Space-Time Duality and Medical Ultrasound

James F. Kelly and Mark M. Meerschaert

A Workshop on Future Directions in Fractional Calculus Research and Applications
Michigan State University

October 21, 2016
1. Power-Law Attenuation in Ultrasound

2. Space-Time Duality

3. General Space-Time Duality and $\alpha = 1$.

4. Conclusions
Acknowledgments

• Robert J. McGough and Xiaofeng Zhao, Department of Electrical and Computer Engineering, MSU
• Harish Sankaranarayanan and Lorick Huang, Department of Probability and Statistics, MSU
• Thomas Szabo, Department of Biomedical Engineering, Boston University
• John P. Nolan, Department of Mathematics and Statistics, American University
Time-Fractional vs. Space-Fractional

Time-Fractional PDEs Models
sub-diffusion via long waiting times ("hold-ups")

\[
\left( \frac{\partial}{\partial t} \right)^\gamma C = A_x C
\]

\[
C(x, t) = \int_0^\infty h_\gamma(x, u)g(u, t) \, du
\]

where \( \partial_t g = A_x g \) and \( h_\gamma(x, u) \) is the inverse stable subordinator density.

Space-Fractional PDEs: Models
super-diffusion via long particle jumps ("fast-paths")

\[
\frac{\partial}{\partial t} C(x, t) = \frac{\partial^\alpha}{\partial x^\alpha} C(x, t)
\]

\[
C(x, t) = f_{\alpha,1}(x, t)
\]

where \( f_{\alpha,\beta}(x, t) \) is a stable density (with scaling parameter \( t \)).
Ultrasound: Two Applications

B-Mode Ultrasound Imaging
(Webb, 2003)

Histotripsy (Maxwell, 2012)
Ultrasound: Power Law Attenuation

- Ultrasound waves attenuate as they travel through tissue.
- Limits maximum imaging depth for B-mode imaging.
- Influences maximum focal pressure for histotripsy.
- Attenuation coefficient $\alpha(\omega)$ fits a power-law

$$\alpha(\omega) = \alpha_0 |\omega|^y.$$
Stokes Wave Equation

- Stokes Wave Equation (1845) is a classical PDE model:

\[
\nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} + \tau \frac{\partial}{\partial t} \nabla^2 p = 0.
\]

- Wave attenuation is proportional to relaxation time \( \tau \) [\( \mu s \)].
- Take FTs with respect to space and time, yielding

\[
\left[ -k^2 + \frac{\omega^2}{c_0^2} + i\omega \tau k^2 \right] \bar{p}(k, \omega) = 0.
\]

- Dispersion relationship is \( k(\omega) = \omega/c_0(1 - i\omega \tau)^{-1/2} \).
- Attenuation is \( \alpha(\omega) = \text{Im}k(\omega) \sim \tau/(2c_0)\omega^2 \) for \( \omega \tau \ll 1 \).
- Phase velocity \( c(\omega) \) is constant for \( \omega \tau \ll 1 \) (no dispersion).
Models for Attenuation in Ultrasound

- Early models (Gurumurthy and Arthur, 1982) modeled attenuation/dispersion in the frequency domain.
- Szabo (1994) proposed a phenomenological model for ultrasound in power law media \((0 \leq y \leq 2)\).

\[
\nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \frac{2\alpha_0}{c_0 \cos(\pi y/2)} \frac{\partial^{y+1} p}{\partial t^{y+1}} = 0.
\]

- Interpolates between the integer-ordered telegrapher’s equation \((y = 0)\) and the (viscous) Blackstock (1967) equation \((y = 2)\) using a time-fractional derivative. Invalid for \(y = 1\).
Power Law Wave Equation (PLWE)

Assume a dispersion relationship:

\[ k(\omega) = \frac{\omega}{c_0} - \frac{\alpha_0 (-i)^{y+1} \omega^y}{\cos(\pi y/2)} \]

for \( \omega \geq 0 \) and \( k(-\omega) = k^*(\omega) \) to ensure real solutions. Imaginary part of the dispersion relationship is

\[ \alpha(\omega) = \alpha_0 |\omega|^y. \]

