Systems of Stochastic Evolution Equations With Constant Coefficients

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What do you think:

- Equation $u_{tt} = u_{txx} + u_{xx}$ is
  (a) hyperbolic (b) parabolic (c) neither
- System $du = u_{xx}dt,\ dv = v_{xx}dt + \sigma u_x dw$ is parabolic for
  (a) all σ (b) no σ (c) $|\sigma| < \sqrt{2}$

Motivation 1

We know that equation $u_t = u_{xx}$ is parabolic and equation $u_{tt} = u_{xx}$ is hyperbolic.

Equation $u_{tt} = u_{txx} + u_{xx}$ is

(a) hyperbolic

(b) parabolic

(c) neither

Motivation 2

We know that the equation $du = u_{xx}dt + \sigma u_x dw(t)$ is parabolic for $|\sigma| \le \sqrt{2}$.

System

$$
\begin{aligned}
du &= u_{xx}dt,\\
dv &= v_{xx}dt + \sigma u_x dw
\end{aligned}
$$

is parabolic for

(a) all σ

(b) no σ

(c) $|\sigma| < \sqrt{2}$

What type is this?

$$u_{tt} = u_{txx} + u_{xx}:$$

Hyperbolic?

$$\|u_t(t)\|^2 + \|u_x(t)\|^2 \le \|u_t(0)\|^2 + \|u_x(0)\|^2.$$

Parabolic? With $v = u_t$ get $v_t = v_{xx} + u_{xx}$.

Neither!

When is this parabolic?

$$du = u_{xx}dt, \quad dv = v_{xx}dt + \sigma u_x dw.$$

(a) For all σ:

$$u_0, v_0 \in L_2 \;\Rightarrow\; u \in L_2((0,T); H^1) \;\Rightarrow\; \sigma u_x \in L_2((0,T) \times \mathbb{R})$$
$$\Rightarrow\; v \in L_2(\Omega \times (0,T); H^1)$$

(b) For $|\sigma| < \sqrt{2}$:

$$\mathbb{E}\int_0^t \left(\|u_x\|_0^2 + \|v_x\|_0^2\right) ds \le \|u_0\|_0^2 + \|v\|_0^2 + \frac{\sigma^2}{2}\mathbb{E}\int_0^t \|u_x\|_0^2 ds$$

Heat Equation

$u_t = au_{xx} + \sigma u_x \dot{w}(t) : \quad 2a \geq \sigma^2$

(a) $\mathbb{E}\|u(t)\|^2 = \mathbb{E}\|u(0)\|^2 - \left(a - \frac{\sigma^2}{2}\right) \int_0^t \mathbb{E}\|u_x(s)\|^2 ds.$

(b) $\widehat{u}_t = -ay^2 \widehat{u} + \mathfrak{i} y \sigma \widehat{u} \dot{w}(t), \quad \mathfrak{i} = \sqrt{-1}.$

$$\widehat{u}(t,y) = \widehat{u}(0,y) \exp\left(-\left(a - \frac{\sigma^2}{2}\right) t y^2 + \mathfrak{i} y \sigma w(t)\right)$$
$$\mathbb{E}|\widehat{u}(t,y)|^2 = \mathbb{E}|\widehat{u}(0,y)|^2 \exp\left(-\left(a - \frac{\sigma^2}{2}\right) t y^2\right)$$

Generalization 1

$V \hookrightarrow H \hookrightarrow V'$, $\mathrm{A} : V \to V'$, $\mathrm{B} : V \to H$.

$\dot{u} = \mathrm{A}u + \mathrm{B}u\,\dot{w}(t)$.

Energy method:

$$\mathbb{E}\|u(t)\|_H^2 = \mathbb{E}\|u(0)\|_H^2 + \int_0^t \mathbb{E}\Big(2[\mathrm{A}u(s), u(s)] + \|\mathrm{B}u(s)\|_H^2\Big)ds.$$

Stochastic Parabolicity Condition:

$$2[\mathrm{A}u, u] + \|\mathrm{B}u\|_H^2 \le -\delta\|u\|_V^2 + c\|u\|_H^2,\ \ \delta > 0.$$

Well-posed: $c \in \mathbb{R}$
(Exponentially) Stable: $c < 0$.

