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# BOUNDARY CONDITIONS FOR FPDE ON A FINITE INTERVAL


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## Introduction

Several works develop numerical methods to solve space fractional PDEs on finite domain **(FPDE)** including variable coefficients. However, even in the constant coefficients case, only a few authors address the issue of well-posedness of FPDE, see Defterli et al[1]. Ervin and Roop[2] showed well-posedness of FPDE with Dirichlet boundary conditions in the $L_2$-setting. Du et al[3] extended the theory to very general boundary conditions.

In this work[4] we formulate FPDE with physically meaningful boundary conditions, show well-posedness in $L_1$ and $C_0$ settings and compute numerical solutions in $L_1$ setting. We only consider the following initial value problem for FPDE on the interval [0,1],

$$\frac{\partial}{\partial t}u(x,t) = Au(x,t); \mathrm{u}(\mathrm{x},0) = u_0(x),$$

where the fractional derivative $A \in \{D_c^\alpha, D^\alpha\}$ is defined below.

## Fractional derivatives

For convenience set $p_\beta(x) = \frac{x^\beta}{\Gamma(\beta+1)}$ for $0 \le x \le 1$ and $\beta > -1$, with the understanding that $x \neq 0$ if $-1 < \beta < 0$.

For $x \in [0,1]$ define:

- Fractional integral of order $\upsilon > 0$,
$$I^\upsilon f(x) = \int_0^x f(s)p_{\upsilon-1}(x-s)ds$$
- Riemann-Liouville fractional derivative of order $1 < \alpha < 2$,
$$D^\alpha f(x) = \frac{d^2}{dx^2}\int_0^x f(s)p_{1-\alpha}(x-s)ds := D^2 I^{2-\alpha} f(x)$$
- First degree Caputo fractional derivative of order $1 < \alpha < 2$,
$$D_c^\alpha f(x) = \frac{d}{dx}\int_0^x \frac{d}{ds}f(s)p_{1-\alpha}(x-s)ds := DI^{2-\alpha}Df(x)$$

## Function spaces

- $C_0 := C_0[0,1]$ denotes the closure in the sup norm of the space of continuous functions with compact support in $[(0,1)]$, with end point(s) excluded for Dirichlet boundary condition.
- $L_1 := L_1[0,1]$ is identified with the closed subspace of Borel measures which consist of measures that possess a density.

## Cauchy problem

We study FPDE on [0,1] as a Cauchy problem,

$$\frac{df}{dt} = Af; f(0) = f_0,$$

where $A$ is a fractional derivative operator on $C_0$ or $L_1$ and its domain $\mathcal{D}(A)$ encode boundary conditions.

## Boundary conditions

The boundary conditions we consider are *zero* boundary conditions and $f \in \mathcal{D}(A)$ satisfies:

- *Dirichlet (absorbing)* boundary condition on the left if $\lim_{x\downarrow 0} f(x) = 0$ and on the right if $\lim_{x\uparrow 1} f(x) = 0$
- *Neumann (no flux)* boundary condition on the left if $\lim_{x\downarrow 0} Ff(x) = 0$ and on the right if $\lim_{x\uparrow 1} Ff(x)$ where the fractional flux $F \in \{D_c^{\alpha-1}, D^{\alpha-1}\}$ and $Af = DFf$

## Domains of fractional derivative operators

Fractional derivative operators $A$ on $C_0$ and $L_1$ are defined for $A \in \{D_c^\alpha, D^\alpha\}$. Here we only list the domains of $A$ on $L_1$. To highlight the boundary conditions, for example, fractional derivative operator $D_c^\alpha$ with left and right Dirichlet boundary conditions, we write $(D_c^\alpha, DD)$.

- $\mathcal{D}(D_c^\alpha, DD) = \{f \in L_1 : f = I^\alpha g - \Gamma(\alpha)I^\alpha g(1)p_{\alpha-1}, g \in L_1\}$
- $\mathcal{D}(D_c^\alpha, DN) = \{f \in L_1 : f = I^\alpha g - Ig(1)p_{\alpha-1}, g \in L_1\}$
- $\mathcal{D}(D_c^\alpha, ND) = \{f \in L_1 : f = I^\alpha g - I^\alpha g(1)p_0, g \in L_1\}$
- $\mathcal{D}(D_c^\alpha, NN) = \{f \in L_1 : f = I^\alpha g - Ig(1)p_\alpha + cp_0, g \in L_1\}$
- $\mathcal{D}(D^\alpha, ND) = \{f \in L_1 : f = I^\alpha g - \Gamma(\alpha-1)I^\alpha g(1)p_{\alpha-2}, g \in L_1\}$
- $\mathcal{D}(D^\alpha, NN) = \{f \in L_1 : f = I^\alpha g - Ig(1)p_\alpha + cp_{\alpha-2}, g \in L_1\}$

Note that there are only six different operators as $D^\alpha f = D_c^\alpha f$ when $f(0) = 0$. The constant $c \in \mathbb{R}$ that appears above is free.