Compute the phase speed as

\[ \frac{1}{c(\omega)} = \frac{\text{Re} \ k(\omega)}{\omega} = \frac{1}{c_0} + \alpha_0 \tan \left( \frac{\pi y}{2} \right) |\omega|^{y-1}, \]

which is predicted by the Kramers-Krönig relationships and supported by measurements.
Square the dispersion relationship and multiply by FT $\bar{p}(k, \omega)$

$$
\left[-k^2 + \frac{\omega^2}{c_0^2} - \frac{2\alpha_0 (-i\omega)^{y+1}}{c_0 \cos(\pi y/2)} - \frac{\alpha_0^2 (-i\omega)^{2y}}{\cos^2(\pi y/2)}\right] \bar{p}(k, \omega) = 0.
$$

Perform an inverse FTs (space and time), yielding the PLWE (Kelly et. al., 2008)

$$
\nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \frac{2\alpha_0}{c_0 \cos(\pi y/2)} \frac{\partial^{y+1} p}{\partial t^{y+1}} - \frac{\alpha_0^2}{\cos^2(\pi y/2)} \frac{\partial^{2y} p}{\partial t^{2y}} = 0,
$$

which satisfies the dispersion relationship exactly for $y \neq 1$. For mammalian tissue, power-law exponent $y$ is very close to one!
PLWE: 3D Green’s Function (1)

Solve PLWE subject to an impulse point-source with zero initial conditions in free-space

\[ \nabla^2 g - \frac{1}{c_0^2} \frac{\partial^2 g}{\partial t^2} - \frac{2\alpha_0}{c_0 \cos(\pi y/2)} \frac{\partial^{y+1} g}{\partial t^{y+1}} - \frac{\alpha_0^2}{\cos^2(\pi y/2)} \frac{\partial^{2y} g}{\partial t^{2y}} = -\delta(R)\delta(t), \]

where \( R \) is the relative displacement between the source and the observer and \( R = |R| \). Take Fourier transform wrt \( t \)

\[ \nabla^2 \hat{g} + k^2(\omega) \hat{g} = -\delta(R), \]

where \( k(\omega) \) is our dispersion relationship. The Green’s function for this Helmholtz equation is a spherical wave

\[ \hat{g}(R, \omega) = \frac{e^{ik(\omega)R}}{4\pi R}. \]
PLWE: 3D Green’s Function (2)

Inserting the dispersion relationship into the spherical wave solution yields

$$\hat{g}(R, \omega) = \left[ \frac{\exp(i\omega R/c_0)}{4\pi R} \right] \left[ \exp \left( -\alpha_0 R(|\omega|^y - i\tan(\pi y/2)|\omega|^y) \right) \right],$$

where the first factor solves the lossless Helmholtz equation. Evaluate inverse Fourier transform and apply the convolution theorem, yielding

$$g(R, t) = \mathcal{F}^{-1} [\hat{g}(R, \omega)].$$

$$g(R, t) = g_D(R, t) \ast g_L(R, t)$$

where $g_D(R, t) = \delta(t - R/c_0)/4\pi R$ is the Green’s function (transient spherical wave) for the lossless wave equation.
Interlude: Stable Parameterizations

1. ST parameterization (Samoradnitsky and Taquu, 1994):

\[ f_{\alpha, \beta}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \exp(i\mu k + \sigma_{\alpha, \beta}(k)) \, dk \]

\[ \psi_{\alpha, \beta}(k) = -|k|^\alpha \left( 1 - i\beta \text{sgn}(k) \tan \left( \frac{\pi \alpha}{2} \right) \right) \quad \text{for } \alpha \neq 1 \]

\[ \psi_{\alpha, \beta}(k) = -|k| \left( 1 + \frac{2i \text{sgn}(k)}{\pi} \ln |k| \right) \quad \text{for } \alpha = 1 \]

\[ \sigma_{\alpha} = |\cos(\pi \alpha/2)| \]

