New Heat Kernel Estimates on Riemannian Manifolds with Negative Curvature

(Partial work join with Junfang Li, UAB)

Xiangjin Xu

Department of Mathematical Sciences
Binghamton University-SUNY, Binghamton, NY, USA

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Problem setting

- (M^n, g): a complete Riemannian manifold with $Ricci(M) \geq -k$,

Heat Equation: $$u_t = \Delta_{\mathbf{g}} u, \quad u(x,0) = u_0(x) \quad on\ M; \qquad (1)$$

- **Solution:** $u(x,t) = e^{-t\Delta_{\mathbf{g}}} u_0(x) = \int_M H(x,y,t) u_0(y) dy$
where $H(x,y,t)$: the heat kernel (the fundamental solution) of M.

- **Guassian Kernel:** $H(x,y,t) = \left(\frac{1}{4\pi t}\right)^{n/2} e^{-\frac{dist^2(x,y)}{4t}}$ when $M = \mathbf{R^n}$.

- $H(x,y,t) = \sum_j e^{-t\lambda_j} e_j(x) e_j(y)$ when manifold M is compact and $\{\lambda_j\}$ and $\{e_j\}$ are eigenvalues and eigenfunctions of $-\Delta_g$ on M.

No exact formula for the heat kernel $H(x,y,t)$ for general manifold M!

- **Short Time Asymptotic Expansion of the Heat Kernel $H(x,y,t)$:**

$\exists\ H_i(x,y)$ on $(M\times M)\setminus C(M)$, $C(M)=\{(x,y)|y\in Cut(x)\}$, such that

$$H(x,y,t)\sim\left(\frac{1}{4\pi t}\right)^{n/2}e^{-\frac{dist^2(x,y)}{4t}}\sum_{i=0}^{\infty}H_i(x,y)t^i \tag{2}$$

holds uniformly as $t\to 0$ on compact subsets of $(M\times M)\setminus C(M)$.

Question:(Global bounds for the heat kernel $H(x,y,t)$)

Are there $A(dist(x,y),t)$ and $B(dist(x,y),t)$ such that

$$A(dist(x,y),t)e^{-\frac{dist^2(x,y)}{4t}}\leq H(x,y,t)\leq B(dist(x,y),t)e^{-\frac{dist^2(x,y)}{4t}}? \tag{3}$$

Remark: By Cheeger-Yau('81) lower bound comparison Theorem and Davies-Mandouvalos('88) Heat kernel bounds on space form, one has

$$H(x,y,t)\geq H^k(x,y,t)\geq c(n)^{-1}h_n(\mathbf{dist}(x,y),t)$$

where $H^k(x,y,t)$ is the heat kernel of space form with $Ric=-k$ and $h_n(r,t)$ is a known function.

History

Li-Yau Differential Harnack inequality:

$$(\textbf{Li}-\textbf{Yau}'\textbf{86})\quad \frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}\leq \frac{n\alpha^2 k}{2(\alpha-1)}+\frac{n\alpha^2}{2t},\qquad \forall\alpha>1.$$

When (M,g) with nonnegative Ricci curvature, the sharp estimate:

$$\frac{|\nabla u|^2}{u^2}-\frac{u_t}{u}\leq\frac{n}{2t}.$$

$$(\textbf{Davies}'\textbf{89})\quad \frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}\leq \frac{n\alpha^2 k}{4(\alpha-1)}+\frac{n\alpha^2}{2t},\qquad \forall\alpha>1.$$

Harnack inequality:

Let $u(x,t)$ be a positive solution of the heat equation (1), then $\forall\alpha>1$,

$$u(x_1,t_1)\leq u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{n\alpha}{2}}\cdot\exp\left(\frac{\alpha\, dist^2(x,y)}{4(t_2-t_1)}+\frac{n\alpha k}{4(\alpha-1)}(t_2-t_1)\right).$$

where $x_1,x_2\in M$ and $0<t_1<t_2<\infty$.

Li-Yau's Bound estimates of $H(x, y, t)$

- **Upper bound of** $H(x, y, t)$: $\forall \delta > 0$,

$$H(x,y,t) \leq C(\delta,n) V_x^{-\frac{1}{2}}(\sqrt{t}) V_y^{-\frac{1}{2}}(\sqrt{t}) \cdot \exp\left(-\frac{dist^2(x,y)}{(4+\delta)t} + C_1(n)\delta k t \right).$$

where $V_x(R) = Vol(B_x(R))$ and $C(\delta, n) \sim \exp(\frac{C}{\delta})$ as $\delta \to 0$.

