A stochastic Burgers equation from zero-range interactions

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–joint work with Patricia Goncalves and Milton Jara

Outline

Kardar-Parisi-Zhang '86 (KPZ) proposed a SPDE to govern the fluctuations of the height variable in some types of growing interfaces. In dimension $d = 1$, the study of their equation has advanced in recent years.

KPZ equation

Let $h = h_t(x)$ be the height function at time t and location x. Then,

$$\partial_t h_t \;=\; D\triangle h_t + a\big(\nabla h_t\big)^2 + \sigma\,\mathcal{W}_t$$

where $\mathcal{W}_t$ is a space-time white noise, and the middle term introduces a nonlinear form of slope-dependent growth speed.

- The equation has difficulties in interpretation since the noise is not spatially regular enough for a differentiable solution, and the square of a gradient of h_t does not allow a weak formulation.

Nontrivial classes of behaviors

The KPZ solution has different fluctuation behaviors, based on heuristics given in KPZ '86.

KPZ Class

When $a, D, \sigma \neq 0$,

$$\frac{h_t(x) - Eh_t(x)}{t^{1/3}} \Rightarrow \xi(x)$$

where ξ is nontrivial (in general some type of TW law) which depends on initial conditions.
Also, spatial correlations are on order $t^{2/3}$:

$$\Big\langle \frac{h_t(0) - Eh_t(0)}{t^{1/3}}; \frac{h_t(xt^{2/3}) - Eh_t(xt^{2/3})}{t^{1/3}} \Big\rangle \sim C(x).$$

EW Class

When $a = 0$ and $D, \sigma \neq 0$,

$$\frac{h_t(x) - Eh_t(x)}{t^{1/4}} \Rightarrow \xi(x)$$

which is Gaussian.
Here, spatial correlations are on order $t^{1/2}$.

Trivial Class

When $a = D = 0$ and $\sigma \neq 0$, marginal distributions are Gaussian in usual diffusive scale $t^{1/2}$ and of course no spatial correlations.

General question

How to make rigorous sense of the KPZ equation? Can one derive it from microscopic interactions?

Most of the work has been done in terms of models which allow a microscopic Cole-Hopf formula, namely simple exclusion and directed polymer models.

These models are weakly asymmetric where the weak asymmetry (difference between rates for right and left jumps) decays on a certain order with respect to the scaling parameter. Namely, $O(N^{-1/2})$, as opposed to the asymmetry $O(N^{-1})$ which comes up in large deviations of the density field.

References for simple exclusion

Bertini-Giacomin '97 (starting from the invariant measure ν_ρ)
Amir-Corwin-Quastel '11; Sasamoto-Spohn '10 (starting from the step profile).

Remark

Our focus here will be on a different approach which involves a 'martingale characterization' of an associated Burgers equation.

–With respect to simple exclusion models, a form of this approach has been studied in Goncalves-Jara '10 and Assing '11.

–For more general models, some new ideas/ingredients are needed for the formulation and proof.

On the Cole-Hopf transform

Recall the KPZ equation

$$\partial_t h_t \; = \; D \triangle h_t + a\big(\nabla h_t\big)^2 + \sigma \mathcal{W}_t \, .$$

Define

$$Z_t \; = \; \exp\Big\{\frac{a}{D} h_t(x)\Big\}.$$

Then,

$$\begin{aligned} \partial_t Z_t \; &= \; \frac{a}{D}\big(\partial_t h_t\big) Z_t \\ &= \; D \triangle Z_t + \frac{a\sigma}{D} Z_t \mathcal{W}_t \end{aligned}$$

which is a (well-posed) stochastic heat equation.

References

Hairer '11 has given a 'rough paths' interpretation of the KPZ equation, which approximates the nonlinear term in a certain way. In particular, he showed that $\log Z_t$, where Z_t solves the stochastic heat equation, satisfies this intepreted KPZ equation (on a torus).