2. Zolotarev C-Parameterization (for duality)

\[ p_{\alpha}(x; \eta, b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \exp \left[ -b|\lambda|^\alpha \exp \left( -\frac{i\pi \eta \lambda}{2|\lambda|} \right) \right] \, d\lambda \]
The second term is a loss function defined as

\[ g_L(R, t) = \mathcal{F}^{-1} \left[ \exp \left( -\alpha_0 R (|\omega|^y - i \tan(\pi y / 2) \omega |\omega|^{y-1}) \right) \right] \]

\[ = \frac{1}{(\alpha_0 R)^{1/y}} f_{y,1} \left( \frac{t}{(\alpha_0 R)^{1/y}} \right). \]

- Nice for engineers, since stable PDFs may be numerically evaluated using STABLE toolbox (Nolan, 1997) or MATLAB 2016a.
- Solution of a time-fractional equation involves a stable density, not an *inverse* stable density. Not what we expected!
- Solution involves a PDF: What is the random variable?
Numerical Results: Green’s Functions

(a) $t = 20 \mu s$

(b) $t = 50 \mu s$

**Figure:** Snapshots of the 3D power law Green’s function for $y = 0.5, 1.5,$ and 2.0 for $\alpha_0 = 0.05 \text{ mm}^{-1}\text{MHz}^{-y}$. Snapshots of the Green’s function are shown for $t = 20$ and 50 $\mu s$. 
Ultrasound Pulse Propagation in Tissue

Given an input pulse $v(t)$, the velocity potential $\phi(r, t)$ is

$$\phi(r, t) = v(t) \ast g(r, t)$$

Pulse experiences a frequency downshift and distortion as depth increases.

(a) $R = 10$ mm

(b) $R = 100$ mm
Causality

Physics demands causality. The Green’s function is causal if $g(R, t) = 0$ for all $t < 0$. PLWE Green’s function is

$$g(R, t) = \frac{1}{4\pi R} \frac{1}{(\alpha_0 R)^{1/y}} f_{y,1} \left( \frac{t - R/c_0}{\alpha_0 R}^{1/y} \right).$$

- If $y < 1$, then $f_{y,1}(z) = 0$ is $z < 0$. Then $g(R, t) = 0$ if $t < R/c_0$, implying causality.
- If $y \geq 1$, then $f_{y,1}(z) > 0$ for all $z$. Then $g(R, t) > 0$ for $t < 0$, violating causality!
- However, $f_{y,1}(z)$ decays with exponential order for $t \to -\infty$:

  $$f_{y,1}(z) \approx A|z|^{\nu} \exp(-B|z|^{\mu}),$$

  where $A$, $B$, $\mu$, and $\nu$ are functions of $y$ only.
- For observation points only one wavelength from the radiating source, the relative magnitude of $g(R, t)$ is less than -136 dB for all $1 < y \leq 2$. 
Many Models: Which is the “Right” One?

The Szabo (1994) wave equation

\[ \nabla^2 p = \frac{1}{c_0^2} \partial_t^2 p + \frac{2\alpha_0}{c_0 b} \partial_t^{y+1} p \]

is a simplified PLWE. Chen and Holm (2004) recommend

\[ \nabla^2 p + \alpha_0 \partial_t (-\nabla^2)^{y/2} p = \frac{1}{c_0^2} \partial_t^2 p \]

using a fractional Laplacian. Caputo (1967) and Wismer (2006) propose

\[ \nabla^2 p = \frac{1}{c_0^2} \partial_t^2 p + \tau^{y-1} \partial_t^{y-1} \nabla^2 p \]

while Treeby and Cox (2010) consider

\[ \nabla^2 p + \alpha_0 \partial_t (-\nabla^2)^{y/2} p + \alpha_1 \partial_t \nabla^{(\beta+1)/2} p = \frac{1}{c_0^2} \partial_t^2 p. \]

All exhibit power law attenuation \( \alpha(\omega) = \alpha_0 |\omega|^\beta \) for \( 1 < y < 2. \)
Solutions

Analytical comparison (Kelly and McGough, 2016)