- **Lower bound of** $H(x, y, t)$ **for** $Ric(M) \geq 0$:

$$H(x,y,t) \geq C^{-1}(\delta,n) V_x^{-\frac{1}{2}}(\sqrt{t}) V_y^{-\frac{1}{2}}(\sqrt{t}) \cdot \exp\left(-\frac{dist^2(x,y)}{(4-\delta)t} \right).$$

Remark: By a slight modified proof of Li-Yau, one could improve as

$$H(x,y,t) \leq C(n) V_x^{-\frac{1}{2}}(\sqrt{\delta t}) V_y^{-\frac{1}{2}}(\sqrt{\delta t}) \cdot \exp\left(-\frac{dist^2(x,y)}{(4+\delta)t} + C_1(n)\delta k t \right).$$

Motivation to improve Li-Yau type estimates

• Short time behavior of the heat kernel $H(t, x, y)$:

$$H(t, x, x) \sim t^{-n/2}(a_0 + a_1 t + a_2 t^2 + \cdots), \ as\ t \searrow 0$$

which suggests

$$-\frac{u_t}{u} \sim \frac{n}{2t}, \ as\ t \searrow 0$$

• Long time asymptotic behavior of upper bound for $\varphi(t)$:
(1) Davies' estimates suggest the lowest upper bound for $\varphi(t)$ may be nk as $t \nearrow \infty$
(2) Yau's gradient estimates for positive harmonic functions on manifolds with negative Ricci lower bound:

$$\frac{|\nabla u|^2}{u^2} \leq (n-1)k$$

(3) Direct computation for the heat kernel $H(t, x, y)$ on hyperbolic spaces suggest the lowest upper bound for $\varphi(t)$ will be at lest $(n-1)k$ too.

Li-Yau type Differential Harnack inequality

Theorem:(Li-X. '11) Let B_{2R} be a geodesic ball with $Ricci(B_{2R}) \geq -k$. let $f = \ln u$, then we get the following Li-Yau type gradient estimate in B_R

$$\sup_{B_R}(|\nabla f|^2 - \alpha f_t - \varphi)(x,t) \leq \frac{nC}{R^2} + \frac{nC\sqrt{k}}{R}\coth(\sqrt{k}\cdot R) + \frac{n^2C}{R^2\tanh(kt)}.$$

where $\alpha(t) = 1 + \frac{\sinh(kt)\cosh(kt) - kt}{\sinh^2(kt)}$ and $\varphi(t) = \frac{nk}{2}\big[\coth(kt)+1\big]$.
Moveover, if $Ric(M) \geq -k$ on the complete manifold, then

$$|\nabla f|^2 - (1 + \frac{\sinh(kt)\cosh(kt) - kt}{\sinh^2(kt)})f_t \leq \quad \frac{nk}{2}\big[\coth(kt)+1\big]. \qquad (4)$$

Remark:
$\alpha(t) \to 1$ and $\varphi(t) \to \frac{n}{2t}$ as $t \searrow 0$,
$\alpha(t) \to 2$ and $\varphi(t) \to nk$ as $t \nearrow \infty$.

Parabolic Harnack inequalities for positive solutions

Theorem:(Li-X. '11) For $\forall x_1, x_2 \in M$, $0 < t_1 < t_2 < \infty$,

$$\frac{u(x_1,t_1)}{u(x_2,t_2)} \leq A_1(t_1,t_2)\cdot \exp\left[\frac{dist^2(x_2,x_1)}{4(t_2-t_1)}(1+A_2(t_1,t_2))\right]$$

where $dist(x_1,x_2)$ is the distance between x_1 and x_2,
$A_1 = \left(\frac{e^{2kt_2}-2kt_2-1}{e^{2kt_1}-2kt_1-1}\right)^{\frac{n}{4}}$, and $A_2(t_1,t_2) = \frac{t_2\coth(kt_2)-t_1\coth(kt_1)}{t_2-t_1}$.

Sketch of Proof: One integrates the following Hamilton-Jacobi inequality

$$f_t \geq -\frac{1}{\alpha}(\phi(t) - |\nabla f|^2),$$

which comes from (4), along the curve $\eta(s) = (\gamma(s), (1-s)t_2 + st_1)$, where γ is a shortest geodesic joining x_1 and x_2.