Remark

This nicely ties in with the BG and ACQ results! We have been looking at how to treat other nearest-neighbor microscopic models, such as zero-range, on which we concentrate in this talk, which have less nice properties.

(Microscopic) Height function

Notation

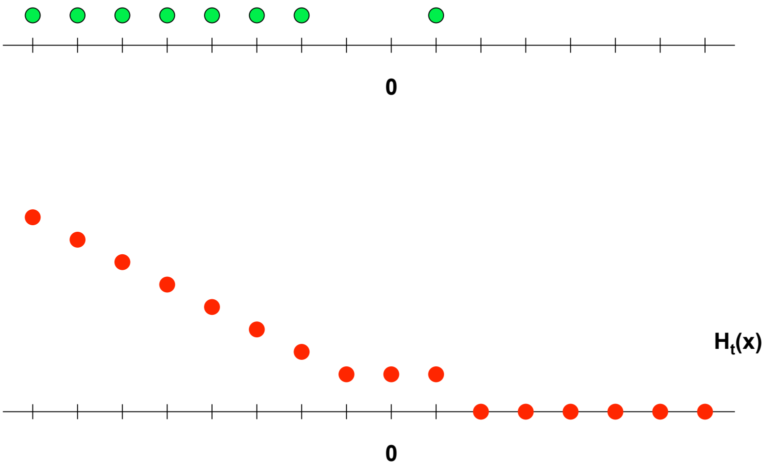
$\eta_t = \{\eta_t(x) : x \in \mathbb{Z}\}$ is the configuration at time t.
$J_x(t)$ is the current through the bond $(x-1, x)$ up to time t.

What is the (microscopic) height function?

$$H_t(x) = \begin{cases} J_0(t) - \sum_{y=0}^{x-1} \eta_t(y) & \text{for } x \geq 1 \\ J_0(t) & \text{for } x = 0 \\ J_0(t) + \sum_{y=x}^{-1} \eta_t(y) & \text{for } x \leq -1. \end{cases}$$

As an example, the much studied Corner-Growth model, corresponds to the height function in simple exclusion starting from the Step Initial Condition.

In the $d = 1$ nearest-neighbor simple exclusion process, continuous-time random walks move on Z except that jumps to occupied vertices are suppressed.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0
$H_t(x)$

A stochastic KPZ-Burgers equation

Since

$$H_t(x) - H_t(x+1) \;=\; \eta_t(x)$$

may be viewed as a 'discrete gradient', then one might consider the equation the gradient $Y_t = \nabla h_t$ satisfies:

$$\partial_t Y_t \;=\; D\triangle Y_t + a\nabla\big(Y_t\big)^2 + \sigma\nabla\mathcal{W}_t$$

which however has the same difficulties, when $a \neq 0$, as the KPZ equation:

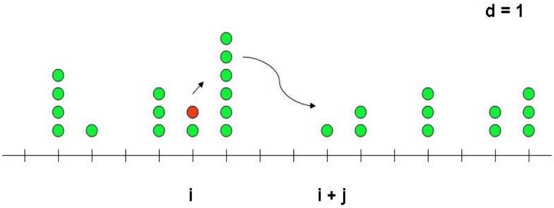
$$\partial_t h_t \;=\; D\triangle h_t + a\big(\nabla h_t\big)^2 + \sigma\mathcal{W}_t$$

Fluctuation field

The microscopic analog of $Y_t = \nabla h_t$, as in BG '97, is the "fluctuation field" of the diffusively scaled process, with a certain weak asymmetry, in a reference frame moving with a process characteristic speed.

Goal: We derive, in a class of interactions, that all limit points of the fluctuation field satisfy a form of the KPZ-Burgers equation. However, among other remarks which we will make, uniqueness of the limit is not established.

ZRP Dynamics

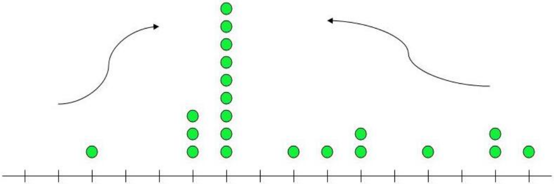


"at a vertex with k particles, one of the particles displaces by j with rate $(g(k)/k)p(j)$."