Numerical comparison (Zhao and McGough, 2016)
Model Unification and Duality

- Many models proposed that agree with experiments, but who's right, and who's wrong?
- The idea of duality: two ways of looking at the same thing (Atiyah, 2008).
- Famous Example: wave-particle duality of light. Light behaves like a particle (Democritus) and a wave (Descartes). These contrary viewpoints were unified by quantum mechanics.
- Perhaps this duality principle can resolve (and unify) these alternative time-fractional and space-fractional models?
A Brief History of Space-Time Duality

• Zolotarev (1961) noted an equivalence between stable densities of index $\alpha$ and $1/\alpha$ in the C parameterization:

Theorem

(Duality Principle) For any pairs of admissible parameters $\alpha \geq 1$, $\theta$ and any $u > 0$

$$p_\alpha (u; \eta, 1) = u^{-(1+\alpha)} p_{\alpha^*} (u^{-\alpha}; \eta^*, 1),$$

where $\alpha^* = 1/\alpha$ and $1 + \eta^* = \alpha(1 + \theta)$.

• Feller (1971) gave a simplified proof of this “curious by-product” using infinite series

• Baeumer et. al. (2009) recognized that (negatively skewed) space-fractional diffusion equations are solved by inverse stable densities, while time-fractional diffusion equations are solved by stable densities:

$$f_{\alpha,-1}(x, t) = \gamma h_\gamma(x, t)$$

where $\gamma = 1/\alpha$. 
A Heuristic Argument

Let $1 < \alpha \leq 2$ and $1/2 \leq \gamma = 1/\alpha < 1$. Consider negatively skewed FDE:

$$\frac{\partial C_0}{\partial t} = \frac{\partial^\alpha C_0}{\partial (-x)^\alpha}.$$ 

Apply the Fourier transform in both variables

$$[(i\omega) - (-ik)^\alpha] \hat{C}_0 = 0.$$ 

Dispersion relationship: $i\omega - (-ik)^\alpha = 0$. Dual dispersion relationship: $(i\omega)^\gamma = (-ik)$. Inverting the FTs leads to the dual equation

$$\frac{\partial^\gamma C_0}{\partial t^\gamma} = -\frac{\partial C_0}{\partial x}.$$ 

- Heaviside (1871) noted this relationship for the classical diffusion equation $(\alpha = 2)$.
- Baeumer et. al. (2009) noted this equivalence from $f_{\alpha,-1}(x, t) = \gamma h_\gamma(x, t)$. 
Some New Results

1. New proof of duality using Fourier-Laplace transforms (FLTs).
2. Duality principle assumes $x > 0$. We extend duality to $x < 0$, thereby covering the real line.
3. Consider problems with drift: fractional advection dispersion equation (FADE)

\[
\frac{\partial C}{\partial t} = -v \frac{\partial C}{\partial x} + D \frac{\partial^{\alpha} C}{\partial (-x)^{\alpha}}.
\]
FLT Approach

Cauchy problem for fractional diffusion/dispersion equation (FDE)

\[ \frac{\partial C_0}{\partial t} = \frac{\partial^\alpha C_0}{\partial (-x)^\alpha} \text{ subject to } C(x, 0) = \delta(x). \]

Apply the Fourier-Laplace transform (FLT)

\[ C_0(k, s) = \int_0^\infty \int_{-\infty}^\infty e^{-st} e^{-ikx} C_0(x, t) \, dx \, dt \]

to get \( sC_0(k, s) - 1 = (-ik)^\alpha C_0(k, s) \). Rearrange as

\[ C_0(k, s) = \frac{1}{s - (-ik)^\alpha}. \]

Apply an inverse LT followed to inverse FT:

\[ C_0(x, t) = \frac{1}{t^{1/\alpha}} f_{\alpha, -1} \left( \frac{x}{t^{1/\alpha}} \right). \]
Why not apply the inverse FT first? The inverse FT can be expressed as (Morse and Feschbach, 1953)

\[ \tilde{C}_0(x, s) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T+i\tau}^{T+i\tau} e^{ikx} \left( s - (-ik)^\alpha \right) dk, \]

where \( \tau > 0 \) is chosen to avoid the branch cut along the negative real axis. Integrand has a single pole at \( k^* = is^{1/\alpha} \) and remains analytic for all other points in the upper half-plane.