On-diagonal upper bound of $H(x,y,t)$:

Theorem:(X. '13): For any $R>0$, we have

$$\begin{aligned} H(x,y,t) \quad \le \quad & V_x^{-\frac{1}{2}}(R)V_y^{-\frac{1}{2}}(R)A_1^2\left(\frac{t}{2},\frac{t}{2}+\delta\left(\frac{t}{2}\right)\right) \\ & \exp\left[\frac{R^2}{2\delta\left(\frac{t}{2}\right)}\left(2+A_2\left(\frac{t}{2},\frac{t}{2}+\delta\left(\frac{t}{2}\right)\right)\right)\right]. \end{aligned}$$

One can obtain different on-diagonal upper bound of the heat kernel by different choice of $\delta(t)$ and R:
Case 1: $\delta(t)=(2t)^2\wedge 1 \triangleq \min\{(2t)^2,1\}$ and $R^2=\delta\left(\frac{t}{2}\right)=t^2\wedge 1$,

$$H(x,y,t)\le V_x^{-\frac{1}{2}}(t\wedge 1)V_y^{-\frac{1}{2}}(t\wedge 1)A_{on}(t)\exp(B_{on}(t)), \qquad (5)$$

where $A_{on}(t)$ and $B_{on}(t)$ are bounded functions such that: $A_{on}(t)\sim 1$ as $t\to 0$, $A_{on}(t)\sim \exp\left[\frac{nk}{2}\right]$ as $t\to\infty$, $B_{on}(t)\sim 1+\frac{k(t+t^2)}{6}$ as $t\to 0$, $B_{on}(t)\sim\frac{3}{2}$ as $t\to\infty$.

Continue

Case 2: $\delta(t) = (2t)^2 \wedge t \triangleq \min\{(2t)^2, t\}$ and $R^2 = \delta\left(\frac{t}{2}\right) = t^2 \wedge t$,

$$H(x,y,t) \leq V_x^{-\frac{1}{2}}(t \wedge \sqrt{t}) V_y^{-\frac{1}{2}}(t \wedge \sqrt{t}) A_{on}(t) \exp(B_{on}(t)), \qquad (6)$$

where $A_{on}(t)$ and $B_{on}(t)$ are smooth functions such that: $A_{on}(t) \sim 1$ as $t \to 0$, $A_{on}(t) \sim \exp\left[\frac{nkt}{2}\right]$ as $t \to \infty$, $B_{on}(t) \sim 1 + \frac{k(t+t^2)}{6}$ as $t \to 0$, $B_{on}(t) \sim 2$ as $t \to \infty$.

Case 3: $\delta(t) = 2t \wedge 1 \triangleq \min\{2t, 1\}$ and $R^2 = \delta\left(\frac{t}{2}\right)$,

$$H(x,y,t) \leq V_x^{-\frac{1}{2}}(\sqrt{t \wedge 1}) V_y^{-\frac{1}{2}}(\sqrt{t \wedge 1}) A(t) \exp(B(t)), \qquad (7)$$

where $A(t) \sim 2^{n/2}$ as $t \to 0$, $A(t) \sim \exp\left[\frac{nk}{2}\right]$ as $t \to \infty$, $B(t) \sim 1 + \frac{kt}{3}$ as $t \to 0$, $B(t) \sim \frac{3}{2}$ as $t \to \infty$.

Continue

if we further assume that $\inf_{x\in M} V_x(1) \geq \epsilon > 0$ for some constant $\epsilon > 0$, we have

$$H(x,y,t) \leq \begin{cases} C_1 t^{-n/2}, & t \leq 1, \\ C_2, & t \geq 1. \end{cases}$$

From A Theorem of Davies-Pang '89. We have

$$H(x,y,t) \leq C \max\left\{ t^{-n/2}\left(1 + \frac{d(x,y)}{\sqrt{t}}\right)^n, 1\right\} \exp\left[-\frac{d^2(x,y)}{4t}\right] \tag{8}$$

where C depends on n, k, ϵ only.

