A Tanaka formula for the derivative of self-intersection local time for fBm

Paul Jung (University of Alabama Birmingham)
Greg Markowsky (Monash University, Melbourne)

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Self-intersection local time

Brownian local time is formally defined by

$$L(t,x) := \int_0^t \delta(x - B_s)\, ds.$$

Two related processes are the Intersection Local Time:

$$\int_0^t \int_0^t \delta(B_s - \tilde{B}_r)\, dr\, ds$$

and Self-intersection Local Time (SLT):

$$\int_0^t \int_0^s \delta(B_s - B_r)\, dr\, ds$$

SLT is used in physics to study polymers, the polaron, and QFT [see X. Chen (2008, 2010)].

The Derivative of Self-intersection Local Time (DSLT)

Our interested is a formal Derivative of SLT:

$$\alpha'(t) := -\frac{1}{2}\int_0^t \int_0^s \delta'(B_s - B_r)\, dr\, ds$$

- Introduced by Rogers/Walsh (90, 91a, 91b) for studying stochastic area integrals w.r.t. local time.
- Rosen (2005) and Markowsky (2008) showed a Tanaka formula:

$$\alpha'(t) = \int_0^t L_s^{B_s}\, dB_s - \frac{1}{2}\int_0^t \mathrm{sgn}(B_t - B_r)\, dr.$$

- Y. Hu and D. Nualart (2009,2010) used $\alpha'(t)$ to show CLTs for the $L^{2,3}$ moduli of continuity for Brownian local time.
- $\alpha'(t)$ has finite nonzero 4/3-variation [Rogers/Walsh (91b) and Hu/Nualart/Song (2012)].

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The DSLT of fBm

Recall that fBm with $H \in (0,1)$ is the centered Gaussian process with covariance

$$\frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).$$

For $H < 2/3$, Yan, Yang, Lu (2008) introduce a version of $\alpha'(t)$ connected with stochastic area integrals w.r.t. local times.
We modify their definition to

$$\alpha'(t) := -H \int_0^t \int_0^s \delta'(B_s^H - B_r^H)(s-r)^{2H-1}\, dr\, ds.$$

Existence of $\alpha'(t)$

For $H < 2/3$, $\alpha'(t)$ exists as a limit in $L^2(\Omega)$ of

$$\alpha'_\varepsilon(t) := -H \int_0^t \int_0^s f'_\varepsilon(B_s^H - B_r^H)(s-r)^{2H-1}\, dr\, ds.$$

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Existence using the Wiener chaos

Theorem

For $H < 2/3$, $\alpha'(t)$ exists in $L^2(\Omega)$. Its Wiener chaos is

$$\alpha'(t) = \sum_{m=1}^{\infty} I_{2m-1}(g(2m-1, t))$$

where

$$g(2m-1, t; v_1, \ldots, v_{2m-1}) = \frac{(-1)^m}{(m-1)!2^{m-1}\sqrt{2\pi}} \int_0^t \int_0^s \frac{\prod_{j=1}^{2m-1} M_H 1_{[r,s]}(v_j)\, dr\, ds}{(s-r)^{H(2m-1)+1}}$$

and

$$\langle M_H 1_{[0,s]}, M_H 1_{[0,t]} \rangle_{L^2(\mathbb{R})} = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Idea of proof

Apply Stroock's formula to $\alpha'_\epsilon(t)$:

$$\frac{1}{n!}\int_0^t\int_0^s (s-r)^{2H-1}\mathbf{E}[D^n f'_\varepsilon(B^H_s - B^H_r)]\,dr\,ds$$

and then use the following:

Lemma (Nualart and Vives (92))

Let F_ε be a family of $L^2(\Omega)$ random variables with chaos expansions $F_\varepsilon = \sum_{n=0}^\infty I_n(f_n^\varepsilon)$. If for all n, f_n^ε converges in $\mathcal{H}^{\otimes n}$ to f_n, and if

$$\sum_{n=0}^\infty \sup_\varepsilon \mathbf{E}[I_n(f_n^\varepsilon)^2] = \sum_{n=0}^\infty \sup_\varepsilon\{n!||f_n^\varepsilon||^2_{\mathcal{H}^{\otimes n}}\} < \infty,$$

then F_ε converges in $L^2(\Omega)$ to $F = \sum_{n=0}^\infty I_n(f_n)$.

