Density Bounds for some Degenerate Stable Driven SDEs.

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Abstract

We consider a system of linear differential equations whose first entry is perturbed by an anisotropic non degenerate (eventually tempered) Stable noise. Assuming some continuity on the noise coefficient, and a Hörmander like condition that allows the propagation of the noise through the system, we prove the uniqueness to the martingale problem for the associated generator, under technical restrictions on the number of oscillators and the dimension. Also, we establish density bounds reflecting the multi-scale behavior of the process.

The equation

We study degenerate stochastic differential equations driven by a finite range stable process, that is:

\[ \begin{align*}
    dX^1_t &= \left( a^{1,1} X_t^1 + \cdots + a^{1,n} X_t^n \right) dt + \sigma(t, X_t) dZ_t \\
    dX^2_t &= \left( a^{2,1} X_t^1 + \cdots + a^{2,n} X_t^n \right) dt \\
    dX^n_t &= \left( a^{n,1} X_t^1 + \cdots + a^{n,n-1} X_t^{n-1} \right) dt \\
    dX^n_t &= \left( a^{n,n-1} X_t^{n-1} + a^{n,n} X_t^n \right) dt, \quad X_0 = x \in \mathbb{R}^n,
\end{align*} \]  

(1)

where \( \sigma \in \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^d \), \( a^{i,j} \in \mathbb{R}^{n \times n} \) for \( 1 \leq i, j \leq n \), \( n \in [1, n] \), and \( \sigma \) is an \( \mathbb{R}^n \) valued symmetric \( \alpha \) stable (possibly tempered) process. Also, \( L \) is an \( \mathbb{R}^d \) valued symmetric \( \alpha \) stable (possibly tempered) process whose Lévy measure is truncated:

\[ \nu(dx) = C_d \omega(|x|) \frac{dz}{|z|^{d+1}}, \quad z = |z| \in \mathbb{R}_+ \times S^{d-1}. \]

Assumptions

- **[H1]** (Hölder regularity): \( \exists H > 0, \eta \in (0, 1), \forall x, y \in \mathbb{R}^n \) and \( \forall t \geq 0, \)

\[ ||\sigma(t, x) - \sigma(t, y)|| \leq H|x-y|^{\eta}. \]  

- **[H2]** (Non degeneracy of the stable measure): \( \exists \lambda_1, \lambda_2 \in \mathbb{R}_+, \forall \alpha \in \mathbb{R}^n \)

\[ \lambda_1 |u|^\alpha \leq \int_{|s| \leq 1} (|u|, \omega) \mu(ds) \leq \lambda_2 |u|^\alpha. \]

(2)

- **[H3]**: (Ellipticity): \( \exists x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}^n \) and \( \forall t \geq 0, \)

\[ \xi^\alpha \leq \left( \xi, \sigma(t, x) \xi \right) \leq c |\xi|^{\alpha}. \]

(3)

- **[H4]**: (Hörmander-like condition for (\( \lambda_1 \)l_{\geq 0})): \( \exists \alpha \in \mathbb{R}, \forall \xi \in \mathbb{R}_+, \forall \alpha \in [2, n-1], \forall (i, j) \in [1, n], \alpha_i^2 || \leq \eta \]

That is \( X_t = R_t(x) + \sum_{i \in [2, n]} Z_t^i, \) where \( R_t(x) \) denotes the transport of the initial condition by the Resolvent of the deterministic ODE associated with (1).

We can put all the component at the same scale by normalizing by

\[ T = \begin{pmatrix}
    s^2 t_i^2 & 0 \\
    0 & s^{n-1} t_i^2
  \end{pmatrix}. \]

The Parametrix Series

We approximate the solution of (1) by the solution of the Frozen equation:

\[ dX_t^y = \Lambda_t X_t^y ds + B_s(x, R_t(y)) dZ_s. \]

(4)

Denoting \( \rho_t \), the density of (4), and \( P_t \), the transition of (1), we have for all \( 0 \leq t < T, (x, y) \in (\mathbb{R}^d)^2 \) and any bounded measurable \( f : \mathbb{R}^d \rightarrow \mathbb{R} \),

\[ P_{t,T} f(x) = \mathbb{E}[f(X_T)|X_0 = x] = \int_{\mathbb{R}^d} \left( \sum_{i=0}^\infty \rho_i \otimes H^{i}(t, T, x, y) \right) f(y), \]

where \( H \) is the parametrix kernel:

\[ \forall 0 \leq t < T, (x, y) \in (\mathbb{R}^d)^2, H(t, T, x, y) := (L_t - P_{t,T}) \rho_i (y, t, x, y). \]

The notation \( \otimes \) stands for the time space convolution:

\[ f \otimes g(t, T, x, y) = \int_0^T du d\mathbb{R}^d f(t, u, x, z) g(u, T, z, y). \]

Main Result

Assume \([H]\) holds. When \( d(1-\alpha) + \alpha + 1 > 0 \), for every \( x \in \mathbb{R}^n \), there is a unique solution to the martingale problem associated with the generator of (1). When \( n = 2, d = 1 \), the unique weak solution of (1) has a density with:

\[ \forall n \geq 1, \exists C = C([H], T, K, m, \alpha) \geq 1, \ s.t. \forall 0 \leq t < s \leq T, (x, y) \in (\mathbb{R}^d)^2, \]

\[ p(t, s, x, y) \leq C p(s, t, x, y) \log(K \cdot |(\mathbb{T}^{n}_{s,t})^{-1}(y - R_s(x))|) \]

where

\[ p(t, s, x, y) = C \frac{\det(T_{s,t})^{-1}}{(K \cdot |(\mathbb{T}^{n}_{s,t})^{-1}(y - R_s(x))|)^{2/\alpha}} \cdot \left( C(s-t)^{(1/\alpha)} \left( T^{(n)}_{s,t} \right)^{-1}(y - R_s(x)) \right). \]

Eventually for \( 0 < T \leq T_0 := T_{\mathbb{R}^n([H], K)} \) small enough, the following diagonal lower bound holds:

\[ \forall 0 \leq t < s \leq T, (x, y) \in (\mathbb{R}^d)^2, s.t. \cdot (|\mathbb{T}^{n}_{s,t})^{-1}(y - R_s(x))| \leq K, \ p(t, s, x, y) \geq C^{-1} \det(T_{s,t})^{-1}. \]

Typical Sets

In the degenerate framework, the typical behavior is given by

\[ |(\mathbb{T}^{n}_{s,t})^{-1}(y - R_s(x))| \approx \frac{y^1 - R_s^1(x)}{s^1} + \cdots + \frac{y^n - R_s^n(x)}{s^{n-1}}. \]

Multi-scale Stable Process

From [Watanabe (TAMS) 2007], we know that if the spectral measure is such that \( \mu(B(r, x)) \leq Cr^{-\alpha} \), then:

\[ p_{2,t}(x) \leq C t^{-d/\alpha} \left( 1 + \frac{|x|}{t} \right)^{-\gamma - \alpha}. \]

And \( X_{2,t}^{x,\alpha,\delta} \in \mathbb{R}^d t^\alpha x + (s-t)^{-1} T_{s,t}^{\alpha,\delta}, \) where \( (S_{\alpha})_{\geq 0} \in \mathbb{R}^d, \) is a stable process with degenerate spectral measure, with support of dimension \( d + \alpha + 1 \in \mathbb{R}^d \).