Off-diagonal upper bound of $H(x, y, t)$:

Theorem:(X. '13): Assume $d^2(x, y) > t^2 \wedge 1$, we have the following off-diagonal upper bound of the heat kernel

$$H(x,y,t) \leq V_x^{-\frac{1}{2}}(t \wedge 1) V_y^{-\frac{1}{2}}(t \wedge 1) A_{off}(t) \exp\left[-\frac{d^2(x,y)}{4t} + B_{off}(t) + C_{off}(t) d^2(x,y)\right], \quad (9)$$

where $A_{off}(t)$, $B_{off}(t)$ and $C_{off}(t)$ are bounded functions such that: $A_{off}(t) \sim 1$ as $t \to 0$, $A_{off}(t) \sim e^{nk}$ as $t \to \infty$, $B_{off}(t) \sim 1$ as $t \to 0$, $B_{off}(t) \sim \frac{5}{4}$ as $t \to \infty$, and $C_{off}(t) \sim \frac{5}{4}$ as $t \to 0$, $C_{off}(t) \sim 0$ as $t \to \infty$. And

$$H(x,y,t) \leq V_x^{-\frac{1}{2}}(t \wedge \sqrt{t}) V_y^{-\frac{1}{2}}(t \wedge \sqrt{t}) A_{off}(t) \exp\left[-\frac{d^2(x,y)}{4t} + B_{off}(t) + C_{off}(t) d^2(x,y)\right], \quad (10)$$

where $A_{off}(t)$, $B_{off}(t)$ and $C_{off}(t)$ are smooth functions such that: $A_{off}(t) \sim 1$ as $t \to 0$, $A_{off}(t) \sim e^{nkt}$ as $t \to \infty$, $B_{off}(t) \sim 1$ as $t \to 0$, $B_{off}(t) \sim \frac{5}{4}$ as $t \to \infty$, and $C_{off}(t) \sim \frac{5}{4}$ as $t \to 0$, $C_{off}(t) \sim 0$ as $t \to \infty$.

Upper bound estimate for Heat kernel $H(x, y, t)$

Theorem: (X. '13) Let (M, g) be a complete Riemannian manifold with $Ric(M) \geq -k$, $k \geq 0$. Let $H(x, y, t)$ be the heat kernel of M, then

$$H(x,y,t) \leq V_x^{-\frac{1}{2}}(t \wedge 1) V_y^{-\frac{1}{2}}(t \wedge 1) A(t) \times \exp\left[-\frac{d^2(x,y)}{4t} + B(t) + C(t) d^2(x,y) \right], \tag{11}$$

where $A(t) = \max\{A_{on}(t), A_{off}(t)\}$, $B(t) = \max\{B_{on}(t) + \frac{t^2 \wedge 1}{4t}, A_{off}(t)\}$ and $C(t) = C_{off}(t)$ are positive bounded functions such that:

$$A(t) \sim \begin{cases} 1, & \text{as } t \to 0, \\ e^{nk}, & \text{as } t \to \infty. \end{cases}$$

$$\text{B(t)} \sim \begin{cases} 1, & \text{as } t \to 0, \\ \frac{3}{2}, & \text{as } t \to \infty. \end{cases}$$

$$\text{C(t)} \sim \begin{cases} \frac{5}{4}, & \text{as } t \to 0, \\ 0, & \text{as } t \to \infty. \end{cases}$$

New lower bounds of the heat kernel

Theorem:(Li-X. '11) Let M be a complete (or compact with convex boundary) Riemannian manifold possibly with $Ricci(M) \geq -k$. Let $H(x,y,t)$ be the (Neumann) heat kernel. Then for all $x, y \in M$ and $t > 0$,

$$H(x,y,t) \geq \quad (4\pi t)^{-\frac{n}{2}} \frac{(2kt)^{\frac{n}{2}}}{(2e^{2kt}-2-4kt)^{\frac{n}{4}}} \cdot \exp\left[-\frac{d^2(x,y)}{4t}\left(1+\frac{kt\coth(kt)-1}{kt}\right)\right].$$

In particular when $Ricci(M) \geq 0$, we have

$$H(x,y,t) \geq \quad (4\pi t)^{-\frac{n}{2}} \exp\left[-\frac{d^2(x,y)}{4t}\right].$$

Theorem: (X. '13) Lower bound of $H(x,y,t)$ for $Ric(M) \geq 0$:

$$H(x,y,t) \geq \tilde{A}(t) V_x^{-\frac{1}{2}}(t) V_y^{-\frac{1}{2}}(t) \cdot \exp\left(-\frac{dist^2(x,y)}{4t} - \tilde{B}(t) dist^2(x,y)\right).$$

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Thank you for attention!