process in $\mathbb{R}$ such that for some $\ell \geq 1$, some continuous strictly increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_0 \gamma = 0$, and for all $s, t \in \mathbb{R}_+$,

$$\frac{1}{\ell}\gamma^2(|t-s|) \leq \mathrm{E}[|B(t) - B(s)|^2] \leq \ell\gamma^2(|t-s|). \tag{1}$$

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$$\mathrm{Var} B\,(t) = \gamma^2\,(t)\,. \tag{2}$$

- We use the same notation B to designate a vector of d iid copies of the scalar version of B.
- Note that γ does not define the law of B since distinct processes with the same variance function γ may satisfy (1).

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- Fractional Brownian motion with Hurst parameter $H \in (0, 1)$.
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- Solutions to the fractional stochastic heat equation with additive noise whose space behavior is of Riesz-kernel type :

$$\partial_t u = \frac{1}{2}\Delta u + \dot{W}^{H,\alpha}, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad u(0,x) = 0,$$

  with $\alpha \in [0, d)$. The solution exists if and only if $d < 4H + \alpha$.
  In this case, $\gamma(t) = t^{H - \frac{d-\alpha}{4}}$.

Examples II

- Volterra processes defined as

$$B^{\gamma}(t) := \int_0^t \sqrt{\left(\frac{d\gamma^2}{dt}\right)(t-s)}dW(s),$$

where γ^2 is of class $\mathcal{C}^2(\mathbb{R}_+ \setminus 0)$, $\lim_0 \gamma = 0$, and γ^2 is increasing and concave ($d\gamma^2/dr$ is non-increasing).
In this case, (1) and (2) hold with $\ell = 2$ and γ.

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- Nualart and Viens'09 : Hitting probabilities for the stochastic heat equation :

$$\partial_t u = \frac{1}{2}\Delta u + \dot{W}^{H}, \quad t \geq 0, \quad x \in S^1.$$

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$$\gamma(r) = \gamma_{H,\beta}(r) := r^H \log^\beta(\frac{1}{r}),$$

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- Let B in $\mathbb{R}^d$ satisfying (1) and (2).

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- Theorem 1 : If γ is concave near the origin and $\gamma'(0+) = +\infty$, then for all $0 < a < b$ and $M > 0$, there exists $C(a, b, M, B) > 0$ such that for any Borel set $A \subset [-M, M]^d$

$$C\mathcal{C}_{\mathrm{K}}(A) \leq \mathrm{P}(B([a, b]) \cap A \neq \varnothing),$$

where $\mathcal{C}_{\mathrm{K}}(A)$ denotes the capacity of the set A with respect to K

$$\mathcal{C}_{\mathrm{K}}(A) := \left[\inf_{\mu \in \mathcal{P}(A)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{K}(|x - y|)\, \mu(dx)\, \mu(dy)\right]^{-1},$$

where $\mathcal{P}(A)$ denotes the set of probability measures with support in A, and K is the potential kernel

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- Remark : If $\gamma(r) = r^H$, then $dH > 1$ if and only if $\mathrm{K}(x) \asymp x^{-d+\frac{1}{H}}$.

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Proof of Theorem 1 : Two-point local nondeterminism

- Hypothesis 1 : For all $0 < a < b < \infty$, there exists $\varepsilon > 0$ and $c_0 \in (0, 1/\sqrt{\ell})$, such that for all $s, t \in [a, b]$ with $0 < t - s \leq \varepsilon$,

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- Second moment argument similar as in Biermé, Lacaux and Xiao'09. See also Testard'86.

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Then there exist constants $L(\gamma, y_0)$ and $t_0(B)$ such that for all $z \in \mathbb{R}$ and for all a, b such that $0 < a < b$ and $b - a \leq t_0$,
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- Example : $\mathrm{ind}\,\gamma_{H,\beta} = H$ for all $\beta \in \mathbb{R}$.

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$$\mathrm{P}(B([a,b]) \cap B(z\,,\varepsilon) \neq \emptyset) \leq \left(\varepsilon \frac{2\kappa\gamma\,(b)}{\gamma^2\,(a)} \left(1 + \frac{1}{F\,(|z|)}\right) + \frac{L\sqrt{\ell}}{\gamma\,(a)}\gamma\,(b-a) \right)^d,$$

where $\kappa := \mathrm{P}\left[\inf_{[a,b]} B > \gamma\,(b)\right]$, $F\,(z) := 1$ for $z \leq \gamma\,(b)$ and $F\,(z) := 1 - e^{-2\mathrm{arctanh}\left(\gamma^2(b)/z^2\right)}$ for $z > \gamma\,(b)$

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- Note that $\kappa > 0$ because B is Gaussian and $a > 0$.

