A Phase Transition for Measure-valued SIR Epidemic Processes

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Based on Joint Work with Steve Lalley and Ed Perkins

Outline

Spatial SIR model in $\mathbb{Z}^d$

- Each site $x \in \mathbb{Z}^d$ represents a village
- N individuals on each site x — N: village size
- People can be *susceptible*, *infected* or *recovered*
- Recovery $\Rightarrow$ immunity — e.g., measles
- Infected recovers after one unit of time
- Infected infects *neighboring* susceptibles with certain probability p_N
- Critical case: $p_N = 1/((2d+1)N)$

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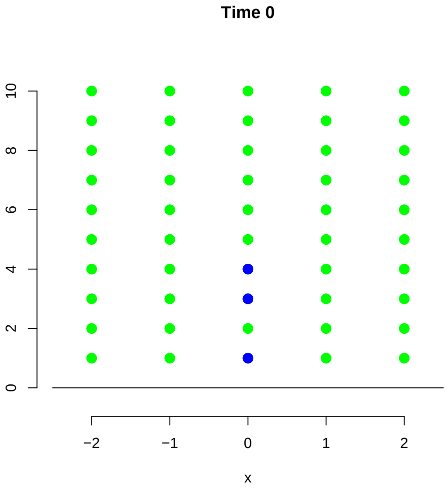
SIR in $\mathbb{Z}^1$

Village size $N = 10$: susceptible, infected or recovered



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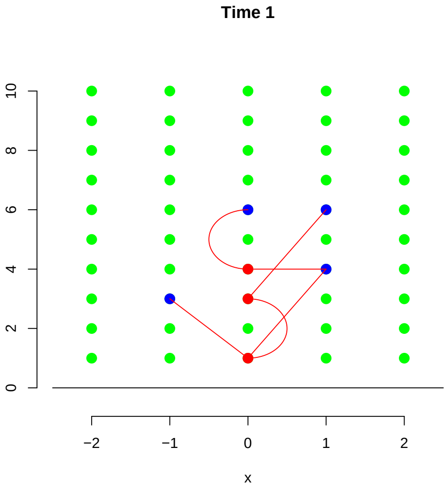

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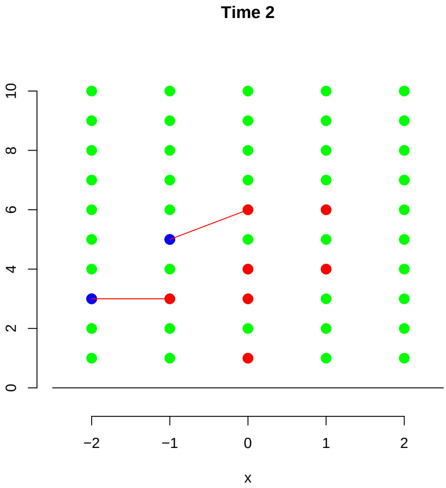

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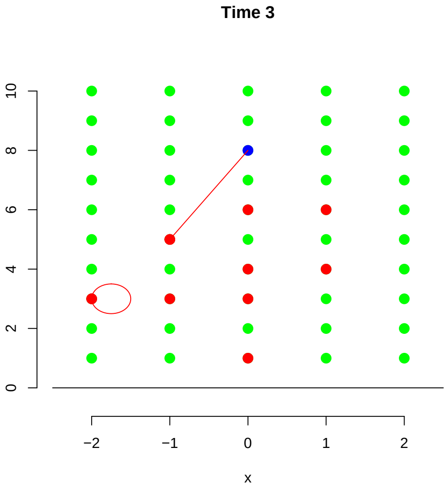

Convergence of SIR When $d \leq 3$

Theorem

[Lalley(2009), Lalley and Zheng(2010)] If for $\alpha = 2/(6-d)$, the initial infections $\mu^N := X_0^N$ are such that

$$\frac{\mu^N(\sqrt{N^\alpha}\cdot)}{N^\alpha} \Rightarrow \mu \text{ with compact support}, \quad \text{as } N \to \infty,$$

Then under some regularity assumptions, as $N \to \infty$,

(i)

$$\left(\frac{R^N_{N^\alpha t}(\sqrt{N^\alpha}\cdot)}{N^{(4-d)/(6-d)}}\right) \Longrightarrow (L_t(\cdot)) \text{ in } D([0,\infty); C(\mathbb{R}^d)),$$

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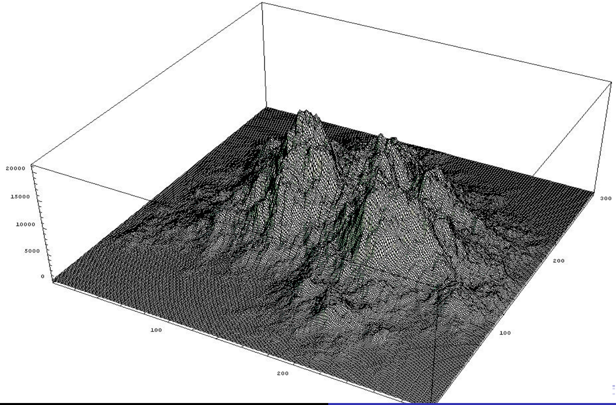
What the $L_t(\cdot)$ Looks Like: the Distribution of Recovered Individuals

Village size $N = 10^8$, IC = one infected individual/site in $[-50, 50]^2$.
Distribution of recovered at time = 10^4: number of recovered per site is of order $10^4 = \sqrt{N}$



