Hitting probabilities for systems of stochastic partial differential equations: an overview

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Based on joint works with:

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Overview

- Introduction to the problem of hitting probabilities
- Benchmark results in the Gaussian case
- Conditions for non-Gaussian random fields
- Our results for systems of stochastic heat and wave equations
- Handling the critical dimension?

Fundamental problem in probabilistic potential theory

Let $U = (U(x),\ x \in \mathbb{R}^k)$ be an $\mathbb{R}^d$-valued continuous stochastic process.

Fix $I \subset \mathbb{R}^k$, compact with positive Lebesgue measure.

The range of U over I is the random compact set

$$U(I) = \{U(x),\ x \in I\}.$$

Question. For $A \subset \mathbb{R}^d$, what are bounds on

$$P\{U(I) \cap A \neq \emptyset\}?$$

Hitting points. Fix $z \in \mathbb{R}^d$. Is $P\{\exists x \in I : U(x) = z\} > 0$?

Which sets $A \subset \mathbb{R}^d$ are polar: $P\{U(I) \cap A \neq \emptyset\} = 0$?

What is the Hausdorff dimension of the range of U?

What is the Hausdorff dimension of level sets of U?

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First example: the Brownian sheet

Let $(W(x),\ x \in \mathbb{R}_+^k)$ denote an k-parameter $\mathbb{R}^d$-valued Brownian sheet, that is, a centered continuous Gaussian random field

$$W(x) = (W_1(x), \dots, W_d(x))$$

with covariance

$$E[W_i(x)W_j(y)] = \prod_{\ell=1}^{k} \min(x_\ell, y_\ell)\delta_{i,j}, \qquad i,j \in \{1, \dots, d\},$$

where $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$.

The case $k = 1$: Brownian motion $B = (B(t),\ t \in \mathbb{R}_+)$.

The case $k > 1$: multi-parameter extension of Brownian motion.

A few references: Orey & Pruitt (1973), R. Adler (1978), W. Kendall (1980), J.B. Walsh (1986), D. & Walsh (1992), Khoshnevisan & Shi (1999)

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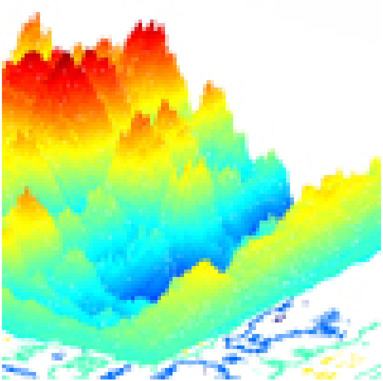
A sample path of the Brownian sheet $N = 2$, $d = 1$



Hitting probabilities for the Brownian sheet

Let $(W(x),\ x \in \mathbb{R}_+^k)$ denote a k-parameter $\mathbb{R}^d$-valued Brownian sheet.

Theorem 1 (Khoshnevisan and Shi, 1999)

Fix $M > 0$ and $0 < a_\ell < b_\ell < \infty$ $(\ell = 1, \dots, k)$. Let

$$I = [a_1, b_1] \times \cdots \times [a_k, b_k] \quad (\subset \mathbb{R}^k).$$

There exists $0 < C < \infty$ such that for all compact sets $A \subset B(0, M)$ $(\subset \mathbb{R}^d)$,

$$\frac{1}{C}\ Cap_{d-2k}(A) \leqslant P\{W(I) \cap A \neq \emptyset\} \leqslant C\ Cap_{d-2k}(A).$$

(see also results of F. Hirsch and S. Song (1991, 1995).