## Grünwald Approximations

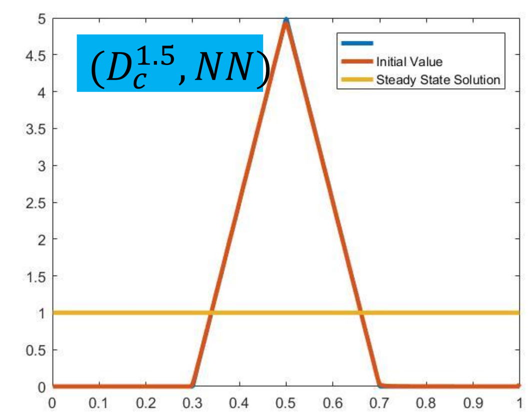
The well known Grünwald formula is modified to incorporate boundary conditions. As numerical solutions are computed on a finite number of grid points, the Grünwald formula can be written using the $n \times n$ matrix,

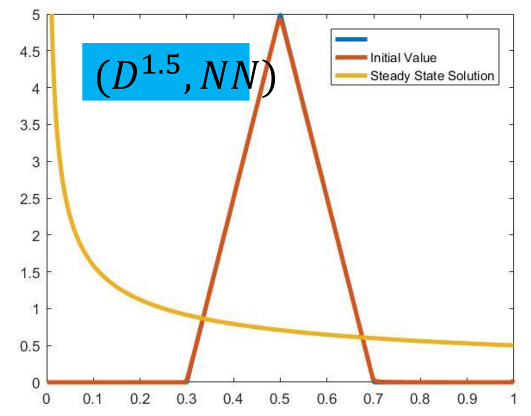
$$M_n = n^\alpha \begin{pmatrix} b_1^l & 1 & 0 & \cdots & 0 & 0 \\ b_2^l & -\alpha & 1 & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ b_{n-1}^l & g_{n-1}^\alpha & g_{n-2}^\alpha & \cdots & -\alpha & 1 \\ b_n & b_{n-1}^r & b_{n-2}^r & & b_2^r & b_1^r \end{pmatrix}.$$

Viewing $M_n$ as a transition rate matrix of a (sub-)Markov process, the entries in first column and last row $(b_\cdot^l, b_\cdot^r, b_n)$ are used to incorporate the boundary conditions. Grünwald approximation operators $G^n$ on $C_0$ and $L_1$ are also constructed using the matrices $M_n$ by way of continuous embedding.

## Steady State Solutions

Note that $D_c^\alpha p_0 = \mathbf{0} = D^\alpha p_{\alpha-2}$ and the steady state solutions for $(D_c^\alpha, NN)$ and $(D^\alpha, NN)$ given below correspond to the functions, $p_0$ and $p_{\alpha-2}$, that appear with the free constant $c$ in the respective domains given on the left.




## Time Evolution Plots

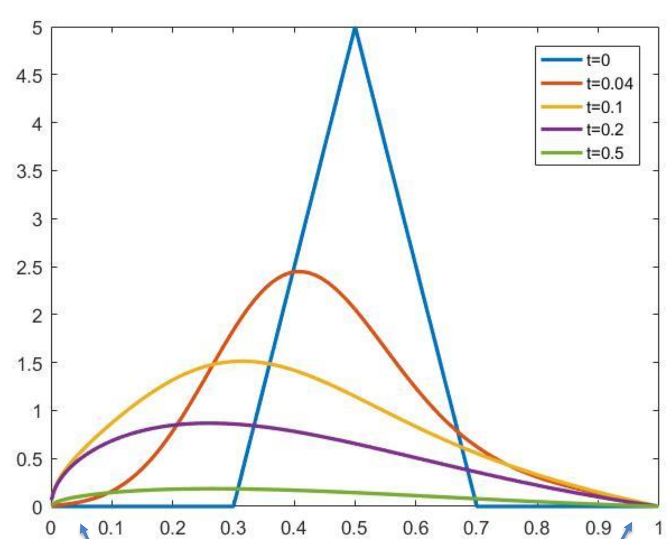
Each of the six figures below show time evolution plots of numerical solution to FPDE in $L_1$-setting,