A digression

Scalar PDO A with constant coefficients and order 2:

$$\mathrm{A}f(x) = \sum_{i,j=1}^{\mathrm{d}} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{\mathrm{d}} b_i \frac{\partial f}{\partial x_i} + cf$$

$$\widehat{\mathrm{A}f}(y) = \Big(- \sum_{i,j=1}^{\mathrm{d}} a_{ij} y_i y_j + \mathfrak{i} \sum_{i=1}^{\mathrm{d}} b_i y_i + c \Big) \widehat{f}(y) = A(y) \widehat{f}(y).$$

Matrix ΨDO A with constant coefficients and order m:

$\widehat{\mathrm{A}\boldsymbol{f}}(y) = A(y)\widehat{\boldsymbol{f}}(y)$ (constant coefficients),
$|A(y)| \sim |y|^m,\ m > 0,\ |y| \to +\infty$ (order m),
$A = A(y)$ is continuous (for technical reasons).
Sobolev spaces $H^\gamma(\mathbb{R}^{\mathrm{d}})$:
$\|\boldsymbol{f}\|_\gamma^2 = \int_{\mathbb{R}^{\mathrm{d}}} (1 + |y|^2)^\gamma |\widehat{\boldsymbol{f}}(y)|^2 dy < \infty.$

Generalization 2

$u_t = \mathrm{A}u + \mathrm{B}u\,\dot{w}(t)$, A and B are PDO or ΨDO.
Fourier transform:

$$\widehat{u}_t = A(y)\widehat{u} + B(y)\widehat{u}\,\dot{w}$$

$$\widehat{u}(t,y) = \widehat{u}_0(y)\exp\Big(\big(A(y) - \frac{1}{2}B^2(y)\big)t + B(y)w(t)\Big)$$

$$|\widehat{u}(t,y)| = |\widehat{u}_0(y)|\exp\Big(\Re\big(A(y) - \frac{1}{2}B^2(y)\big)t + \Re\big(B(y)\big)w(t)\Big)$$

$$\mathbb{E}|\widehat{u}(t,y)|^2 = \mathbb{E}|\widehat{u}_0(y)|^2\exp\bigg(\Big(\Re\big(2A(y) - B^2(y)\big) + 2\big(\Re B(y)\big)^2\Big)t\bigg).$$

Well-posed: $2\Re\big(A(y)\big) + |B(y)|^2 \le c.$

Neutrally Stable: $2\Re\big(A(y)\big) + |B(y)|^2 \le 0.$

Exponentially Stable: $2\Re\big(A(y)\big) + |B(y)|^2 \le \delta < 0.$

The (A, B) system: definitions

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \big(\mathrm{A}\boldsymbol{u}(s) + \boldsymbol{f}(s)\big) ds + \int_0^T \big(\mathrm{B}\boldsymbol{u}(s) + \boldsymbol{g}(s)\big) dw(s),\ 0 \le t \le T.$$

Well posed: $\boldsymbol{u}_0, \boldsymbol{f}, \boldsymbol{g} \in H^r \ \Rightarrow \boldsymbol{u}(t) \in L_2(\Omega; H^\gamma)$ and

$$\mathbb{E}\|\boldsymbol{u}(t)\|_\gamma^2 \le C(T)\left(\|u_0\|_r^2 + \int_0^t \|f(s)\|_r^2 ds + \int_0^t \|g(s)\|_r^2 ds\right).$$

Stable: $C(T) \le C_0$;

Neutrally Stable: $\mathbb{E}\|\boldsymbol{u}(t)\|_\gamma^2 \le C_0\|u_0\|_r^2$ when $\boldsymbol{f} = \boldsymbol{g} = 0$.

Theorem

The (A,B) system is well posed if and only if

$$\Re\lambda\left(\overline{A(y)}\otimes I+I\otimes A(y)+\overline{B(y)}\otimes B(y)\right)\leq C_0\ln(2+|y|).$$

The (A,B) system is stable if and only if

$$\Re\lambda\left(\overline{A(y)}\otimes I+I\otimes A(y)+\overline{B(y)}\otimes B(y)\right)\leq -\delta<0.$$

What is $\otimes$?

If $A, B \in \mathbb{C}^{n\times n}$, then $A \otimes B$ is n^2-by-n^2 block matrix with blocks $A_{ij}B$. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b & 2a & 2b \\ c & d & 2c & 2d \\ 3a & 3b & 4a & 4b \\ 3c & 3d & 4c & 4d \end{pmatrix}$$

What for? $BXA^\top = C \Leftrightarrow (A \otimes B)\mathrm{vec}(X) = \mathrm{vec}(C)$;

$$\mathrm{vec}(X) = \boldsymbol{X} = (X_{11}, \ldots, X_{n1}, X_{12}, \ldots, X_{n2}, \ldots, X_{1n}, \ldots, X_{nn})^\top.$$

That is,

$$\mathrm{vec}\big(BXA^\top\big) = (A \otimes B)\mathrm{vec}(X).$$

Why $\otimes$?

Consider

$$\dot{\boldsymbol{v}}(t,y) = A(y)\boldsymbol{v} + \widehat{\boldsymbol{f}}(t,y) + \big(B(y)\boldsymbol{v} + \widehat{\boldsymbol{g}}(t,y)\big)\dot{w}.$$

Hope: $\boldsymbol{v} = \widehat{\boldsymbol{u}}$.

