Asymptotics of Heat Equation with Large, Highly Oscillatory, Random Potential

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Introduction

PDE with highly heterogeneous random coefficients

$$P(x, \frac{x}{\varepsilon}, \partial_x)u_\varepsilon = 0$$

1. Homogenization and error estimate
   - $u_\varepsilon \to u_0$? $\quad u_\varepsilon - u_0 \sim \varepsilon^\gamma$?
   - $(u_\varepsilon - u_0)/\varepsilon^\gamma \Rightarrow$ universal distribution?
2. Dependence of limiting equation on random coefficients
   - short-range-correlated coefficients: homogenization
   - long-range-correlated coefficients: convergence to SPDE
3. Application: uncertainty quantification, inverse problem

Some references

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Problem setup

Equation:

$$\begin{aligned} \partial_t u_\varepsilon(t,x) &= \frac{1}{2}\Delta u_\varepsilon(t,x) + i\frac{1}{\varepsilon^\beta}V(\frac{x}{\varepsilon},\omega)u_\varepsilon(t,x) \\ u_\varepsilon(0,x) &= f(x) \end{aligned}$$

- $V(x)$: stationary random field
- $\varepsilon \ll 1$: heterogeneity of small scales
- imaginary unit: stability
- $d \geq 3$, $\beta > 0$ to be determined

Question:

- limiting equation of u_ε
- dependence on the statistical property of $V(x)$
- quantify the error $u_\varepsilon - u_0$ when the limit u_0 is deterministic

Probabilistic representation and weak convergence approach

Equation:

$$\begin{aligned}\partial_t u_\varepsilon(t,x) &= \frac{1}{2}\Delta u_\varepsilon(t,x) + i\frac{1}{\varepsilon^\beta}V(\frac{x}{\varepsilon},\omega)u_\varepsilon(t,x)\\ u_\varepsilon(0,x) &= f(x)\end{aligned}$$

Feynman-Kac formula:

- $u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\varepsilon^{-\beta}\int_0^t V((x+B_s)/\varepsilon)ds)\}$
- scaling property of B_t: $u_\varepsilon(t,x) \sim \tilde{u}_\varepsilon(t,x)$

$$\tilde{u}_\varepsilon(t,x) = \mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i\varepsilon^{2-\beta}\int_0^{t/\varepsilon^2} V(B_s)ds)\}$$

Weak convergence:

$$(\varepsilon B_{t/\varepsilon^2}, \varepsilon^{2-\beta}\int_0^{t/\varepsilon^2} V(B_s)ds) \Rightarrow ?$$

Brownian motion in random scenery

Weak convergence:

$$\varepsilon^{2-\beta} \int_0^{t/\varepsilon^2} V(B_s)ds \Rightarrow ?$$

Two independent random sources:

- $V(x)$: random coefficients from the PDE
- B_t: Brownian motion from the Feyman-Kac formula

Two different weak convergences:

- quenched: for fixed realization of $V(x)$,
  $\varepsilon^{2-\beta} \int_0^{t/\varepsilon^2} V(B_s)ds \Rightarrow ?$
- annealed: in product probability space,
  $\varepsilon^{2-\beta} \int_0^{t/\varepsilon^2} V(B_s)ds \Rightarrow ?$

Kipnis&Varadhan's approach

Medium seen from an observer

- $(\Omega, \mathcal{F}, \pi)$: *random medium* associated with a group of measure-preserving, ergodic transformation $\{\tau_x, x \in \mathbb{R}^d\}$
- $V(x, \omega) = \mathbb{V}(\tau_x \omega)$ for $\mathbb{V} \in L^2(\pi)$ with mean zero
- $y_s = \tau_{B_s}\omega$: stationary Markov process, ergodic with respect to π
- $\varepsilon^{2-\beta} \int_0^{t/\varepsilon^2} V(B_s, \omega) ds = \varepsilon^{2-\beta} \int_0^{t/\varepsilon^2} \mathbb{V}(y_s) ds$

Corrector equation and martingale CLT

- solve $(\lambda - \frac{1}{2}\Delta)\phi_\lambda = \mathbb{V}$, decompose $\varepsilon^{2-\beta} \int_0^{t/\varepsilon^2} \mathbb{V}(y_s) ds = R_t^\varepsilon + M_t^\varepsilon$ with R_t^ε small and M_t^ε martingale
- prove $R_t^\varepsilon \to 0$ and apply martingale CLT to M_t^ε

