Answers of Master's Exam, Fall, 2009

1) $1 - 3e^{-2}$

2) 9/207

3) a) $M_n(t) = e^{t/n}(n+1)/(5n) + e^{(1-1/n)t}(4n+1)/(5n)$

b) $\lim_{n\to\infty} M_n(t) = 1/5 + (4/5)e^t$, for all t. This is the mgf of the Bernoulli distribution with parameter p = 4/5 = 0.8. By the continuity theorem for mgf's, Z_n converges in distribution to this Bernoulli distribution.

4) a) $E(Z_1) = p_1 + p_3$., $Cov(Z_1, Z_2) = Var(Y_3) = p_3(1 - p_3)$

b) (Z_1, Z_2) takes the following values with probabilities as given:

(0,0)	$q_1 q_2 q_3$	(0, 1)	$q_1 p_2 q_3$	(0, 2)	0
(1,0)	$p_1 q_2 p_3$	(1,1)	$q_1 q_2 p_3 + p_1 p_2 q_3$	(1,2)	$q_1 p_2 p_3$
(2,0)	0	(2,1)	$p_1 q_2 p_3$	(2,2)	$p_1 p_2 p_3$

5) a) $X_3 = F_3^{-1}(F_1(X_1)) = X_1^{-1/3}$

b) $f_M(m) = 6m^6$ for $0 \le m \le 1$, since each of $Y_1 = X_1$, $Y_2 = X_2^2$, $Y_3 = X_3^3$ has cdf $G(y) = y^2$ on [0,1]. .

c) Find $H(y) = P(Y_1 - Y_2 . > y)$ for each y, then integrate H(y) g(y) from 0 to one, where g is the derivative of G. Answer: 1/6

6) a) Let $\lambda(x_1, \ldots, x_n)$ be the likelihood ratio with $f(x_1, \ldots, x_n; \theta = \eta)$ in the numerator,

$f(x_1, \ldots, x_n; \theta = 1)$ in the denominator. Then $\lambda(x_1, \ldots, x_n) = \eta^n (\Pi\ x_i)^{\eta-1}$. We should reject for large $\log(\lambda) = n\log(\eta) + (\eta - 1)\log(\Pi\ x_i)$. For $\eta > 1$, this is an increasing function of the product W. Thus, we should reject for W > c for some constant c. We want $P(W > c \mid \theta = 1) = 1 - c$ for n = 1. Thus, c should be $1-\alpha$.

Comment: The power function of this test for any θ is $P(W > c \mid \theta) = 1 - c^{\theta}$. The general case for any n can be solved using the transformation $Y_i = -\log(X_i)$ and taking advantage of the fact that the sum of independent exponential rv's has the gamma distribution with shape parameter n. \

7) Could use the sign test. p-value = 58/128 Two-sided test.

8) a) $\hat{\theta} = (1 - \bar{X})/\bar{X}$

b) $\hat{\theta}_{MLE} = \bar{Y}$, where $Y_i = -\log(X_i)$ for each i.

c) Each Y_i has the exponential distribution with mean θ. Thus $E(\bar{Y}) = \theta$. Answer: Yes

9) a) Let $Q(\beta_1, \beta_2) = \Sigma (Y_{1j} - \beta_1)^2 + \Sigma (Y_{2j} - \beta_2)^2$. Differentiating wrt to β_i we get $\hat{\beta}_i$ for i = 1, 2.

b) $E(\hat{\beta}_1 - \hat{\beta}_2) = \beta_1 - \beta_2$, $Var(\hat{\beta}_1 - \hat{\beta}_2) = \sigma^2(1/n_1 + 1/n_2)$.

c) For $\sigma^2 = 10$, the Z statistic is $-2.6/[10(1/5 + 1/5)] = -1.3$, so the p-value for a one sided test is $P(Z \leq -1.3) \doteq 0.10$. Do not reject H_0.