- $g : \mathbb{N}_0 \to \mathbb{R}_+$, $g(0) = 0$, $g(k) > 0$ for $k \geq 1$.
- $p(\cdot)$ is a (translation-invariant) jump probability on lattice $\mathbb{Z}^d$.

Cases

- g is on "linear order", e.g. $g(k) = k$ corresponds to independent particles.
- g is "sublinear", e.g. $g(k) = k^\gamma$ for $\gamma \in (0, 1)$
- g is bounded, e.g. $g(k) = 1_{[k \geq 1]}$.
- g corresponds to "aggregation" phenomena, $g(k) \downarrow$ as $k \uparrow \infty$, e.g. $g(k) = 1_{[k \geq 1]}(1 + c/k)$, $c > 2$ (Beltran-Landim '12)



Configurations and Generator

The process $\eta_t^N = \eta_t$ is defined on space $\Omega = \{0, 1, 2, \ldots\}^{\mathbb{Z}}$

A configuration $\eta_t = \{\eta_t(i) : i \in \mathbb{Z}\}$
where $\eta_t(i) = \#$ particles at i at time t

We consider a ZRP with 'weakly asymmetric nearest-neighbor' jumps, and Markov generator where time is 'speeded up' by N^2:

$$
\begin{aligned}
(L_N\phi)(\eta) \;=\; N^2 \sum_x \Big\{ & p_N g(\eta(x))\big(\phi(\eta^{x,x+1}) - \phi\big) \\
& + q_N g(\eta(x+1))\big(\phi(\eta^{x+1,x}) - \phi\big)\Big\}
\end{aligned}
$$

Invariant measures

There exists a family of invariant measures indexed by density.

Invariant measures

For $0 \le \alpha < \alpha^*$,

$$\bar{\nu}_\alpha = \prod_{i \in \mathbb{Z}^d} \bar{\mu}_\alpha \qquad \text{(Andjel '81)}$$

where

$$\begin{aligned} \bar{\mu}_\alpha(k) &= \frac{1}{Z_\alpha} \frac{\alpha^k}{g(1)\cdots g(k)} \quad \text{for } k \ge 1; \\ &= \frac{1}{Z_\alpha} \quad \text{for } k = 0. \end{aligned}$$

Reparametrization

It will be convenient to reparametrize in terms of 'density'

$$\rho(\alpha) \ = \ \frac{1}{Z_\alpha}\sum_{k\geq 0} k\frac{\alpha^k}{g(k)!}$$

Can show $\rho = \rho(\alpha) : [0,\alpha^*) \to [0,\infty)$ is invertible, so can write

$$\begin{aligned} \alpha &= \alpha(\rho) \\ \mu_\rho &= \bar{\mu}_{\alpha(\rho)} \\ \nu_\rho &= \bar{\nu}_{\alpha(\rho)} = \prod_{i\in\mathbb{Z}^d} \mu_\rho. \end{aligned}$$

Mean departure rate

In fact,

$$\alpha(\rho) \;=\; E_{\nu_\rho}[g(\eta)],$$

is the 'mean-rate' of departure from a site, which will appear in later results.

- $g(k) = k$. Then, $\alpha(\rho) = \rho$.
- g is "linear". Then, $c_1\rho \leq \alpha(\rho) \leq c_2\rho$.
- $g(k) = k^\gamma$ for $\gamma \in (0,1)$. Then $\alpha(\rho) \uparrow \infty$ and $\alpha(\rho)/\rho \downarrow 0$
- $g(k) = 1_{[k\geq 1]}$. Then, $\alpha(\rho) = \rho/(1+\rho)$.

