Master's Exam - Fall 2010
October 28, 2010
1:00 pm - 5:00 pm

NUMBER:________________

A. The number of points for each problem is given.

B. There are problems with varying numbers of parts.
Problems 1 - 5 Probability (60 points)
Problems 6 - 9 Statistics (60 points)

C. Write your answers on the exam paper itself. If you need more room you may use the extra sheets provided. Answer as many questions as you can on each part. For Problems 2, 3 and 7 you can answer Part (b) even if you cannot answer Part (a). Tables are provided.

Good Luck!

Problem 1. (10 pts.) A class consists of 60% men and 40% women. Of the men, 25% are blond, while 45% of the women are blond. If a student is chosen at random and is found to be blond, what is the probability that student is a man?

Problem 2. A population is made-up of items of three types: Type 1, Type 2 and Type 3 in proportions $\pi_1 > 0$, $\pi_2 > 0$ and $\pi_3 > 0$ where $\pi_1 + \pi_2 + \pi_3 = 1$. An item is drawn at random and replaced. The process is repeated until for the first time all types have been observed. Let N denote the number of selections until all types have been observed for the first time. (For example, $N = 7$ for the outcome (Type 1, Type 2, Type 2, Type 1, Type 1, Type 1, Type 3) and $N = 4$ for the outcome (Type 3, Type 2, Type 2, Type 1).)

2(a) (10 pts.) Show that $P(N > n) = \sum_{i=1}^{3}(1-\pi_i)^n - \sum_{i=1}^{3}\pi_i{}^n$, $n = 3, 4, 5, \ldots$.

Hint: $(N > n) = (X_1 = 0) \cup (X_2 = 0) \cup (X_3 = 0)$ where X_i = number of Type i observed in the first n selections, $i = 1, 2, 3$.

Problem 2.(b) (10 pts.) Use (a) and the formula $E(N) = \sum_{n=0}^{\infty} P(N > n)$ to compute $E(N)$ when $\pi_1 = 0.5$, $\pi_2 = 0.3$ and $\pi_3 = 0.2$.

Problem 3 Suppose (X, Y) follows a uniform distribution on $R:=\{(x, y): x > 0, x + |y| < 1\}$. This means that the joint probability density function of (X, Y) is constant on R and zero outside R.

3 (a) (10 pts.) Calculate $E(X \mid Y = 0.2)$.

Problem 3. (b) (10 pts.) Compute $P(Y > 3X - 1)$.

Problem 4. (10 pts.) Let X_n be a random variable that follows uniform distribution on $(-1/n, 1/n)$. Does $\{X_n\}$ converge in probability? If yes, what does it converge to? Please justify your answer.

Problem 5. Let $X_1, X_2, \ldots, X_n$ be *iid* uniform on the interval $(0, \theta)$ where $\theta > 0$.

5 (a) (7 pts.) Determine the distribution of $Y = \max\{X_1, X_2, \ldots, X_n\}$.

Problem 5. (b) (4 pts.) Consider a test of size 0.05 for testing H_0: $\theta = 1$ versus H_1: $\theta > 1$ based on Y and with rejection region $Y > c$. Determine the value of c.

Problem 5. (c) (4 pts.) Determine the power of the test as a function of θ, $\theta > 1$.

Problem 6. (15 pts.) Let $X_1, X_2, \ldots, X_n$ be independent random variables with common probability density function

$$f_\theta(x) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}, \quad x \geq \mu$$

$$= 0, \quad x < \mu.$$

where the parameter is $\theta = (\mu, \sigma)$ with $-\infty < \mu < \infty$, $\sigma > 0$. Find the maximum likelihood estimator of θ.

Problem 7. In a large population of trees, the trees have characteristics X and Y where X is at two levels and Y is at 3 levels. A random sample of $n = 100$ trees is cross-classified into the levels of X and Y with this resulting 2 x 3 contingency table.

	Characteristic Y			
Characteristic X	**1**	**2**	**3**	**Total**
1	10	20	30	60
2	10	20	10	40
Total	20	40	40	100

(a) (7 pts.) Compute an approximate 95% confidence interval for the population proportion of trees with characteristic Y at level 3.

(b) (8 pts.) Carry-out a chi-square test of independence for the characteristics X and Y. Is the hypothesis of independence rejected at level 0.05? Why or why not?

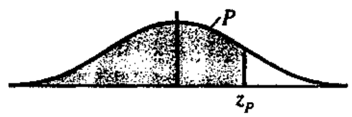
Problem 8. Consider the model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, $i = 1, 2, \ldots, n$ where the x_i are constants and the ϵ_i are uncorrelated random variables with common expectation 0 and common variance σ^2. Suppose that one is supplied with the additional information that $\beta_0 = 2 + \beta_1$.

8 (a) (7 pts.) What is the least squares estimate of $\hat{\beta}_1$ of β_1 under this constraint?

Problem 8. (b) (8 pts.) What is its expectation $E(\hat{\beta}_1)$ and what is its variance $V(\hat{\beta}_1)$?

TABLE 2

Cumulative Normal Distribution—Values of P Corresponding to z_p for the Normal Curve

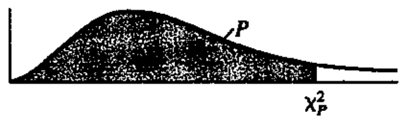


z is the standard normal variable. The value of P for $-z_p$ equals 1 minus the value of P for $+z_p$; for example, the P for -1.62 equals $1 - .9474 = .0526$.

z_p	*.00*	*.01*	*.02*	*.03*	*.04*	*.05*	*.06*	*.07*	*.08*	*.09*
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

TABLE 3

Percentiles of the χ^2 Distribution—Values of χ^2_P Corresponding to P



df	$\chi^2_{.005}$	$\chi^2_{.01}$	$\chi^2_{.025}$	$\chi^2_{.05}$	$\chi^2_{.10}$	$\chi^2_{.90}$	$\chi^2_{.95}$	$\chi^2_{.975}$	$\chi^2_{.99}$	$\chi^2_{.995}$
1	.000039	.00016	.00098	.0039	.0158	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.1026	.2107	4.61	5.99	7.38	9.21	10.60
3	.0717	.115	.216	.352	.584	6.25	7.81	9.35	11.34	12.84
4	.207	.297	.484	.711	1.064	7.78	9.49	11.14	13.28	14.86
5	.412	.554	.831	1.15	1.61	9.24	11.07	12.83	15.09	16.75
6	.676	.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.73	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
120	83.85	86.92	91.58	95.70	100.62	140.23	146.57	152.21	158.95	163.64

For large degrees of freedom,

$$\chi^2_P = \tfrac{1}{2}(z_P + \sqrt{2v - 1})^2 \text{ approximately,}$$

where v = degrees of freedom and z_P is given in Table 2.