![Contour C = L_T + C_T for x > 0.](image-url)
FLT Approach (Cont’d)

Evaluate the contour integral, yielding

\[ \tilde{C}_0(x, s) = \gamma s^{\gamma-1} \exp(-xs^\gamma) \quad \text{for} \ x > 0, \]

where \( \gamma = 1/\alpha \in [1/2, 1) \). Invert using

\[ \tilde{h}_{\gamma,+}(x, s) = s^{\gamma-1} \exp(-xs^\gamma) \]

for the LT of the inverse stable subordinator density (see Meerschaert and Sikorskii, 2012)

\[ h_{\gamma,+}(x, t) = \frac{t}{\gamma x^{\gamma 1+1/\gamma}} f_{\gamma,1} \left( tx^{-1/\gamma} \right). \]

Compare and use the uniqueness of the LT to get

\[ C_0(x, t) = \gamma h_{\gamma,+}(x, t) \quad \text{for all} \ x > 0. \]

For \( x > 0 \), the negatively skewed diffusion (dispersion) equation is solved by a positively skewed stable PDF with index \( \gamma = 1/\alpha \).
FLT Approach (Cont’d)

Take FT of $\tilde{h}_{\gamma,+}(x, s) = H(x)s^{\gamma-1}\exp(-xs^\gamma)$, yielding

$$\tilde{h}_{\gamma,+}(k, s) = \frac{s^{\gamma-1}}{ik + s^\gamma}.$$ 

Rewrite $s^{\gamma}\tilde{h}_{\gamma,+}(k, s) - s^{\gamma-1} = -(ik)\tilde{h}_{\gamma,+}(k, s)$ and invert

$$\left(\frac{\partial}{\partial t}\right)^\gamma h_{\gamma,+}(x, t) = -\frac{\partial}{\partial x} h_+(x, t); \quad h_{\gamma,+}(x, 0) = \delta(x).$$

Since $C_0(x, t)$ is proportional to $h_{\gamma,+}(x, t)$ for all $x > 0$ and $t > 0$,

$$\left(\frac{\partial}{\partial t}\right)^\gamma C_0(x, t) = -\frac{\partial}{\partial x} C_0(x, t) \quad \text{for } x > 0 \text{ and } t > 0.$$ 

Agrees with heuristic argument and Baeumer et. al. (2009) result.
Duality for $x < 0$

Apply the reflection property $p_\alpha(-x; \eta, b, 0) = p_\alpha(x; -\eta, b, 0)$ for stable densities for $x < 0$:

$$p_\alpha (x; \eta, 1, 0) = p_\alpha (-|x|; \eta, 1, 0)$$
$$= p_\alpha (|x|; -\eta, 1, 0)$$
$$= |x|^{-1-\alpha} p_\gamma (|x|^{-\alpha}; \eta^*, 1, 0)$$

with $\gamma = 1/\alpha$ and $\eta^* = 2 - 3\gamma$. In ST parameterization

$$f_{\alpha,-1}(x, 0) = |x|^{-1-1/\gamma} f_{\gamma,\beta^*} (|x|^{-1/\gamma}).$$

Hence, $C_0(x, t) = \gamma h_{-}\gamma(-x, t)$ for $x < 0$ where

$$h_{\gamma,-}(x, t) = \frac{t}{\gamma x^{1+1/\gamma}} f_{\gamma,\beta^*} \left( tx^{-1/\gamma} \right) H(x).$$
Duality for FADE

Consider the negatively-skewed FADE (Benson et. al., 2000)

$$\frac{\partial C}{\partial t} = -v \frac{\partial C}{\partial x} + D \frac{\partial^\alpha C}{\partial (-x)^\alpha}$$

on the real line. Then $C(x, t)$ has a traveling wave solution

$$C(x, t) = C_0(x - vt, Dt)$$

where $C_0(x, t)$ solves the FDE. Apply duality on the positive and negative axes:

$$C(x, t) = \gamma h_{\gamma,+}(x - vt, Dt)H(x - vt) + \gamma h_{\gamma,-}(x - vt, Dt)H(vt - x)$$

where $H(x)$ is the Heaviside function.
Dual Solution for FADE

\[ C(x, t) = \gamma h_{\gamma,+}(x - vt, Dt)H(x - vt) + \gamma h_{\gamma,-}(x - vt, Dt)H(vt - x). \]