Weak convergence of Brownian motion in random scenery

Assumption

- $R(x) = \mathbb{E}\{V(0)V(x)\}$ satisfies

$$\int_{\mathbb{R}^d} R(x)|x|^{2-d}dx \sim \int_{\mathbb{R}^d} \hat{R}(\xi)|\xi|^{-2}d\xi < \infty$$

- CLT scaling: $\beta = 1$

Proposition (**Weak convergence in probability**)

$$(\varepsilon B_{t/\varepsilon^2}, \varepsilon \int_0^{t/\varepsilon^2} V(B_s)ds) \Rightarrow (W_t^1, \sigma W_t^2)$$

Remark

- W_t^1, W_t^2 independent Brownian motions, $\sigma^2 = 4(2\pi)^{-d} \int_{\mathbb{R}^d} \hat{R}(\xi)|\xi|^{-2}d\xi$
- $\mathbb{E}_B\{F(\varepsilon B_{t/\varepsilon^2}, \varepsilon \int_0^{t/\varepsilon^2} V(B_s)ds)\} \to \mathbb{E}\{F(W_t^1, \sigma W_t^2)\}$ in $\pi-$probability for $F \in C_b(\mathbb{R}^{d+1})$
- weaker than quenched and stronger than annealed convergence

Quantitative martingale CLT: Kantorovich distance

- $\varepsilon \int_0^{t/\varepsilon^2} V(B_s)ds = R_t^\varepsilon + M_t^\varepsilon \sim M_t^\varepsilon \to \sigma W_t$
- $u_\varepsilon(t,x) - u_0(t,x) \sim \mathit{dist}(R_t^\varepsilon, 0) + \mathit{dist}(M_t^\varepsilon, \sigma W_t)$

Theorem (Mourrat 12)

Let M_t be continuous, square-integrable martingale and W_t standard Brownian motion, then

$$d_{1,k}(M_1, W_1) \le (k \vee 1)\mathbb{E}|\langle M\rangle_1 - 1|$$

where $\langle M\rangle_t$ quadratic variation associated with M_t and

$$d_{1,k}(X,Y) := \sup\{|\mathbb{E}\{f(X) - f(Y)\}|, f \in \mathcal{C}_b^2(\mathbb{R}), \|f'\|_\infty \le 1, \|f''\|_\infty \le k\}$$

Random coefficient: homogenization setting

- **Assumption 1**. Finiteness of asymptotic variance
$$\hat{R}(\xi)|\xi|^{-2} \in L^1(\mathbb{R}^d)$$
- **Assumption 2**. Integrability condition
$$\mathbb{E}\{V(x)^6\} < \infty$$
- **Assumption 3**. Strongly mixing property
mixing coefficient $\rho(r) \leq C_n(1 \wedge r^{-n})$

Definition: $V(x)$ is strongly mixing with coefficient $\rho(r)$ if $\mathbb{E}_\pi\{\phi_1(V)\phi_2(V)\} \leq \rho(r)$ for any two compact sets K_1, K_2 with $d(K_1, K_2) \geq r$ and any random variables $\phi_1(V), \phi_2(V)$ with $\phi_i(V)$ being $\mathcal{F}_{K_i}-$measurable and $\mathbb{E}_\pi\{\phi_i(V)\} = 0, \mathbb{E}_\pi\{\phi_i^2(V)\} = 1$.