Measuring the size of sets: capacity

Capacity. $\text{Cap}_\beta(A)$ denotes the Bessel-Riesz capacity of A:

$$\text{Cap}_\beta(A) = \frac{1}{\inf_{\mu \in \mathcal{P}(A)} \mathcal{E}_\beta(\mu)},$$

$$\mathcal{E}_\beta(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k_\beta(x-y)\mu(dx)\mu(dy)$$

and

$$k_\beta(x) = \begin{cases} \|x\|^{-\beta} & \text{if } 0 < \beta < d, \\ \ln(\frac{1}{\|x\|}) & \text{if } \beta = 0, \\ 1 & \text{if } \beta < 0. \end{cases}$$

Examples. If $A = \{z\}$, then:

$$\text{Cap}_\beta(\{z\}) = \begin{cases} 1 & \text{if } \beta < 0, \\ 0 & \text{if } \beta \geqslant 0. \end{cases}$$

If A is a subspace of $\mathbb{R}^d$ with dimension $\ell \in \{1, \ldots, d-1\}$, then:

$$\text{Cap}_\beta(A) \begin{cases} > 0 & \text{if } \beta < \ell, \\ = 0 & \text{if } \beta \geqslant \ell. \end{cases}$$

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Another measure of the size of sets: Hausdorff measure

For $\beta \geqslant 0$, the β-dimensional Hausdorff measure of A is defined by

$$\mathcal{H}_\beta(A) = \lim_{\epsilon \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} (2r_i)^\beta : A \subseteq \bigcup_{i=1}^{\infty} B(x_i\,, r_i),\ \sup_{i \geq 1} r_i \leqslant \epsilon \right\}.$$

When $\beta < 0$, we define $\mathcal{H}_\beta(A)$ to be infinite.

Note. For $\beta_1 > \beta_2 > 0$,

$$\mathrm{Cap}_{\beta_1}(A) > 0 \Rightarrow \mathcal{H}_{\beta_1}(A) > 0 \Rightarrow \mathrm{Cap}_{\beta_2}(A) > 0.$$

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Anisotropic Gaussian fields (Xiao, 2008)

Let $(V(x),\ x \in \mathbb{R}^k)$ be a centered continuous Gaussian random field with values in $\mathbb{R}^d$ with i.i.d. components: $V(x) = (V_1(x), \dots, V_d(x))$. Set

$$\sigma^2(x, y) = E[(V_1(x) - V_1(y))^2].$$

Let I be a "rectangle". Assume the two conditions:

(C1) There exists $0 < c < \infty$ and $H_1, \dots, H_k \in]0, 1[$ such that for all $x \in I$,

$$c^{-1} \leqslant \sigma^2(0, x) \leqslant c,$$

and for all $x, y \in I$,

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(H_j is the Hölder exponent for coordinate j).

(C2) There is $c > 0$ such that for all $x, y \in I$,

$$\mathrm{Var}(V_1(y) \mid V_1(x)) \geqslant c \sum_{j=1}^{k} |x_j - y_j|^{2H_j}.$$

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Anisotropic Gaussian fields

Theorem 2 (Biermé, Lacaux & Xiao, 2007)

Fix $M > 0$. *Set*

$$Q = \sum_{j=1}^{k} \frac{1}{H_j}.$$

Assume $d > Q$. *Then there is* $0 < C < \infty$ *such that for every compact set* $A \subset B(0, M)$,

$$C^{-1} \; Cap_{d-Q}(A) \leqslant P\{V(I) \cap A \neq \emptyset\} \leqslant C\mathcal{H}_{d-Q}(A).$$

Special case obtained by D., Khoshnevisan and E. Nualart (2007)

This results tells us what sort of inequality to aim for when we have a non-Gaussian process and information about its Hölder exponents.

Notice the Hausdorff measure appearing on the right-hand side.

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Hitting points in the critical dimension

For the Brownian sheet W:

$$P\{W(I) \cap A \neq \emptyset\} \leqslant C \ \mathrm{Cap}_{d-2k}(A).$$

If $d = 2k$ and $A = \{z\}$ ($z \in \mathbb{R}^d$), then $\mathrm{Cap}_0(A) = 0$, so

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and therefore, points are polar.

For an anisotropic Gaussian random field V:

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If $d = Q$ and $A = \{z\}$ ($z \in \mathbb{R}^d$), then $\mathcal{H}_0(A) = 1$, so polarity of points remains unclear.