Figure: Comparison of FADE solution (solid) with dual solution (markers) with parameters are \( \alpha = 3/2, \nu = 1, t = 2, \) and \( D = 1. \)
The Governing Equation

For $x > vt$, we can show the FLT relationship (Kelly and Meerschaert, 2016)

$$\bar{C}(k, s) = \frac{\gamma (s + ikv)^{\gamma - 1}}{D^\gamma ik + (s + ikv)^\gamma}.$$ 

Invert using the FLT formula (Meerschaert et. al., 2002)

$$\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right)^\gamma f(x, t) \mapsto (s + ikv)^\gamma \tilde{f}(k, s)$$

and the LT formula $t^{-\gamma}/\Gamma(1 - \gamma) \mapsto s^{\gamma - 1}$, yielding a coupled space-time fractional governing equation for $x > vt$

$$\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right)^\gamma C(x, t) = -D^\gamma \frac{\partial}{\partial x} C(x, t) + \gamma \delta(x - vt) \frac{t^{-\gamma}}{\Gamma(1 - \gamma)}.$$ 

This space-time operator is a fractional material derivative (Sokolov and Metzler, 2003).
Physical Explanation

• Negatively skewed FADE models large negative (upstream) jumps. Zhang (2009) noted this is unphysical!

• The dual space-time fractional equation resolves this problem. Consider the fractional material derivative:

\[
\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right)^\gamma
\]

• Material derivative is the time-rate of change in a moving coordinate system.

• The Caputo derivative models waiting times (retention) in this moving frame.
Can we extend these results?

1. **Space-fractional PDEs:** \( \partial_t u(x, t) = p \partial_x^\alpha u(x, t) + q \partial_{-x}^\alpha u(x, t) \).

2. **Tempered FDEs:** \( \partial_t u(x, t) = \partial_{-x}^{\alpha, \lambda} u(x, t) \), where \( \partial_{-x}^{\alpha, \lambda} \) is the tempered fractional RL derivative (Baeumer and Meerschaert, 2010) and (Li et. al, 2015).

3. **FDEs with boundary conditions:** \( \partial_t u(x, t) = \partial_{-x}^\alpha u(x, t) \) on \( x > 0 \) with \( \partial_{-x}^{\alpha-1} u(0, t) = 0 \).
General space-FDEs

Consider FPDE for $x > 0$

$$\frac{\partial}{\partial t} C(x, t) = p \frac{\partial^\alpha}{\partial x^\alpha} C(x, t) + q \frac{\partial^\alpha}{\partial (-x)^\alpha} C(x, t),$$

(1)

where $p + q = 1$, $\beta = p - q$, and the fractional derivatives are Riemann-Liouville. Solution is

$$C(x, t) = \frac{1}{t^{1/\alpha}} f_{\alpha, \beta} \left( \frac{x}{t^{1/\alpha}} \right).$$

(2)

Rewrite in Zolotarev’s C parameterization, apply duality, and transform back to ST parameterization:

$$C(x, t) = \frac{t}{x^{1+1/\gamma}} \frac{1}{x^{1/\gamma}} f_{\gamma, \beta^*} \left( \frac{t}{x^{1/\gamma}} \right) H(x)$$

(3)

with $\gamma = 1/\alpha$ and skew $\beta^* = \beta^*(\beta, \alpha)$. 
General space-FDEs

This dual solution solves a time-fractional PDE:

\[
p^* \frac{\partial^\gamma}{\partial t^\gamma} C(x, t) + q^* \frac{\partial^\gamma}{\partial (-t)^\gamma} C(x, t) = -\frac{\partial}{\partial x} C(x, t) + p^* \delta(x) b(t),
\]

where \( \beta^* = p^* - q^* \) and \( b(t) \) is a source term. Several questions:

