Answers of Master's Exam, Spring, 2006

1) a) $\dfrac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}} + \dfrac{\binom{4}{1}\binom{6}{2}}{\binom{10}{3}} = 80/120$

b) (7/10)(6/9)(5/8)(4/7)(3/6)

c) 0.00278 (using ½ correction)

d) 0.115 (using ½ correction)

e) 0.9970

2) a) $f_Y(y) = 3 y^2$ for $0 \le y \le 1$.

b) $F_M(m) = m^6$ for $0 \le m \le 1$, 0 for $m < 0$, 1 for $m > 1$

c) 0

d) $E(Y_1/ Y_2) = E(Y_1) E(1 / Y_2) = (3/4)(3/2) = 9/8$.

3) a) Let $X_1, X_2, \ldots$ be independent and identically distributed with mean μ.

Let $\bar{X}_n = \dfrac{X_1 + \ldots + X_n}{n}$ Then for any $\varepsilon > 0$

$$\lim_{n \to \infty} P(| \bar{X}_n - \mu | > \varepsilon) = 0$$

b) Suppose that $Var(X_1) = \sigma^2 < \infty$. Then $P(| \bar{X}_n - \mu| > \varepsilon) \le Var(\bar{X}_n)/\varepsilon^2 = (\sigma^2 / n)/\varepsilon^2$, whose limit as n approaches infinity is zero.

4) a) $f_{XY}(x, y) = 4 x^3 / x = 4 x^2$ for $0 \le y \le x$, $0 < x < 1$.

Therefore, $f_Y(y) = \int_y^1 4 x^2 \, dx = (4/3)(1 - y^3)$ for $0 \le y < 1$, 0 otherwise.

b) $U^{1/4}$

c) Given X = x, Y is uniformly distributed on [0, x], so $E(Y | X = x) = x/2$. $E(Y | X)$ is therefore X/2.

d) See a).

5) a) $B = (A_1 \cup A_2 \cup A_3) \cap (A_4 \cup A_5)$

b) Let $q_k = 1 - p_k$ for each k. $P(B) = (1 - q_1 q_2 q_3)(1 - q_4 q_5)$

c) $P(A_1 \cap B) = p_1 (1 - q_4 q_5)$. Divide this by $P(B)$ to get $P(A_1 \mid B)$
$= p_1/(1 - q_1 q_2 q_3)$

Statistics Part

6) a) The function given is not a density. Multiply it by 2. $\hat{\lambda} = n / \Sigma X_i^2$.

   b) Then $\mu = E(X_1) = 1/\lambda$, so the method of moments estimator is $1/\bar{X}$.

7) a) Let D_i = (Weight loss of low-card diet) – (Weight loss of high-card diet) , $i = 1, 2, \ldots, 6$

   Suppose that the D_i constitute a random sample from the $N(\mu_D, \sigma_D^2)$ distribution.

   Let H_0: $\mu_D \leq 0$, H_a: $\mu_D > 0$

   Reject H_0 for $t \geq 2.015$. We observe $t = 2.9077$, so we reject H_0.

   b) We could use the sign test. We observe $X = 5$ positive values. For median = 0, $X \sim$ binomial(8, ½), so the p-value is $P(X \geq 5) = 6/64 > 0.05$ so we do not reject H_0 at the $\alpha = 0.05$ level.
   Wilcoxon's signed rank statistic is $W_+ = 20$, so p-value = $P(W_+ = 20 \text{ or } 21) = 6/64$, so we reject H_0.

8) a) Let $\hat{p}_1$ and $\hat{p}_2$ be the sample proportions X_1/n_1 and X_2/n_2. $\hat{\Delta} = \hat{p}_1 - \hat{p}_2$. Then $Var(\hat{\Delta}) = p_1 (1 - p_1)/n_1 + p_2 (1 - p_2)/n_2$ and we can estimate this variance by replacing the p_i by their estimates. Call this estimator $\hat{\sigma}^2$. A 95% Confidence Interval:

$$[\hat{\Delta} \pm 1.96\ \hat{\sigma}]$$

   b) $[0.0833 \pm 0.0590]$

   c) In a large number, say 10000, repetitions of this experiments, always with samples sizes 1000 and 800, about 95 % of the intervals obtained would contain the true parameter Δ.
   d) 19208

9) a) 0.15625

   b) The Neyman-Pearson Theorem states that the most powerful test of H_0: $p = 0.25$ vs H_a: $p = p_0$, where $p_0 > 0.25$ rejects for

$\lambda = p_0^2 (1 - p_0)^{s-2} / 0.25^2(1 - .2)^{s-2} \geq (p_0/0.25)^2 ((1 - p_0)/0.25)^{s-2} \geq k$ for some k , where $s = x_1 + x_2$, the number of successes. Since $p_0/0.25 > 1$, λ is a decreasing function of s, so, equivalently, we should reject for $s \leq k^*$, where k^* is some constant. In this case we take $k^* = 2$.

c) Power = 3/8.

10) a) Let $Q(\beta) = \Sigma (Y_i - \beta x_i)^2$. Taking the partial derivative wrt to β and setting the result equal to zero, we get $\hat{\beta}$ as given.

b) Replacing each Y_i by $\beta x_i + \varepsilon_i$, we get $E(\hat{\beta}) = \beta + E(\Sigma \varepsilon_i x_i)/(\Sigma x_i^2) = \beta$.
c) $Var(\hat{\beta}) = \sigma^2 / \Sigma x_i^2$.
d) d) $\hat{\beta} = 2\ 333$, $S^2 = 2/9$, [2.333 ± 0.656]

10) a) $P(X_{(1)} \leq w) = 1 - P(X_{(1)} > w) = 1 - (e^{-w/\lambda})^n = 1 - e^{-wn/\lambda}$, so $E(X_{(1)}) = \lambda / n$ and $E(\lambda^*) = \lambda$.

b) $Var(\lambda^*) = (\lambda/n)^2$ and $Var(\bar{X}) = \lambda^2/n$, so $e(\lambda^*, \bar{X}) = Var(\bar{X}) / Var(\lambda^*) = n$. Thus, λ^* has greater efficiency for all $n > 1$.