(This issue can be decided on a case-by-case basis for many Gaussian processes.)

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Polarity of points in dimensions > the critical dimension

Case $k = 1$, $d \geqslant 3$: let $(B(t), t \in \mathbb{R}_+)$ be a standard Brownian motion with values in $\mathbb{R}^3$. Want to explain why it does not hit points.

Explanation. Let $t_k = 1 + k2^{-2n}$. Fix $x \in \mathbb{R}^d$. Then

$$
\begin{aligned}
P\{\exists t \in [1,2] : B(t) = x\} &= P\left(\bigcup_{k=1}^{2^{2n}} \{\exists t \in [t_{k-1}, t_k] : B(t) = x\}\right) \\
&\leqslant \sum_{k=1}^{2^{2n}} P\{\exists t \in [t_{k-1}, t_k] : B(t) = x\} \\
&= \sum_{k=1}^{2^{2n}} \left[P\{\|B_{t_k} - x\| \leqslant n2^{-n}\} + e(n)\right] \\
&\leqslant \sum_{k=1}^{2^{2n}} \left[c(n2^{-n})^d + e(n)\right] \\
&= 2^{2n}\left[c(n2^{-n})^d + e(n)\right] \\
&= cn^d 2^{(2-d)n} + 2^{2n} e(n) \\
&\to 0 \qquad \text{as } n \to +\infty \text{ (because } d \geqslant 3).
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Hitting probabilities for non-Gaussian processes: upper bounds

Let $U = \{U(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ be an $\mathbb{R}^d$-valued continuous process.

Theorem 3 (D. & Sanz-Solé: upper bound)

Let $D \subset \mathbb{R}^d$. Assume that:
(1) For any $x \in \mathbb{R}^k$, $U(t,x)$ has a density $p_{(t,x)}$, and

$$\sup_{z \in D^{(2)}} \sup_{(t,x)\in(I\times J)^{(1)}} p_{(t,x)}(z) \le C$$

($D^{(2)}$ is the 2-enlargement of D.)
(2) There exist $\delta_1, \delta_2 \in\,]0,1]$ and a constant C such that, for any $q \in [1,\infty[$, $(t,x),(s,y) \in (I \times J)^{(1)}$,

$$E\left(\|U(t,x) - U(s,y))\|^q\right) \le C(|t-s|^{\delta_1} + \|x-y\|^{\delta_2})^q.$$

Then for any $\eta > 0$, for every Borel set $A \subset D$,

$$P\left\{v(I) \cap A \neq \emptyset\right\} \le C\mathcal{H}_{d-\eta-\frac{1}{\delta_1}-\frac{k}{\delta_2}}(A).$$

Remarks. (a) Condition (2) is essentially a condition on Hölder continuity.
(b) Condition (1) can often be obtained by using Malliavin calculus.
(c) Note that $d-\eta$ appears in the upper bound, but this is otherwise similar to the result of Biermé, Lacaux and Xiao (2007).

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Non-polarity of points in dimensions $<$ the critical dimension

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Want to show that for $d < 2k$, a point $z \in \mathbb{R}^d$ is not polar. Set $I = [1,2]^k$.

$$P\{\exists x \in I : W(x) = z\} = \lim_{\varepsilon \downarrow 0} P\{W(I) \cap B(z,\varepsilon) \neq \emptyset\}.$$

Define $J_\varepsilon = \int_I dx\, 1_{B(z,\varepsilon)}(W(x))$. Then

$$P\{W(I) \cap B(z,\varepsilon) \neq \emptyset\} \geqslant P\{J_\varepsilon > 0\} \geqslant \frac{(E(J_\varepsilon))^2}{E(J_\varepsilon^2)}.$$

Lower bound on $E(J_\varepsilon)$:

$$E(J_\varepsilon) = \int_I dx\ P\{W(x) \in B(z,\varepsilon)\} \geqslant \int_I dx\ \varepsilon^d \inf_{w \in B(z,\varepsilon)} p_x(w) = c\,\varepsilon^d.$$