Main result: homogenization setting

Theorem (Bal-Gu 2013)

Let $\beta = 1$ and u_ε, u_0 solve the following equations respectively with the same initial condition $f \in \mathcal{C}_b(\mathbb{R}^d)$:

$$\begin{aligned} \partial_t u_\varepsilon(t,x) &= \frac{1}{2}\Delta u_\varepsilon(t,x) + i\frac{1}{\varepsilon}V(\frac{x}{\varepsilon})u_\varepsilon(t,x) \\ \partial_t u_0(t,x) &= \frac{1}{2}\Delta u_0(t,x) - \frac{1}{2}\sigma^2 u_0(t,x) \end{aligned}$$

then we have

- *under Assumption 1, $u_\varepsilon(t,x) \to u_0(t,x)$ in probability*
- *under Assumption 1, 2, 3 and if we further assume $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,*

$$\mathbb{E}_\pi\{|u_\varepsilon(t,x) - u_0(t,x)|\} \lesssim \begin{cases} \sqrt{\varepsilon} & d = 3 \\ \varepsilon\sqrt{|\log \varepsilon|} & d = 4 \\ \varepsilon & d > 4 \end{cases}$$

Main result: homogenization setting

Remark:

- finiteness of asymptotic variance $\Rightarrow$ homogenization

$$\int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi \sim \int_{\mathbb{R}^d} \frac{R(x)}{|x|^{d-2}} dx < \infty$$

  i.e., $R(x) \sim |x|^{-\alpha}$ with $\alpha > 2$
  both short-range-correlated($\alpha > d$) and long-range correlated($2 < \alpha \leq d$) lead to homogenization!

- integrability condition and strongly mixing property $\Rightarrow$ error estimate
- similar results hold for elliptic equation

$$(-\Delta + 1)u_\varepsilon(t,x) + i\frac{1}{\varepsilon}V(\tfrac{x}{\varepsilon})u_\varepsilon(t,x) = f(x)$$

Question:

1. asymptotic distribution of rescaled error $\frac{u_\varepsilon - u_0}{\varepsilon^\gamma} \Rightarrow ?$
2. the case when $R(x) \sim |x|^{-\alpha}$ with $\alpha \in (0,2)$

Random coefficient: SPDE setting

Assumptions: $V(x) = \Phi(g(x))$:

- $g(x)$: stationary Gaussian field, $R_g(x) = \mathbb{E}\{g(0)g(x)\}$ with $|R_g(x)| \lesssim \prod_{i=1}^d 1 \wedge |x_i|^{-\alpha_i}$ and $R_g(x) \sim c_d \prod_{i=1}^d |x_i|^{-\alpha_i}$ $\alpha_i \in (0,1)$ and $\alpha = \sum_{i=1}^d \alpha_i \in (0,2)$.
- Φ has Hermite rank 1, i.e., $V_k := \mathbb{E}\{\Phi(g)H_k(g)\}$ for Hermite polynomial H_k and $V_0 = 0, V_1 \neq 0$ [Taqqu 75]

Properties:

- $R(x) = \mathbb{E}\{V(0)V(x)\}$ satisfies $R(x)|x|^{2-d} \notin L^1(\mathbb{R}^d)$
- $R(x) \sim V_1^2 c_d \prod_{i=1}^d |x_i|^{-\alpha_i}$.

Weak convergence of Brownian motion in random scenery

Brownian motion in Gaussian noise:

- $W(dx)$: generalized Gaussian random field with

$$\mathbb{E}\{W(dx)W(dy)\} = \prod_{i=1}^{d} |x_i - y_i|^{-\alpha_i} dxdy$$

- Brownian motion in Gaussian noise:

$$\int_0^t \dot{W}(B_s)ds := \lim_{\delta \to 0} \int_0^t \int_{\mathbb{R}^d} \phi_\delta(x - B_s)W(dx)ds$$

ϕ_δ: approximation to identity

Proposition (annealed weak convergence)

Let $\beta = \alpha/2 (\alpha = \sum_{i=1}^{d} \alpha_i)$,

$$\frac{1}{\varepsilon^{\alpha/2}} \int_0^t V\left(\frac{x + B_s}{\varepsilon}\right)ds \Rightarrow V_1 \sqrt{c_d} \int_0^t \dot{W}(B_s)ds$$

Main result: SPDE setting

Theorem (Bal-Gu 2013)

Let $\beta = \alpha/2$ and u_ε, u_0 solve the following equations respectively with the same initial condition $f \in \mathcal{C}_b(\mathbb{R}^d)$:

$$\begin{aligned}
\partial_t u_\varepsilon(t,x) &= \frac{1}{2}\Delta u_\varepsilon(t,x) + i\frac{1}{\varepsilon^{\alpha/2}}V(\frac{x}{\varepsilon})u_\varepsilon(t,x) \\
\partial_t u_0(t,x) &= \frac{1}{2}\Delta u_0(t,x) + iV_1\sqrt{c_d}\dot{W}(x)u_0(t,x)
\end{aligned}$$

then we have for fixed (t,x), $u_\varepsilon(t,x) \to u_0(t,x)$ in distribution.