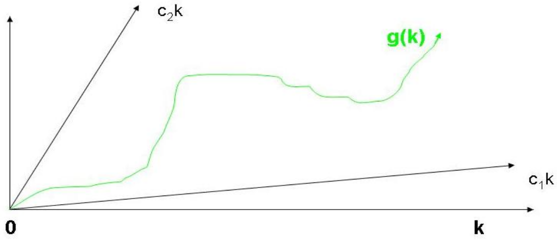
One of the reasons, ZRP is interesting is that it possesses different equilibriation behavior with respect to different types of g, which can be measured by "mixing" or "spectral gap" estimates.

Spectral gap

Let $W_\ell(k)$ be the inverse of the spectral gap of the process defined on the cube $\Lambda_\ell = \{-\ell, \ldots, \ell\}^d$ with k particles, when the transition probability p is symmetric and nearest-neighbor

Different types of spectral gaps

- g is on linear order: $W_\ell(k) \sim C\ell^2$. S. - Landim - Varadhan '96
  Here, 'linear' means $\sup_k |g(k+1) - g(k)| \le a_0$, and
  $g(k + b_0) - g(k) \ge b_1 > 0$



- g is sublinear: For the example mentioned $(g(k) = k^\gamma)$,
  $W_\ell(k) = \ell^2(1+\rho)^{1-\gamma}$ where $\rho = k/\ell$. Nagahata '10
- g is bounded: For the case given $(g(k) = 1_{[k \ge 1]})$,
  $W_\ell(k) = \ell^2(1+\rho)^2$. Morris '06

ZRP Assumptions

WA For $a \in \mathbb{R}$ and $\gamma > 0$, which control the 'weak asymmetry',

$$p_N = 1/2 + \frac{a}{2N^\gamma} \quad \text{and} \quad q_N = 1/2 - \frac{a}{2N^\gamma}.$$

On the function g, we assume the following.

LG $\sup_k |g(k+1) - g(k)| \leq a_0$.

SP The spectral gap satisfies

$$E_{\nu_\rho}\Big[W_\ell\Big(\sum_{x\in\Lambda_\ell} \eta(x)\Big)^2\Big] \ \leq \ C(\rho)\ell^4.$$

Remarks

- The [LG] assumption is useful in construction.
- The [SP] assumption is satisfied by the g's mentioned on previous slide.

Hydrodynamics

Before getting to the fluctuation results, we mention the "hydrodynamics" for the scaling limit of the "density field:" Converging in probability,

$$X_t^n(H) = \frac{1}{N}\sum_{x\in\mathbb{Z}} H(\frac{x}{N})\eta_t^N(x) \to \int_{\mathbb{R}} H(u)\rho(t,u)du.$$

The hydroynamic equation for $\rho = \rho(t,u)$, when the asymmetry $p_N - q_N = O(N^{-1})$ and $\gamma = 1$, is

$$\partial_t\rho + a\nabla\alpha(\rho(t,u)) = \frac{1}{2}\Delta\alpha(\rho(t,u)).$$

Equilibrium fluctuations

For the remainder of the talk, to simplify the discussion, we start under the invariant measure ν_ρ.

Let $\gamma = 1$ and the asymmetry $p_N - q_N = O(N^{-1})$. Define a "fluctuation" field

$$Y_t^N(H) \; = \; \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} H\big(\frac{x}{N}\big)\big(\eta_t^N(x) - \rho\big).$$

Then, starting from the invariant measure ν_ρ, Y_t^N converges in $D([0,T], \mathbb{S}'(\mathbb{R}))$ to the Ornstein-Uhlenbeck process given by

$$dY_t \; = \; \frac{1}{2}\alpha'(\rho)\Delta Z_t dt + a\alpha'(\rho)\nabla Y_t dt + \sqrt{\alpha(\rho)}\nabla \mathcal{W}_t$$

where $\mathcal{W}_t$ is a space-time white noise.

The drift term, as is well known, can be understood in terms of the characteristic velocity $a\alpha'(\rho)$ from linearizing the hydrodynamic equation about $\rho(t,u) \equiv \rho$.