1. Is this time-fractional equation the scaling limit of some CTRW? For example, a time-reversed subordinator? (Lorick Huang)

2. Is it possible to transform only the negative jumps into a positive time-fractional derivative, yielding a governing equation without the negatively-skewed time-fractional derivative?
Tempered FDEs

Truncated power-laws can be modeled using tempered time derivatives or tempered space derivatives. Consider

\[ \partial_t u = \partial_{\lambda}^\alpha u \text{ where } u(x, 0) = \delta(x). \]

where \( \partial_{\lambda}^\alpha \) has Fourier symbol \( \psi(k) = (\lambda - ik)^\alpha - \lambda^\alpha, 1 < \alpha \leq 2, \) and \( \lambda > 0. \) Solve using FLT’s and apply Zolotarev duality, yielding

\[ u(x, t) = e^{\lambda x} e^{-\lambda^\alpha t} f_{\alpha,-1}(x, t) \]
\[ = \gamma e^{\lambda x} e^{-\lambda^\alpha t} h_\gamma(x, t), \]

Solves

\[ \left( \frac{\partial}{\partial t} \right)^{\gamma, \lambda} \quad u(x, t) = -\partial_x u(x, t) + b(x, t) \]
FDEs with boundary conditions

- Boundary-value problems for space-fractional PDEs are difficult.
- Is it possible to transform a space FDE with boundary conditions to an equivalent time-fractional FDE with boundary conditions?

\[ \partial_t u(x, t) = \partial_{-x}^\alpha u(x, t) \]

on \( x > 0 \) subject to a fractional flux boundary condition

\[ \partial_{-x}^{\alpha-1} u(0, t) = 0 \]
“Fractional Derivative” of order 1

What is the governing equation of Lévy motion of order one and skewness one? Define an operator

\[ D_+^1 f(x) = \mathcal{F}^{-1} \left[ \psi_{1,1}(-k) \hat{f}(k) \right], \]

where \( \psi_{1,1}(k) \) is the log characteristic function of a stable law with \( \alpha = 1 \) and \( \beta = 1 \):

\[ \psi_{1,1}(k) = -|k| \left( 1 + \frac{2i \text{sgn}(k)}{\pi} \ln |k| \right). \]

By Lemma 7.3.9 in (Meerschaert and Scheffler, 2001)

\[ \psi_{1,1}(k) = \frac{2}{\pi} \int_0^\infty \left( e^{iky} - 1 - ik \sin y \right) y^{-2} dy. \]

Invert FT, yielding the generator form:

\[ D_+^1 f(x) = \int_0^\infty \left( f(x - y) - f(x) + f'(x) \sin y \right) y^{-2} dy. \]
Caputo Form and an example

Integrate by parts with $u = f(x - y) - f(x) + f'(x)\sin y$ and $dv = y^{-2}dy$, to yield the Caputo form

$$D_+^1 f(x) = \frac{2}{\pi} \int_{0}^{\infty} \left[ f'(x) \cos y - f'(x - y) \right] y^{-1} dy.$$ 

Example

Let $f(x) = e^{\lambda x}$, where $\lambda > 0$.

$$D_+^1 f(x) = \frac{2}{\pi} \int_{0}^{\infty} \left[ \lambda e^{\lambda x} \cos y - \lambda e^{\lambda(x-y)} \right] y^{-1} dy$$

$$= \frac{2\lambda}{\pi} e^{\lambda x} \int_{0}^{\infty} \left( \cos y - e^{-\lambda y} \right) y^{-1} dy$$

$$= \frac{2}{\pi} \lambda \ln \lambda e^{\lambda x}$$

If $\lambda = 1$, this “derivative” is zero!
Summary

- Fractional wave equations (e.g. PLWE) are used to model attenuation and dispersion in biomedical ultrasound.
- Both TF and SF power-law models exist, prompting the question: "What is the correct model?"
- Space-time duality, which links SF and TF PDEs, allows models to be unified.
- We have applied duality to the negatively-skewed FDE and the spatial FADE.
- Many questions remain regarding general FDEs, FDEs with boundary conditions, etc.