Main result: SPDE setting

Result:

$$\begin{aligned}\partial_t u_\varepsilon(t,x) &= \frac{1}{2}\Delta u_\varepsilon(t,x) + i\frac{1}{\varepsilon^{\alpha/2}}V(\frac{x}{\varepsilon})u_\varepsilon(t,x)\\ \partial_t u_0(t,x) &= \frac{1}{2}\Delta u_0(t,x) + iV_1\sqrt{c_d}\dot{W}(x)u_0(t,x)\end{aligned}$$

$u_\varepsilon(t,x) \Rightarrow u_0(t,x)$ in distribution

Remark:

- solution to the limiting SPDE [Hu-Nualart-Song 11]: $u_0(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(iV_1\sqrt{c_d}\int_0^t \dot{W}(B_s)ds)\}$
- Hermite rank equals one $\Rightarrow$ Gaussian noise in the limit
- proof based on moment convergence
- $\dot{W}(x)$ with other type of covariance structure, e.g., $\mathbb{E}\{W(dx)W(dy)\} = |x-y|^{-\alpha}dxdy$ with $\alpha \in (0,2)$

Homogenization vs Convergence to SPDE

1. **Brownian motion in random scenery**
   - *homogenization*:

     $\varepsilon^{-1} \int_0^t V(B_s/\varepsilon)ds \Rightarrow \sigma W_t$ for Brownian motion W_t
   - *SPDE*:

     $\varepsilon^{-\alpha/2} \int_0^t V(B_s/\varepsilon)ds \Rightarrow \int_0^t \dot{W}(B_s)ds$ for Gaussian noise $\dot{W}$
2. **Assumptions** on random potentials:
   - *homogenization*:

     stationarity, ergodicity, $\hat{R}(\xi)|\xi|^{-2} \in L^1(\mathbb{R}^d)$
   - *SPDE*:

     stationarity, functional of Gaussian process

Discussions

1. Without stability: $\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \frac{1}{\varepsilon} V(\frac{x}{\varepsilon}) u_\varepsilon$
   - uniform integrability of $\exp(\varepsilon^{-1} \int_0^t V(B_s/\varepsilon) ds)$?
   - small time restriction for $V(x)$ Gaussian: $\exists T > 0$, $\forall t \in (0, T)$, $u_\varepsilon(t,x) \to u_0(t,x)$ in probability with $\partial_t u_0 = \frac{1}{2}\Delta u_0 + \frac{1}{2}\sigma^2 u_0$ [Bal 10]
2. *Homogenization setting*: asymptotic distribution of corrector $\varepsilon^{-\gamma}(u_\varepsilon - u_0)$?
3. *SPDE setting*: non-Gaussian noise in the limit?
   Hermite rank equals two, $V(x) = \Phi(g(x)) \sim \frac{V_2}{2}(g^2(x) - 1)$
   $d = 1$, Rosenblatt distribution [Taqqu 75]
4. Low dimension cases:
   - $d = 2$: similar results hold with $\varepsilon^{-1} \to (\varepsilon |\log \varepsilon|)^{-1}$
   - $d = 1$: SPDE in the limit;
     $\varepsilon^{-\frac{1}{2}} \int_0^t V(B_s/\varepsilon) ds \to \int_{\mathbb{R}} L_t(x) W(dx)$; local time exists in 1−d
     [Pardoux-Piatnitski 06]

Summary

1. Homogenization or convergence to SPDE

$$i\frac{1}{\varepsilon^{\beta}}V(\frac{x}{\varepsilon}) \to \begin{cases} -\sigma^2/2 \\ iV_1\sqrt{c_d}\dot{W}(x) \end{cases}$$

2. Limiting equation depending on the correlation property of random coefficients; integrability of $\hat{R}(\xi)|\xi|^{-2}$
3. Stationarity+ergodicity $\Rightarrow$ homogenization; error estimate requires more, e.g., strongly mixing property