One can remove it, however, by observing the field in the reference frame shifted, according to process characteristic velocity, by

$$\frac{1}{N}(p_N - q_N)\alpha'(\rho)tN^2 \ = \ \frac{a\alpha'(\rho)tN}{N^\gamma}.$$

Define

$$\mathcal{Y}_t^{N,\gamma}(H) \ = \ \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}} H\Big(\frac{x}{N} - \frac{a\alpha'(\rho)tN}{N^\gamma}\Big)\big(\eta_t^N(x) - \rho\big).$$

If $\gamma = 1$, the last result is equivalent to $\mathcal{Y}_t^{N,\gamma}$ converging to the unique solution of

$$d\,\mathcal{Y}_t \ = \ \frac{1}{2}\alpha'(\rho)\Delta\,\mathcal{Y}_t\,dt + \sqrt{\alpha(\rho)}\nabla\,\mathcal{W}_t\,.$$

'Crossover'

Although we will be primarily interested in when the asymmetry $p_N - q_N = O(N^{-1/2})$ corresponds to $\gamma = 1/2$, one might ask what happens when $1/2 < \gamma \leq 1$.

It turns out, there is no effect–the asymmetry is not strong enough to influence the fluctuations. For simple exclusion, this was observed in Sasamoto-Spohn '10, Goncalves-Jara '12.

Theorem. When $1/2 < \gamma \leq 1$, we have that $\mathcal{Y}_t^{N,\gamma}$ converges to the solution of the equation for $\gamma = 1$.

When $\gamma = 1/2$, however, the asymmetry has nontrivial influence on the limit of $\mathcal{Y}_t^{N,\gamma}$.

In particular, a form of the nonlinear term, $\nabla(\mathcal{Y}_t)^2$, in the stochastic KPZ-Burgers equation is picked up in general.

Martingale derivations

Let $\gamma = 1/2$. We will try to show that limits of $\mathcal{Y}_t^N := \mathcal{Y}_t^{N,\gamma}$ solve the following form of the stochastic KPZ-Burgers equation:

$$\partial_t \mathcal{Y}_t \;=\; \frac{\alpha'(\rho)}{2} \triangle \mathcal{Y}_t \, dt + a\alpha''(\rho) \nabla \big(\mathcal{Y}_t \big)^2 + \sqrt{\alpha(\rho)} \nabla \mathcal{W}_t \,.$$

Then, the scaled fluctuation field, observed along process characteristics, is

$$\mathcal{Y}_t^N(G) \;:=\; \frac{1}{\sqrt{N}} \sum_x G\Big(\frac{x}{N} - a\alpha'(\rho) N^{1/2} t\Big) \big(\eta_t^N(x) - \rho\big).$$

Note: We will abbreviate $G\Big(\frac{x}{N} - a\alpha'(\rho) N^{1/2} t\Big) = G_{a,t}\Big(\frac{x}{N}\Big)$.

Stochastic differential

Write

$$d\,\mathcal{Y}_t^N(G) \;=\; \Big[\frac{\partial}{\partial_t} + L_N\Big]\,\mathcal{Y}_t^N(G)dt + d\,\mathcal{M}_t^N(G).$$

Together, after a calculation,

$$\begin{aligned}
&\Big[\frac{\partial}{\partial_t} + L_N\Big]\,\mathcal{Y}_t^N(G)\\
&\sim\; \frac{1}{2\sqrt{N}}\sum_x \triangle G_{a,t}\big(\frac{x}{N}\big)\big[g(\eta_t(x)) - \alpha(\rho)\big]\\
&\quad + \frac{a}{2}\sum_x \nabla G_{a,t}\big(\frac{x}{N}\big)\big[g(\eta_t(x)) - \alpha(\rho) - a\alpha'(\rho)(\eta_t(x) - \rho)\big]
\end{aligned}$$

The martingale $\mathcal{M}_t^N(G)$ has quadratic variation

$$
\begin{aligned}
&\langle \mathcal{M}_t^N(G) \rangle \\
&= \frac{N^2}{\sqrt{N}^2} \int_0^t \sum_x \frac{\left(\nabla G_{a,t}\left(\frac{x}{N}\right)\right)^2}{N^2} \left[p_N g(\eta_s(x)) + q_N g(\eta_s(x+1))\right] ds \\
&\to t\alpha(\rho) \|\nabla G\|^2_{L^2(\mathbb{R})}
\end{aligned}
$$

in probability, since we start from an ergodic measure ν_ρ.

