Master's Exam - Fall 2009

October 15, 2009
1:00 pm – 5:00 pm

NUMBER:____

A. The number of points for each problem is given.

B. There are problems with varying numbers of parts.

1 – 5	Probability (80 points)
6 – 9	Statistics (80 points)

C. Write your answers on the exam paper itself. If you need more room you may use the extra sheets provided. Answer as many questions as you can on each part. Tables are provided.

PROBABILITY PART. **NUMBER:**____________________

1. (10 pts.) A deck consists of ten cards with numbers $1, 1, 2, 3, 3, 3, 3, 4, 4, 4$. Two cards are drawn randomly without replacement and then put back into the deck. This is repeated independently 45 times. Using an approximation, compute the chance that two or more of the 45 trials result in one card with 1 and one card with 2.

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2. (10 pts.) Suppose that only 1% of a population has a rare disease. If a person has the disease, medical tests can detect it correctly with probability 0.9. On the other hand, if he/she does not have the disease, medical tests will give correct results with probability 0.8. If medical tests show that a person has the disease, what is the probability that he/she does not have it?

3. Let Z_n be the discrete random variable which puts a mass of $\frac{n+1}{5n}$ on $\frac{1}{n}$ and a mass of $\frac{4n-1}{5n}$ on $1 - \frac{1}{n}$.

(a) (4 pts.) Calculate the moment generating function of Z_n.

(b) (6 pts.) Using (a) or otherwise, show that Z_n converges in distribution to a Bernoulli random variable with parameter 0.8.

4. Consider random variables Y_i with probabilities

$$\begin{cases} P(Y_i = 1) = p_i \\ P(Y_i = 0) = 1 - p_i \end{cases}$$

for $i = 1, 2, 3$. Assume that $0 < p_i < 1$ and Y_i are mutually independent for $i = 1, 2, 3$. Let $Z_1 = Y_1 + Y_3$ and $Z_2 = Y_2 + Y_3$.

(a) (5 pts.) Find $\mathbf{E}(Z_1)$ and $\mathbf{cov}(Z_1, Z_2)$.

(b) (15 pts.) Find the joint probability mass function of (Z_1, Z_2).

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5. Let X_1, X_2, X_3 be independent random variables with cdfs $F_1(x) = x^2, F_2(x) = x^4, F_3(x) = x^6$ for $0 \leq x \leq 1$ correspondingly. $F_i(x) = 0, i = 1, 2, 3,$ for $x < 0$ and $F_i(x) = 1, i = 1, 2, 3,$ for $x > 1$.

(a) (5 pts.) Given a random variable X_1 generated from F_1, how could such a random variable X_3 be generated from X_1?

(b) (10 pts.) Let $M = \max(X_1, X_2^2, X_3^3)$. Find the density of M.

(c) (15 pts.) Find $P(X_2^4 - X_1^2 > X_3^6)$. (Hint: graph the area of integration).

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STATISTICS PART. **NUMBER:**____________________

6. Let $X_1, \ldots, X_n$ be a random sample from the density $f(x, \theta) = \theta x^{\theta - 1}$, $0 < x < 1$.

(a) (10 pts.) Consider testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$. Prove that the rejection region of the uniformly most powerful (UMP) test has the form $\{\prod_{i=1}^{n} X_i > c\}$.

(b) (10 pts.) When $n = 1$, find the the exact value of the the critical value c such that the test in part (a) has level $\alpha = 0.05$.

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7. (10 pts.) A comparative experiment testing the effect of hallucinogenic drugs on rats running through a maze is performed. There are 7 rats involved in the experiment. Each rat goes through the maze two times - once after taking the hallucinogenic drugs, and once after taking a placebo. The running time (in seconds) are listed below.

Mouse	1	2	3	4	5	6	7
Time with Hallucinogenic drugs (X_i)	24	42	25	21	30	41	54
Time without Hallucinogenic drugs (Y_i)	22	37	22	25	22	42	44

Suppose you want to use the given data to test the hypothesis that the hallucinogenic drugs have effect on the average running time. What nonparametric test should you use? Calculate the exact p-value of this test.

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8. Let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x, \theta) = \frac{1}{\theta} x^{\frac{1-\theta}{\theta}}$ with support $0 < x < 1$ and with $0 < \theta < 1$.

(a) (10 pts.) Find the method of moments estimator of θ.

(b) (10 pts.) Find the maximum likelihood estimator of θ.

(c) (10 pts.) Is the mle an unbiased estimate of θ?

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9. Let sample $Y_{ij}; i = 1, 2, j = 1, \ldots, n$ satisfy the following model (I) $Y_{ij} = \beta_i + \varepsilon_{ij}$, where ε_{ij}; $i = 1, 2, j = 1, \ldots, n$, are i.i.d. with distribution $N(0, \sigma^2)$.

(a) (5 pts.) Show that the least squares estimator of β_i is $\hat{\beta}_i = \frac{1}{n}\sum_{j=1}^{n} Y_{ij}$. Do **NOT** use matrix or vector space methods.

(b) (5 pts.) Find $E(\hat{\beta}_1 - \hat{\beta}_2)$ and $Var(\hat{\beta}_1 - \hat{\beta}_2)$.

(c) (10 pts.) In a study of the effect of drugs on high blood pressure, 10 patients were randomly divided into two groups of size 5 and patients in one group were treated using drug A and patients in the other group were treated using drug A. Followings are blood pressure of each patient recorded after treatment.

drug A (X_i)	113	134	127	136	129	$\sum_{i=1}^{5} X_i = 639$
drug B (Y_i)	135	133	127	131	126	$\sum_{i=1}^{5} Y_i = 652$

Apply the model (I) to this data, and carry out the 0.05 level test that drug A is more effective than drug B by assuming $\sigma^2 = 10$. Explicitly explain how you apply the model and state appropriate null and alternative hypotheses.

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