$$\frac{\partial}{\partial t}u(x,t) = Au(x,t); \mathrm{u}(\mathrm{x},0) = u_0(x),$$

where $A \in \{D_c^{1.5}, D^{1.5}\}$ with boundary conditions as indicated on the figures and $u_0(x) = \begin{cases} 25x - 7.5, & 0.3 < x \le 0.5 \\ -25x + 7.5, & 0.5 < x < 0.7. \\ 0, & \text{otherwise} \end{cases}$

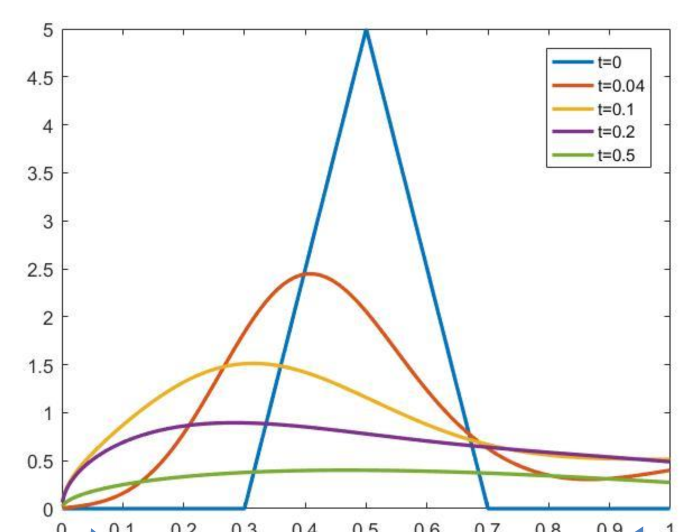
The numerical solutions were computed using (modified) Grünwald formula with $n = 1000$ grid points along with MATLAB ode15s solver.