Remark

The idea, as usual, now is to close these equations in terms of the fluctuation field itself.

Boltzmann-Gibbs principles

One needs to generalize Brox-Rost's '84 Boltzmann-Gibbs principle:

$$E_{\nu_\rho}\Big|\int_0^t \frac{1}{\sqrt{N}}\sum_x \triangle G(x/N)\big[g(\eta_t(x))-\alpha(\rho)\big] - \frac{1}{\sqrt{N}}\sum_x \triangle G(x/N)\alpha'(\rho)\big[\eta_t(x)-\rho\big]ds\Big|^2 \ = \ o(1).$$

This is enough for one of the terms. But, the nonlinearity gives a term with no normalization.

We need to replace

$$\int_0^t \sum_x \nabla G_{a,t}(x/N)\big\{g(\eta_s(x))-\alpha(\rho)-\alpha'(\rho)\big[\eta_s(x)-\rho\big]\big\}ds.$$

A generalized Boltzmann-Gibbs principle

We show that

$$E_{\nu_\rho}\Big|\int_0^t \sum_x \nabla G(x/N)\big[g(\eta_s(x)) - \alpha(\rho) - \alpha'(\rho)\big(\eta_s(x)-\rho\big)\big]$$
$$- \frac{\alpha''(\rho)}{2}\sum_x \nabla G(x/N)\Big[\big(\eta_s^{(\ell)}(x)-\rho\big)^2 - \frac{\sigma^2(\rho)}{\ell}\Big]ds\Big|^2$$
$$\leq \; c(G)\Big[\frac{t\ell}{N} + \frac{t^2N^2}{\ell^3}\Big].$$

Here, $\sigma^2(\rho) = \mathrm{Var}_{\nu_\rho}(\eta(0))$ and

$$\eta^{(\ell)}(x) \; = \; \frac{1}{2\ell+1}\sum_{|y|\leq\ell}\eta(y+x).$$

Remarks

- Although there is no space average, we are taking advantage of the time average. The main tool is analysis of H_{-1} norm bounds.
- We will take $\ell = \epsilon N$, introducing another scale.

We can show that $\{\mathcal{Y}_t^N\}$ and $\{\mathcal{M}_t^N\}$ are tight on $D([0,T];\mathcal{S}')$ and limit points are concentrated on continuous paths.

Then, subsequentially,

$$\mathcal{M}_t^{N'}(G) \;\to\; \mathcal{M}_t(G)$$

which has quadratic variation $t\alpha(\rho)\|\nabla G\|^2_{L^2(\mathbb{R})}$
(a BM by Levy's thm).

Looking at the term in the generalized Boltzmann-Gibbs estimate, with $\ell = \epsilon N$,

$$\int_0^t \sum_x \nabla G(x/N)\big[\big(\eta_s^{(\ell)}(x) - \rho\big)^2 - \frac{\sigma^2(\rho)}{\ell}\big]ds,$$

define now

$$\mathcal{A}_t^{N,\epsilon}(G) \;=\; \int_0^t \frac{1}{N}\sum_x \nabla G(x/N)\big[\,\mathcal{Y}_s^N\left((2\epsilon)^{-1}1_{[x/N-\epsilon,x/N+\epsilon]}\right)\big]^2 ds$$

One has, subsequentially,

$$\begin{aligned}\lim_{N'\uparrow\infty}\mathcal{A}_t^{N',\epsilon}(G) \;&=\; \int_0^t\int_{\mathbb{R}}\nabla G(x)\big[\,\mathcal{Y}_s\left((2\epsilon)^{-1}1_{[x-\epsilon,x+\epsilon]}\right)dxds \\ &:=\; \mathcal{A}_t^{\epsilon}(G)\end{aligned}$$