Upper bound on $E(J_\varepsilon^2)$:

$$E(J_\varepsilon^2) = E\left[\int_I dx\, 1_{B(z,\varepsilon)}(W(x)) \int_I dy\, 1_{B(z,\varepsilon)}(W(y))\right]$$
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Need: (1) a lower bound on the probability density function of $W(x)$;
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Non-polarity of points in dimensions < the critical dimension

Let $(W(x),\ x \in \mathbb{R}_+^k)$ be a k-parameter $\mathbb{R}^d$-valued Brownian sheet.
Want to show that for $d < 2k$, a point $z \in \mathbb{R}^d$ is not polar. Set $I = [1,2]^k$.

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Hitting probabilities for non-Gaussian processes: lower bounds

Let $U = \{U(x),\ x \in \mathbb{R}^k\}$, $k \in \mathbb{N}^*$, be an $\mathbb{R}^d$-valued continuous process.

Theorem 4 (D. & Sanz-Solé)

Fix $N > 0$, $I \subset \mathbb{R}^k$ compact with positive Lebesgue measure, and assume:
(1) The density p_x of $U(x)$ is continuous, bounded, and positive.
(2) For any $x, y \in I$ with $x \neq y$, $(U(x), U(y))$ has a density $p_{x,y}$ w.r.t. Lebesgue measure in $\mathbb{R}^{2d}$, and there exist $\gamma, \alpha \in]0, \infty[$ such that for any $z_1, z_2 \in [-N, N]^d$

$$p_{x,y}(z_1, z_2) \leqslant \frac{C}{\|x-y\|^{\gamma}} \left[\frac{\|x-y\|^{\alpha}}{\|z_1 - z_2\|} \wedge 1 \right]^p,$$

where $p > (\gamma - k)\frac{2d}{\alpha} \vee 2$. Then there exists $c > 0$ such that for all Borel sets $A \subset [-N, N]^d$,

$$P\{U(I) \cap A \neq \emptyset\} \geq c\ \mathrm{Cap}_{\frac{1}{\alpha}(\gamma - k)}(A).$$

Remark. The r.h.s. in (2) is not of Gaussian type. It is a weaker condition.

Non-linear systems of stochastic p.d.e.'s

Let L be a partial differential operator (e.g. $L = \frac{\partial}{\partial t} - \Delta$).

Let $u(t,x) = (u^1(t,x), \dots, u^d(t,x)) \in \mathbb{R}^d$ be the solution of

$$\begin{cases} Lu^1(t,x) & = b^1(u(t,x)) + \sum_{j=1}^d \sigma_{1,j}(u(t,x)) \dot{W}_j(t,x), \\ & \vdots \\ Lu^d(t,x) & = b^d(u(t,x)) + \sum_{j=1}^d \sigma_{d,j}(u(t,x)) \dot{W}_j(t,x), \end{cases}$$

$$t \in]0,T], \quad x \in \mathbb{R}^k.$$

Lipschitz non-linearities: b^i, $\sigma_{i,j} : \mathbb{R}^d \to \mathbb{R}$, $\qquad i = 1, \dots, d$

Initial conditions: e.g. $u(0,x) = u_0(x)$ given.

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Cases considered

Wave equation, $k = 1$ (D. & E. Nualart, 2004):

$$Lu^i(t,x) = \frac{\partial^2 u^i}{\partial t^2}(t,x) - \frac{\partial^2 u^i}{\partial x^2}(t,x)$$

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The spatial dimension is $k \geqslant 1$ $(x \in \mathbb{R}^k)$.

k = 1. Usually, $\dot{W}(t,x)$ is space-time white noise with values in $\mathbb{R}^d$:

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$$E\,[\dot{W}_i(t,x)\dot{W}_j(s,y)] = \delta(t-s)\,\delta(x-y)\,\delta_{ij}.$$

k ⩾ 1. $\dot{W}(t,x)$ is spatially homogeneous Gaussian noise that is white in time:

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Upper bounds

- Existence (+ uniq.) of a random field solution $(u(t,x))$. Walsh (1986), ...
  Condition $0 < \beta < (2 \wedge k)$ is necessary and sufficient.
- Moments of increments (Hölder continuity), optimal exponents (!):

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- existence of a uniformly bounded density. Uses additional smoothness and uniform ellipticity hypotheses on b and σ and Malliavin calculus.