Fact 1: If $\displaystyle\int_{\mathbb{R}^{\mathrm{d}}} (1+|y|^2)^{\gamma}\ \mathbb{E}|\boldsymbol{v}(t,y)|^2 dy < \infty$,
then $\boldsymbol{v} = \widehat{\boldsymbol{u}}$.

Fact 2: if $\boldsymbol{f} = \boldsymbol{g} = 0$ and $\boldsymbol{U} = \mathrm{vec}\big(\mathbb{E}\boldsymbol{v}\boldsymbol{v}^*\big)$ then

$$\dot{\boldsymbol{U}} = \Big(\overline{A(y)} \otimes I + I \otimes A(y) + \overline{B(y)} \otimes B(y)\Big)\boldsymbol{U}.$$

Without $\otimes$

$$\dot{\boldsymbol{v}} = A\boldsymbol{v} + B\boldsymbol{v}\dot{w} \quad (\boldsymbol{f} = \boldsymbol{g} = 0).$$

Try $\mathfrak{v} = \mathbb{E}|\boldsymbol{v}|^2 = \mathbb{E}\boldsymbol{v}^*\boldsymbol{v}$:

$$\dot{\mathfrak{v}} = \boldsymbol{v}^*\Big(A(y) + A^*(y) + B^*(y)B(y)\Big)\boldsymbol{v}.$$

Not an equation for $\mathfrak{v}$, so will not get a necessary condition.
Can get a sufficient one, though:

$$\lambda\Big(A(y) + A^*(y) + B^*(y)B(y)\Big) \leq C.$$

More or less the same as you would get integrating by parts.

Bottom line

For the (A, B) system

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \big(\mathrm{A}\boldsymbol{u}(s) + \boldsymbol{f}(s)\big)ds + \int_0^T \big(\mathrm{B}\boldsymbol{u}(s) + \boldsymbol{g}(s)\big)dw(s),$$

the *obvious* characteristic matrix is

$$A(y) + A^*(y) + B^*(y)B(y),$$

but the *correct* characteristic matrix is

$$\mathcal{M}_{A,B}(y) = \overline{A(y)} \otimes I + I \otimes A(y) + \overline{B(y)} \otimes B(y).$$

Stochastic Parabolic System

The (A, B) system is `parabolic` of order $2p$ if there exists a positive number p such that

$$0 < \lim_{|y|\to\infty} \frac{\|A(y)\| + \|B^*(y)B(y)\|}{|y|^{2p}} < \infty;$$

$$\lambda\big(\mathcal{M}_{A,B}(y)\big) \le -\varepsilon|y|^{2p} + L,\ \varepsilon > 0,\ L \in \mathbb{R}.$$

A modification of the original example

$u_t = u_{xx},\ v_t = v_{xx} + \mathrm{B}u\,\dot{w},$
$B^2(y) = \sigma|y|^{2\beta},\ \sigma \in \mathbb{R},\ \beta \geq 0.$ Then

$$\mathcal{M}_{A,B}(y) = \begin{pmatrix} -2y^2 & 0 & 0 & 0 \\ 0 & -3y^2 & 0 & 0 \\ 0 & 0 & -3y^2 & 0 \\ \sigma|y|^{2\beta} & 0 & 0 & -4y^2 \end{pmatrix}.$$

The system is

- □ Well posed for all β and σ;
- □ Parabolic of order 2 for $\beta \leq 1$ and all σ.

Why no $\otimes$ in deterministic case?

There is:

$$\mathcal{M}_{A,0}(y) = \overline{A(y)} \otimes I + I \otimes A(y),$$

known as Kronecker sum, but

$$\lambda_{i,j}\left(\overline{A(y)} \otimes I + I \otimes A(y)\right) = \overline{\lambda_i(A)} + \lambda_j(A),$$

AND know from linear algebra that an upper bound on $\Re\lambda(A)$ is weaker than a similar bound on $\lambda(A + A^*)$.

Routh-Hurwitz criterion: real case

$p(z) = c_0 z^n + c_1 z^{n-1} + \ldots + c_{n-1} z + c_n$, $c_k \in \mathbb{R}$, $c_0 > 0$.

Objective: if $p(z) = 0$, then $\Re z < 0$.

Result: need positive principal minors of

$$\begin{vmatrix} c_1 & c_3 & c_5 & \ldots \\ c_0 & c_2 & c_4 & \ldots \\ 0 & c_0 & c_2 & \ldots \\ 0 & c_1 & c_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

that is, determinants of orders 1, 2, $\ldots$, n.