Then, plugging into the Boltzmann-Gibbs estimate, with $\ell = \epsilon N$, we have

$$E_{\nu_\rho}\Big| \int_0^t \sum_x \nabla G_{a,s}(x/N)\big[g(\eta_s(x)) - \alpha(\rho) - \alpha'(\rho)\big(\eta_s(x) - \rho\big)\big]ds - \frac{\alpha''(\rho)}{2}\mathcal{A}_t^{N,\epsilon}(G)\Big|^2 \leq c(G)\Big[t\epsilon + \frac{t^2}{N\epsilon^3}\Big].$$

Importantly, one also may conclude, after some calculations, that $\{\mathcal{A}_t^\epsilon(G)\}$ is Cauchy in $L^2(\nu_\rho)$ as $\epsilon \downarrow 0$.

Let $\mathcal{A}_t(G)$ be the $L^2(\nu_\rho)$ limit.

Now,

$$\begin{aligned}\mathcal{M}_t^N(G) &= \mathcal{Y}_t^N(G) - \mathcal{Y}_0^N(G) - \frac{\alpha'(\rho)}{2}\int_0^t \mathcal{Y}_s^N(\triangle G)ds \\ &\quad - \frac{a\alpha''(\rho)}{2}\mathcal{A}_t^{N,\epsilon}(G) + o(1).\end{aligned}$$

After taking limit on $N' \uparrow \infty$ and $\epsilon \downarrow 0$, we obtain the following limit characterization.

Theorem. All limit points are such that

$$\mathcal{M}_t(G) \;=\; \mathcal{Y}_t(G) - \mathcal{Y}_0(G) - \frac{\alpha'(\rho)}{2}\int_0^t \triangle\,\mathcal{Y}_s(G) - \frac{a\alpha''(\rho)}{2}\,\mathcal{A}_t(G)$$

is a continuous martingale with quadratic variation $\alpha(\rho)t\|\nabla G\|^2_{L^2(\mathbb{R})}$.

Formally,

$$\partial_t\,\mathcal{Y}_t \;=\; \frac{\alpha'(\rho)}{2}\triangle\,\mathcal{Y}_t + \frac{a\alpha''(\rho)}{2}\,\mathcal{A}_t + \sqrt{\alpha(\rho)}\nabla\,\mathcal{W}_t\,.$$

Remarks

1. The Cauchy limit $\mathcal{A}_t$ represents $\nabla(\mathcal{Y}_t)^2$.

2. When g is such that $\alpha''(\rho) = 0$, then the limit is an O-U process (not in KPZ class), e.g. 'independent particles'.

3. Uniqueness of limit points, or of the martingale characterization has not been shown, which would allow one to use results for simple exclusion (the 'Cole-Hopf' solution satisfies the martingale characterization) to prove 'universality'.

4. The same result, with possibly different initial condition, holds when starting from μ^N which have uniformly bounded entropy with respect to ν_ρ.

Proof ideas

The main point is the generalized Boltzmann-Gibbs principle with respect to function

$$f(\eta(x)) \; = \; g(\eta(x)) - \alpha(\rho) - \alpha'(\rho)[\eta(x) - \rho].$$

In a nutshell, we approximate

$$f(\eta_s(x)) \; \sim \; E[f(\eta_s(x))|\eta_s^{(\ell)}(x)]$$

which in turn can be written

$$E[f(\eta_s(x))|\eta_s^{(\ell)}(x)] \quad \sim \quad \frac{\alpha''(\rho)}{2}\Big[\frac{1}{2\ell+1}\sum_{|y|\leq \ell}\eta_s(y) - \rho\Big]^2 + O(\ell^{-3/2}).$$