A Tanaka formula for the DSLT of fBm

Theorem

For $H < 2/3$, the following holds in $L^2(\Omega)$ for all t:

$$H\,\alpha'(t) = \int_0^t L_s^{B_s^H}\, dB_s^H - \frac{1}{2}\int_0^t \mathrm{sgn}(B_t^H - B_r^H)\, dr\ .$$

Formally apply a fractional Itô formula [e.g., Bender (03)] to $B_s^H - B_r^H$ with s going from r to t:

$$\begin{aligned} & 1_{[(0,\infty)}(B_t^H - B_r^H) - 1_{[(0,\infty)}(0) \\ = & \int_r^t \delta(B_s^H - B_r^H) dB_s^H + H\int_r^t \delta'(B_s^H - B_r^H)(s-r)^{2H-1}\, ds \end{aligned}$$

Then integrate dr from 0 to t and apply Fubini:

$$\begin{aligned} & \int_0^t \frac{1}{2}\mathrm{sgn}(B_t^H - B_r^H)\, dr \\ = & \int_0^t L_s^{B_s^H} dB_s^H + H\int_0^t\int_0^s \delta'(B_s^H - B_r^H)(s-r)^{2H-1}\, dr\, ds. \end{aligned}$$

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More rigorously, the first term on the LHS is the limit of

$$\int_0^t \int_r^t f_\varepsilon(B_s^H - B_r^H)\, dB_s^H\, dr$$

To justify Fubini, use the chaos expansion:

$$f_\varepsilon(B_s^H - B_r^H) = \sum_{n\geq 0} I_n \left(\frac{1}{n!} \mathbf{E}[(\frac{d^n}{dx^n} f_\varepsilon)(B_s^H - B_r^H)](M_H 1_{[r,s]})^{\otimes n} \right).$$

We refine this to a Hermite chaos expansion using (tensor products of) Hermite functions as a basis of $\mathbf{H}^{\otimes n}$:

$$c_\beta(s,r) = \frac{1}{n!} \mathbf{E}[(\frac{d^n}{dx^n} f_\varepsilon)(B_s^H - B_r^H - y)] \langle \xi^{\odot \beta}, (M_H 1_{[r,s]})^{\otimes n} \rangle_{\mathbf{H}^{\otimes n}}.$$

If $\sum \beta! c_\beta < \infty$ then we are in $L^2(\Omega)$. For more stringent summability, we get Hida test functions (and thus distributions).

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A Fubini theorem

An extension of Fubini theorems in Cheridito/Nualart (2005) and Mishura (2008) to integrals of Hida distributions.

Lemma (Fubini-Tonelli theorem)

Let

$$F_{s,r} = \sum_{\beta \in \Lambda} c_\beta(s,r)\mathcal{H}_\beta$$

be an $(\mathcal{S})^*$*-valued process indexed by* $(s,r) \in \mathbb{R} \times [0,t]$. *If, for each* (β, k) *pair,* $c_\beta(s,r)M_H\xi_k(s)$ *is bounded above or below by an* $L^1([r,t] \times [0,t])$ *function, then*

$$\int_0^t \int_r^t F_{s,r}(\omega)\, dB_s^H\, dr = \int_0^t \left(\int_0^s F_{s,r}(\omega)\, dr \right) dB_s^H. \qquad (1)$$

The equality in (1) *is in the sense that if one side is in* $(\mathcal{S})^*$, *then the other is as well, and they are equal.*

Back to existence of $\alpha'(t)$

Set

$$\begin{aligned} \mathcal{D}_t &:= \{0 \le r \le s \le t\} \\ \lambda &:= \mathrm{Var}(B^H_s - B^H_r) \\ \rho &:= \mathrm{Var}(B^H_{s'} - B^H_{r'}) \\ \mu &:= \mathrm{Cov}(B^H_s - B^H_r, B^H_{s'} - B^H_{r'}) \end{aligned}$$

Existence in $L^2(\Omega)$ is implied by

$$\int_{\mathcal{D}_t^2} \frac{\mu (s-r)^{2H-1}(s'-r')^{2H-1}}{(\lambda\rho - \mu^2)^{3/2}} \, dr\, dr'\, ds\, ds' < \infty.$$

Comes from looking at $\mathbf{E}[(\alpha'_\epsilon(t))^2]$.