Convergence of SIR When $d \leq 3$, *ctd*

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[Lalley(2009), Lalley and Zheng(2010)] (ii) Moreover,

$$\frac{X^N_{N^\alpha t}(\sqrt{N^\alpha}\cdot)}{N^\alpha} \Longrightarrow X_t \text{ in } D([0,\infty);\mathcal{M}_F(\mathbb{R}^d)),$$

where X_t satisfies that for each $\varphi \in C^2_b(\mathbb{R}^d)$,

$$X_t(\varphi) - \mu(\varphi) = \frac{1}{2}\int_0^t X_s(\Delta\varphi)\,ds - \int_0^t X_s(L_s\varphi)\,ds + M_t(\varphi);$$

(iii) $L_t(\cdot)$ is the local time density of X_t, i.e., the density of $\int_0^t X_s\,ds$, and $M_t(\varphi)$ is a martingale with quadratic variation $[M(\varphi)]_t = \int_0^t X_s(\varphi^2)\,ds$.

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- Q: Can the process survive forever?
- Total mass bounded by a drift-less Feller diffusion $\Rightarrow$ almost sure extinction
- If the infection probability were slightly bigger, $= (1+\theta/N^{2/(6-d)})/((2d+1)N)$ for some $\theta > 0$, then the limit process would be

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- Also arises as a scaling limit of certain stochastic reaction-diffusion systems [Mueller and Tribe(2011)]
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- Constant drift term: θ
- *Increasing* killing term: L_t
    - Recall L_t is the density of $\int_0^t X_s\, ds$
- Bigger θ gives bigger drift term, but also *accelerates the accumulation of the killing term!*
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A Phase Transition

Theorem

[Lalley, Perkins, and Zheng(2013+)] For $d = 2$ or 3, there exist critical values $\theta_c = \theta_c(d) > 0$ such that the following holds.

(i) *if $\theta > \theta_c$, then*

$$P(X \text{ survives forever}) > 0.$$

(ii) *if $\theta < \theta_c$, then*

$$P(X \text{ dies out}) = 1.$$

For $d = 1$, for any θ, X dies out almost surely.

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Universality of Critical Values

- In dimensions $d = 2, 3$, the critical values θ_c do not depend on the initial mass μ
- Usually this follows from the Markov property, and the absolute continuity of laws of X_1 under different P^μ's
- In our case the Markov property only holds for (X_t, L_t):

$$X_t(\varphi) - \mu(\varphi)$$
$$= \frac{1}{2}\int_0^t X_s(\Delta\varphi)\,ds + \theta\int_0^t X_s(\varphi)\,ds - \int_0^t X_s(L_s\varphi)\,ds + M_t(\varphi),$$

  whose law is *not* absolutely continuous under different P^μ's
  - To see this, consider the supports of L_1 and X_0
- Alternatives?

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Weak Local Extinction

Proposition

There exists $\kappa < \infty$ *such that for any* $\theta \in \mathbb{R}, \gamma > 0$*, and any* $R \geq 1$*,*

$$E\langle L_\infty, \mathbf{1}_{B_R(0)}\rangle \leq \frac{2|\mu|}{\kappa + 2\theta^+} + V_d(\kappa + 2\theta^+)(R+1)^d < \infty.$$

In particular, $X_t(B_R(0)) \to 0$ *almost surely.*

Proof.

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Proposition

The critical value $\theta_c = \theta_c(d, \mu)$ depends only on the dimension d and not on the choice of $0 \neq \mu$.

Proof.

- Local extinction $\Rightarrow$ introducing a compactly supported killing term K won't affect survival or not:
$$X_t(\varphi) = \mu(\varphi) + \int_0^t X_s\left(\Delta\varphi/2 + \theta\varphi - K\varphi - L_s(X)\varphi\right)\, ds + M_t(\varphi)$$
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Proof.

- Local extinction $\Rightarrow$ introducing a compactly supported killing term K won't affect survival or not:
$$X_t(\varphi) = \mu(\varphi) + \int_0^t X_s\left(\Delta\varphi/2 + \theta\varphi - K\varphi - L_s(X)\varphi\right) ds + M_t(\varphi)$$
- L_1 is compactly supported $\Rightarrow$ survival only depends on X_1, and is independent of L_1
- Absolute continuity of $\mathcal{L}(X_1)$ under different P^μ's.

□

Extinction in Dimension One

Proposition

If $d = 1$ then for every $\theta \in \mathbb{R}$ and every initial measure μ, X_t dies out almost surely.

Proof.

- In order to survive forever, $|X_t|$ must blow up
- Also can show that X_t spreads out at most linearly, in other words, almost surely, X_t is contained in $[-Ct, Ct]$ for all t sufficiently large
- Hence, if survival,

$$\langle L_N, \mathbf{1}_{[-CN,CN]} \rangle = \int_0^N \langle X_t, \mathbf{1}_{[-CN,CN]} \rangle \, dt \approx \int_0^N |X_t| \, dt$$

  must grow in N faster than linear rate
- This contradicts the Local extinction lemma, which says that when $d = 1$, $E\langle L_\infty, \mathbf{1}_{[-CN,CN]} \rangle$ grows at most linearly in N

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Proofs of Extinction/Survival

- Extinction for $\theta > 0$ small: By comparison to subcritical branching following [Mueller and Tribe(1994)] who proved a phase transition for a SPDE arising as the limit of one-dimensional contact process;
- Survival for θ large ($d = 2, 3$): By comparison with supercritical oriented percolation.
  - Key is to control the growth and spread of local time L to show infection can propagate ahead of local time wave;
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Thank you!

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