To address errors in approximation, we use the H_{-1} norm lemma:

$$E_{\nu_\rho}\Big(\int_0^t f(\eta_s)ds\Big)^2 \ \leq \ \frac{ct}{N^2}\|f\|_{-1}^2$$

where

$$\|f\|_{-1} \ = \ \sup_{\varphi}\Big\{\frac{E_{\nu_\rho}[f\varphi]}{D_{\nu_\rho}(\varphi)^{1/2}}\Big\}$$

and

$$D_{\nu_\rho}(\phi) \ = \ \frac{1}{2}\sum_x E_{\nu_\rho}\big[g(\eta(x))(\phi(\eta^{x,x+1}) - \phi)^2\big].$$

The spectral gap assumption will also play a role here.

More specifically, the argument can be separated into three steps.

Step 1. [1-block estimate] We approximate, in $L^2(\nu_\rho)$,

$$\int_0^t \sum_x h(x) f(\eta_s(x)) ds \ \sim \ \int_0^t \sum_x h(x) E[f(\eta_s(x)) | \eta_s^{(\ell_0)}(x)] ds.$$

Step 2. [2-block estimate] Approximate, in $L^2(\nu_\rho)$,

$$\int_0^t \sum_x h(x) E[f(\eta_s(x)) | \eta_s^{(\ell_0)}(x)] ds$$
$$\sim \ \int_0^t \sum_x h(x) E[f(\eta_s(x)) | \eta_s^{(\ell)}(x)] ds$$

where $\ell >> \ell_0$.

Step 3. [Equivalence of ensembles] Note that

$$E_{\nu_\rho}[f] \;=\; \frac{d}{dz}E_{\nu_z}[f]|_{z=\rho} \;=\; 0.$$

Then, approximate in $L^4(\nu_\rho)$, using local central limit theorems,

$$\int_0^t \sum_x h(x) E[f(\eta_s(x))|\eta_s^{(\ell)}(x)]ds$$

$$\sim \int_0^t \sum_x h(x)\frac{\alpha''(\rho)}{2}\big[(\eta_s^{(\ell)}(x)-\rho)^2 - \frac{\sigma^2(\rho)}{2\ell+1}\big]ds.$$

More on Step 2

We sketch briefly the argument for

$$E_{\nu_\rho}\Big[\int_0^t \sum_x h(x)\big\{E[f(\eta_s(x))|\eta_s^{(\ell_0)}(x)] - E[f(\eta_s(x))|\eta_s^{(\ell)}]\big\}ds\Big]^2$$
$$\leq \frac{ct\ell}{N^2}\sum_x h(x)^2 = O\Big(\frac{\ell}{N}\Big)$$

Express

$$E[f(\eta(x))|\eta^{(\ell_0)}(x)] - E[f(\eta(x))|\eta^{(\ell)}]$$
$$= \sum_r E[f(\eta(x))|\eta^{(\ell_r)}(x)] - E[f(\eta(x))|\eta^{(\ell_{r+1})}]$$

where $\ell_{r+1} = 2\ell_r$.

Now, we can write

$$H_{r,r+1} := E[f(\eta(x))|\eta^{(\ell_r)}(x)] - E[f(\eta(x))|\eta^{(\ell_{r+1})}] \;=\; S_{r+1}u_{r+1}$$

since the function is mean-zero with respect to the canonical invariant measure on the block with width ℓ_{r+1}. Here, S_{r+1} is the symmetric generator on the block.

Then, after some calculation, we get the bound

$$\|H_{r,r+1}\|_{-1} \;\leq\; E_{\nu_\rho}[W^2(\ell_{r+1}, \ell_{r+1}\eta^{\ell_r+1}(x))]^{1/4}\|H_{r,r+1}\|_{L^4(\nu_\rho)}.$$

By the spectral gap assumption, this is less than $C\ell_{r+1}\cdot\|H_{r,r+1}\|_{L^4}$. With equivalence of ensembles, $\|H_{r,r+1}\|_{L^4}\sim\ell_{r+1}^{-1}$. These are the main ingredients to obtain the desired bound.