Upper bound : probability hitting small balls

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- Proposition 2 : Under assumption (3), for all $0 < a < b < \infty$, with $b - a$ small enough, and for all $z \in \mathbb{R}^d$ and $\epsilon > 0$,

$$\mathrm{P}(B([a,b]) \cap B(z,\varepsilon) \neq \emptyset) \leq \left(\varepsilon \frac{2\kappa\gamma(b)}{\gamma^2(a)} \left(1 + \frac{1}{F(|z|)}\right) + \frac{L\sqrt{\ell}}{\gamma(a)} \gamma(b-a) \right)^d ,$$

where $\kappa := \mathrm{P}\left[\inf_{[a,b]} B > \gamma(b)\right]$, $F(z) := 1$ for $z \leq \gamma(b)$ and $F(z) := 1 - e^{-2\operatorname{arctanh}\left(\gamma^2(b)/z^2\right)}$ for $z > \gamma(b)$

- Note that $\kappa > 0$ because B is Gaussian and $a > 0$.
- Note that, since $\operatorname{arctanh}(x)$ is equivalent to x for x small, for large z, $1/F(z)$ is equivalent to $z^2/\left(2\gamma^2(b)\right)$.

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- To prove this, it would be sufficient to show that, Z has a bounded density on $(0, +\infty) \longrightarrow$ Malliavin calculus !

Nourdin-Viens'09 criterion

- Let X be a centered random variable in $\mathbb{D}^{1,2}$. Define

$$G_X := \int_0^{+\infty} du e^{-u} \mathrm{E}\left[\int_{\mathbb{R}_+} D_r X \; D_r^{(u)} X^{(u)} dr \middle| \mathcal{F} \right] \left(= \langle DX, -DL^{-1}X \rangle_H \right)$$

  where $X^{(u)}$ denotes a random variable with the same law as X, but constructed using a copy B^u of B such that $\mathrm{Corr}\,(B, B^u) = e^{-u}$.

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- Proposition 3 : Then $\mathrm{supp}(X) = [\alpha, \beta]$, $-\infty \leq \alpha < 0 < \beta \leq +\infty$. Assume there exists $\alpha' \in (\alpha, 0)$ such that $g_X(x) > 0$ for all $x \in [\alpha', \beta)$. Then X has a density ρ on $[\alpha', \beta)$, and for almost every $z \in [\alpha', \beta)$,

$$\rho(x) = \frac{\mathrm{E}\left[|X|\right]}{2g_X(x)} \exp\left(-\int_0^x ydy/g_X(y)\right).$$

Application of Nourdin-Viens'09 criterion

- Since Z is positive and B is continuous with a symmetric law, it suffices to study the random variable G_X relative to

$$X := \inf_{s\in[a,b]} (B(s) - z)_+ - \mu, \quad \mu := \mathrm{E}\left[\inf_{s\in[a,b]} (B(s) - z)_+\right].$$

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- Several computations led to

$$\begin{aligned} g_X(x) &\geq \frac{\gamma^2(a)}{2} \int_0^\infty du\, e^{-u} \tilde{\mathrm{P}}\left[\min_{[a,b]} \tilde{B} > z\sqrt{\tanh(u/2)}\right] \\ &\geq \frac{\gamma^2(a)}{2\kappa} F(z). \end{aligned}$$

Hitting probabilities : upper bound

- Theorem 3 : Assume that

Then for all $0 < a < b < \infty$, any $M > 0$, there exists a constant $C(a, b, B, M) > 0$ such that for any Borel set $A \subset [-M, M]^d$,

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- Recall Theorem 1 : $C\mathcal{C}_{\mathrm{K}}(A) \leq \mathrm{P}(B([a,b]) \cap A \neq \varnothing)$

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- Proposition 4 : If $\lim_{r\to 0} \frac{r\gamma'(r)}{\gamma(r)}$ exists, then

$$\mathrm{K} \asymp 1/\varphi \quad \Longleftrightarrow \quad d > 1/\lim_{r\to 0} \frac{r\gamma'(r)}{\gamma(r)}.$$

Necessary condition to hit points

- Lemma 3 : Assume that $\gamma(r) = o(r^{1/d})$ near 0, that $\varphi(s) := s^d/\gamma^{-1}(s)$ is non-decreasing near 0 and Condition (3) holds.

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