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Second moment bounds from Hu (2001)

$\lambda = \mathrm{Var}(B^H_s - B^H_r),\ \rho = \mathrm{Var}(B^H_{s'} - B^H_{r'}),\ \mu = \mathrm{Cov}(B^H_s - B^H_r, B^H_{s'} - B^H_{r'})$

Lemma 3.1 of Hu (2001) asserts

(i) For $r < r' < s < s'$ and $a = r' - r$, $b = s - r'$, $c = s' - s$,

$$\lambda\rho - \mu^2 \geq C\left((a+b)^{2H}c^{2H} + a^{2H}(b+c)^{2H}\right).$$

(ii) For $r < r' < s' < s$ and $a = r' - r$, $b = s' - r'$, $c = s - s'$,

$$\lambda\rho - \mu^2 \geq Cb^{2H}(a+b+c)^{2H}.$$

(iii) For $r < s < r' < s'$ and $a = s - r$, $b = r' - s$, $c = s' - r'$,

$$\lambda\rho - \mu^2 \geq C(a^{2H}c^{2H}).$$

A modification of (ii)

In every instance we have seen, the following suffices:

(ii') For $r < r' < s' < s$ and $a = r' - r$, $b = s' - r'$, $c = s - s'$,

$$\lambda\rho - \mu^2 \geq Cb^{2H}(a^{2H} + c^{2H}).$$

A Conjecture

Conjecture

- The critical parameter is $H_c = 2/3$. At H_c, $\frac{1}{\log(1/\varepsilon)^\gamma}\alpha'_\varepsilon(t)$ converges in distribution to a normal law for some $\gamma > 0$.
- For $H > H_c$, $\varepsilon^{-\gamma(H)}\alpha'_\varepsilon(t)$ converges in distribution to a normal law for some function $\gamma(H) > 0$ which goes to 0 at $H = 2/3$.

- We have shown related results for

$$\tilde{\alpha}'(t) := -H \int_0^t \int_0^s \delta'(B_s^H - B_r^H)\, dr\, ds.$$

- This would mirror the behavior of SLT shown in Y. Hu and D. Nualart (2005). To our knowledge, no such CLT has been proved even for SLT in two dimensions at $H_c = 3/4$.
- This would validate Rosen's statement that "The (DSLT of Brownian motion) in $\mathbb{R}^1$, in a certain sense, is even more singular than self-intersection local time in $\mathbb{R}^2$."

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A generalization to $\alpha'_t(y)$

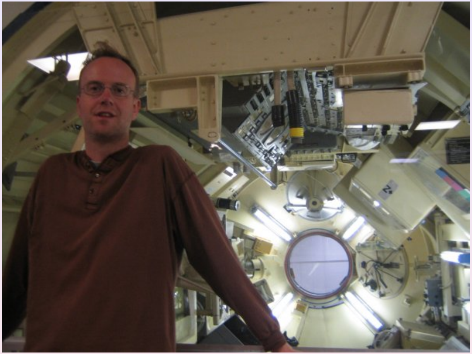
The above generalizes to

$$\alpha'_t(y) := -H \int_0^t \int_0^s \delta'(B_s^H - B_r^H - y)(s-r)^{2H-1}\, dr\, ds.$$

Theorem

For $H < 2/3$, the following holds in $L^2(\Omega)$ for all y and t:

$$H\,\alpha'_t(y) + \frac{1}{2}\operatorname{sgn}(y)t = \int_0^t L_s^{B_s^H - y}\, dB_s^H - \frac{1}{2}\int_0^t \operatorname{sgn}(B_t^H - B_r^H - y)\, dr\,.$$



Thanks for your attention!