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lead to the following upper bound on hitting probabilities:

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where $Q = \frac{1}{H_1} + \frac{k}{H_2}$.

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For the systems of stochastic heat and wave equations, points are polar if $d > Q$.

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Typically, γ depends on d, and the best result would be with $\gamma = d$ (true for Gaussian case $b \equiv 0$, $\sigma \equiv \mathrm{Id}$).

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Results for systems of non-linear equations

Wave equation, $k = 1$, space-time white noise (D. & E. Nualart, 2004): $\gamma = d$, $H_1 = H_2 = \frac{1}{2}$,

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Heat equation, $k = 1$, space-time white noise (D., Khoshnevisan & E. Nualart, 2007, 2009) $\gamma = d + \eta$ $(\eta > 0)$, $H_1 = \frac{1}{4}$, $H_2 = \frac{1}{2}$,

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Heat equation, $k \geqslant 1$, homog. noise β (D., Khoshnevisan & E. Nualart, 2013) $\gamma = d + \eta$ $(\eta > 0)$, $H_1 = \frac{2-\beta}{4}$, $H_2 = \frac{2-\beta}{2}$,

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Wave equation, $k \in \{1, 2, 3\}$, homog. noise β (D., Sanz-Solé, MAMS, 2014?) $\gamma = d + \frac{4d^2}{2-\beta} + \eta$ $(\eta > 0)$, $H_1 = H_2 = \frac{2-\beta}{2}$,

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Results for systems of non-linear equations

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What about the critical dimension?

The previous results state that points are polar if $d > Q$, and are not polar if $d < Q$, where $Q = \frac{1}{H_1} + \frac{k}{H_2}$. What if Q is an integer and $d = Q$?

Even in the Gaussian case, there is no general theory (but the answer is known in various cases).

Mueller & Tribe (2002). Linear stochastic heat equation:

$$\frac{\partial u^i}{\partial t}(t,x) - \frac{\partial^2 u^i}{\partial x^2}(t,x) = \sigma \dot{W}^i(t,x), \qquad t > 0,\ x \in \mathbb{R},$$

but only in the case $\sigma \equiv 1$: the method does not apply to a deterministic function $\sigma = \sigma(t,x)$!

A method of Talagrand (1995, 1998). Based a harmonic representation (of fractional Brownian motion).

Ongoing project with C. Mueller & Y. Xiao.

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A harmonic representation of the solution to the linear stochastic heat equation in spatial dimension 1

Let $W_1(d\tau, d\xi)$ and $W_2(d\tau, d\xi)$ be two independent real-valued space-time white noises. Set

$$\begin{aligned} v(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} (\xi^4+\tau^2)^{-1}[\cos(\xi x)(\cos(\tau t) - e^{-t\xi^2}) - \sin(\xi x)\sin(\tau t)] \\ \times (\xi^2 W_1(d\tau, d\xi) - \tau W_2(d\tau, d\xi)) \\ + \int_{\mathbb{R}}\int_{\mathbb{R}} (\xi^4+\tau^2)^{-1}[\sin(\xi x)(\cos(\tau t) - e^{-t\xi^2}) - \cos(\xi x)\sin(\tau t)] \\ \times (\xi^2 W_2(d\tau, d\xi) - \tau W_1(d\tau, d\xi)) \end{aligned} \tag{1}$$

Observations.

(1) $v(0,x) = 0$ and $\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2}$ is a space-time white noise.
(2) Such formulas can be obtained for many linear spde's, and then Talagrand's method can be applied.
(3) Solution to non-linear spde's can be (locally) approximated by (conditionally) linear spde's, so we expect to handle the critical dimensions for many non-linear systems of spde's.

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Publications on hitting probabilities

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