Routh-Hurwitz criterion: complex case

Complex case: $p(z) = c_0 z^n + c_1 z^{n-1} + \ldots + c_{n-1} z + c_n$, $c_0 \neq 0$.

Step 1. $p(\mathfrak{i}z) = \mathfrak{i}(a_0 z^n + a_1 z^{n-1} + \ldots) + (b_0 z^n + b_{n-1} z^{n-1} + \ldots)$.

Step 2. Need positive principal minors of *even* orders of

$$\begin{vmatrix} a_0 & a_1 & a_2 & \ldots \\ b_0 & b_1 & b_2 & \ldots \\ 0 & a_0 & a_1 & \ldots \\ 0 & b_0 & b_1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

that is, determinants of orders 2, 4, $\ldots$, $2n$.

Cannot decrease complexity: Anderson and Jury, 1977.

Second order: deterministic case

$$u_{tt} + \mathrm{A}u_t + \mathrm{M}u = 0, \quad z^2 + A(y)z + M(y) = 0.$$

Recall: no need for $\otimes$.
Real case: need $A(y) \geq -c$, $M(y) \geq -c$, where
$c \in \mathbb{R}$ to be well posed;
$c < 0$ to be stable.
Example of no stability (effect of the double root).
$u_{tt} - 2u_{txx} + u_{xxxx} = 0$, $z = -y^2$,
$u(0, x) = 0$, $\widehat{u}_t(0, y) = e^{-y^2}$; $u_t(0) \in H^\gamma$ for all $\gamma \in \mathbb{R}$;
$\widehat{u}(t, y) = te^{-y^2(1+t)}$, $\|u(t)\|_\gamma^2 \geq Ct^{3/2}$ for all $\gamma \in \mathbb{R}$.

Complex case: Routh-Hurwitz of order 4.
Example: $u_{tt} - \gamma u_{txx} - c^2 u_{xx} + a u_{xxx} = 0$, $z^2 + \gamma y^2 z + c^2 y^2 - \mathrm{i} a y^3 = 0$.
Well-posed: $\gamma > 0$, Neutrally stable: $|a| < \gamma|c|$. NOT neutrally stable if $|a| = \gamma|c|$.

Second order: stochastic case

Complexity: order 3 for real symbols, order 8 for complex symbols.
Example: $u_{tt} = \gamma u_{txx} + c^2 u_{xx} + \sigma u_{xx} \dot{w}(t)$.

$$\mathcal{M}_{A,B}(y) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -c^2y^2 & -\gamma y^2 & 0 & 1 \\ -c^2y^2 & 0 & -\gamma y^2 & 1 \\ \sigma^2 y^4 & -c^2y^2 & -c^2y^2 & -2\gamma y^2 \end{pmatrix},$$

$$p(z) = (z + \gamma y^2)\big(z^3 + 3\gamma y^2 z^2 + (4c^2y^2 + 2\gamma^2 y^4)z + 2(2\gamma c^2 - \sigma^2)y^4\big)$$

□ Well-posed: $\gamma > 0$;
□ Stable: $2\gamma c^2 > \sigma^2$ (cf. R. Z. Khasminiskii, Stochastic stability of differential equations, 2nd Ed, 2012, Sec. 6.10)
□ Neutrally stable: $2\gamma c^2 = \sigma^2$.

Ultimate goal

Variable and adapted coefficients, adapted input:

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \big(\mathrm{A}\boldsymbol{u}(s) + \boldsymbol{f}(s)\big)ds + \sum_{k\geq 1}\int_0^T \big(\mathrm{B}_k\boldsymbol{u}(s) + \boldsymbol{g}_k(s)\big)dw_k(s),$$

$$\sum_k \|B_k(\omega, t, x; y)\|^2 \leq C(1+|y|)^b.$$

Characteristic matrix:

$$\mathcal{M}_{A,B}(\omega, t, x; y) = \overline{A}\otimes I + I \otimes A + \sum_k \overline{B}_k \otimes B_k.$$

Well posed: $\Re\lambda\mathcal{M}_{A,B} \leq C_0 \ln(2+|y|)$;
Exponentially stable: $\Re\lambda\big(\mathcal{M}_{A,B}\big) \leq -\delta < 0$
Parabolic: $\Re\lambda\big(\mathcal{M}_{A,B}\big) \leq -\varepsilon|y|^{2p} + L$.

Some history

Leopold Kronecker: 1823–1891. Famous quote: " God created the integers, all else is the work of man".

First appearance of $\otimes$:
Johann Georg Zehfuss Ueber eine gewisse Determinante, Zeitschrift fr Mathematik und Physik, 3, 1858.

$\otimes$ also known as direct product or Zehfuss product.

Conclusion

Kronecker product rules!