$(D_c^{1.5}, DD)$



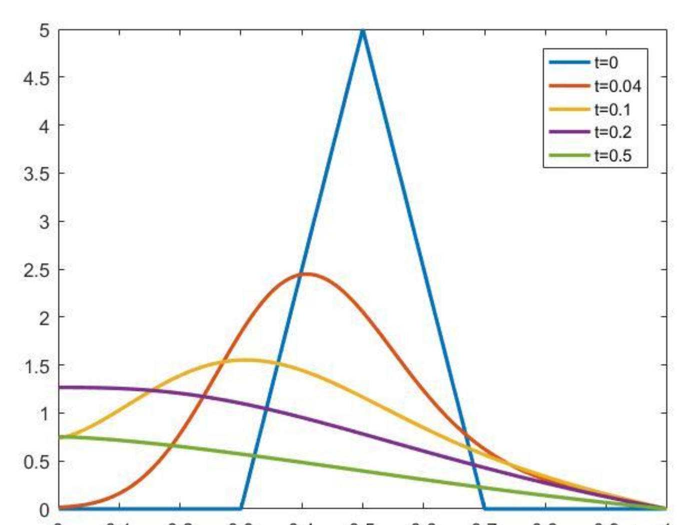

Absorbing boundary — Absorbing boundary

$(D_c^{1.5}, DN)$



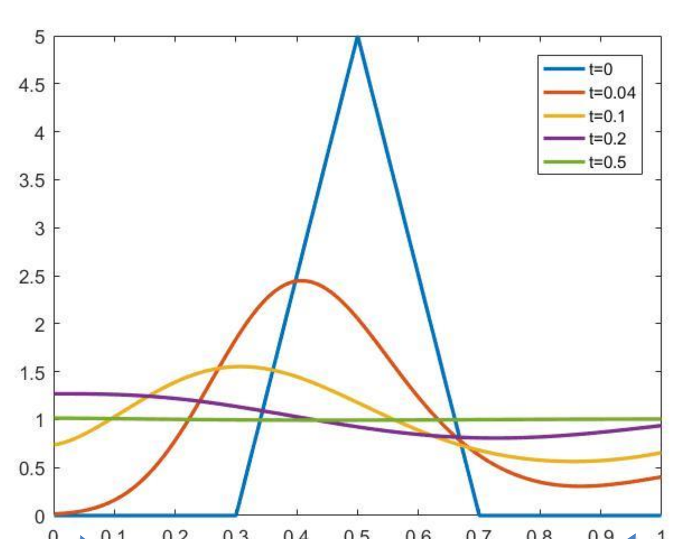

Absorbing boundary — No flux boundary

$(D_c^{1.5}, ND)$



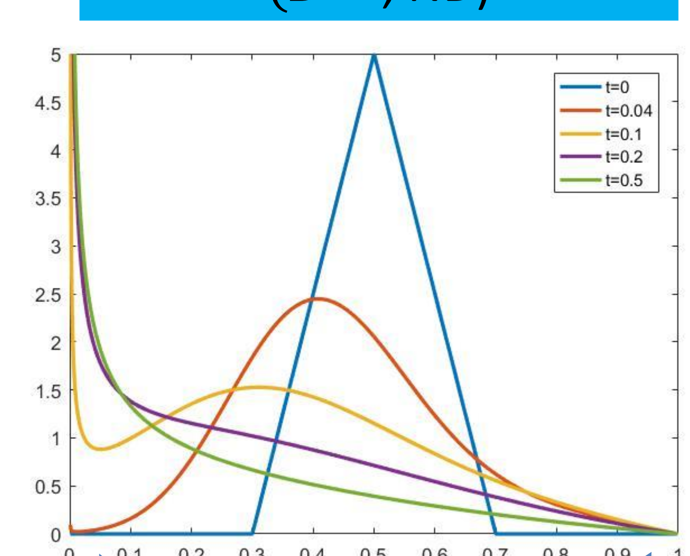

No flux boundary — Absorbing boundary

$(D_c^{1.5}, \mathrm{NN})$



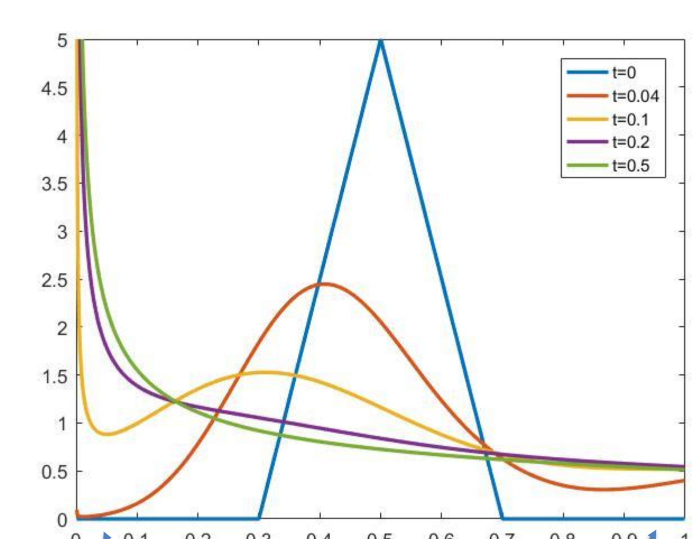

No flux boundary — No flux boundary

$(D^{1.5}, \mathrm{ND})$




No flux boundary — Absorbing boundary

$(D^{1.5}, \mathrm{NN})$




No flux boundary — No flux boundary

## Well-posedness

In line with classical theory, we study the backward and forward equations on $C_0$ and $L_1$, respectively. Grünwald approximations are shown to generate positive, strongly continuous, contraction semigroups on $C_0$ and $L_1$. Further, we show that the fractional derivative operators are dissipative, densely defined and closed with dense range$(I - A)$. Lumer-Phillips theorem[5] implies that $A$ generate strongly continuous, contraction semigroups on $C_0$ and $L_1$. This implies that the Cauchy problems associated with operators $A$ and initial value $f_0$ are well-posed[5], that is, the respective FPDE are well-posed.

## Process convergence

For each $f \in Core(A)$ we show that there exists $f_n \in C_0$(and $L_1$) such that $G^n f_n \to Af$ in the respective norms. Trotter-Kato theorem[5] implies that the semigroups generated by $G^n$ converge to the semigroups generated by $A$. Convergence of Feller semigroups (on $C_0$) further implies that the processes associated with Grünwald approximations converge to the Feller processes governed by the fractional derivative operators in the Skorokhod topology[6].

## Future Research

- Non-zero boundary conditions
- Identify the limiting processes explicitly
- Boundary conditions for FPDE with two-sided fractional derivatives
- Boundary conditions for FPDE on finite domains in $\mathbb{R}^d$

## References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