Preliminary Exam: Probability
9:00am – 2:00pm, August 27, 2004

Question 1. (10 points) Let $\{X_n\}$ be a Cauchy sequence of random variables in $L^p(\Omega)$ ($p \geq 1$), i.e., for any $\varepsilon > 0$, there exists an integer n_0 such that $\|X_n - X_m\|_p \leq \varepsilon$ for all $n, m \geq n_0$. Prove the following:

(i) For any sequences $\{a_k\}$ and $\{b_k\}$ of positive numbers, there is a sequence of increasing positive integers $\{n_k\}$ so that

$$\mathbb{P}\Big\{|X_{n_{k+1}} - X_{n_k}| > a_k\Big\} \leq \frac{b_k}{a_k}.$$

(ii) There is a sequence $n_k \uparrow \infty$ of integers and a random variable $X \in L^p(\Omega)$ such that X_{n_k} converges to X both in L^p and almost surely.

(iii) $X_n \to X$ in $L^p(\Omega)$ [that is, $L^p(\Omega)$ is complete].

Question 2. (15 points) Let $X_1, \ldots, X_n, \ldots$ be a sequence of i.i.d. random variables with $\mathbb{E}|X_1| < \infty$ and $\mathbb{E}(X_1) = 0$. Let $S_n = \sum_{i=1}^n X_i$. For any $\varepsilon > 0$, define

$$N_\varepsilon = \sum_{n=1}^\infty \mathbb{1}_{\{|S_n| > \varepsilon n\}}.$$

(i) Prove that, with probability 1, $N_\varepsilon < \infty$ for all $\varepsilon > 0$, i.e. $\mathbb{P}\{N_\varepsilon < \infty \text{ for all } \varepsilon > 0\} = 1$.

(ii) We further assume $\mathbb{E}(X_1^2) = \sigma^2$, find

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^\infty \mathbb{P}\{|X_1| > \varepsilon\sqrt{n}\}.$$

[Hint: Start by calculating $\int_0^\infty \mathbb{P}\{|X_1| > \varepsilon\sqrt{x}\}dx$].

(iii) Prove that if $X_1 \sim N(0,1)$, then

$$\lim_{\varepsilon \to 0} \varepsilon^2\, \mathbb{E}(N_\varepsilon) = 1.$$

Question 3. (10 points) Let $X_1, \ldots, X_n, \ldots$ be a sequence of i.i.d. random variables such that

$$\mathbb{P}\Big\{|X_1| > x\Big\} \sim x^{-\alpha} \qquad \text{as } x \to \infty,$$

where $\alpha \in (0,1)$ is a constant. Prove the following statements:

(i) If the real sequence $\{a_n\}$ satisfies $\sum_{n=1}^{\infty} |a_n|^\alpha < \infty$, then $\sum_{n=1}^{\infty} a_n X_n$ converges almost surely.

(ii) For any $0 < p < \alpha$, we have

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n^{1/p}} = 0 \quad \text{a.s.}$$

Question 4. (15 points) Let $X_1, \ldots, X_n, \ldots$ be a sequence of i.i.d. random variables such that $\mathbb{P}\{X_1 > x\} = \mathbb{P}\{X_1 < -x\}$ and

$$\mathbb{P}\{|X_1| > x\} = \begin{cases} 1 & \text{if } 0 \le x < e, \\ \frac{1}{x^2 \log x} & \text{if } x \ge e. \end{cases}$$

Prove the following statements:

(i) $\mathbb{E}(X_1^2) = \infty$.

(ii) Let $Y_{n,m} = X_m \mathbb{1}_{\{|X_m| \le \sqrt{n}\}}$. As $n \to \infty$,

$$\sum_{m=1}^{n} \mathbb{P}\{Y_{n,m} \ne X_m\} \to 0.$$

(iii) As $n \to \infty$, $\mathbb{E}(Y_{n,m}^2) \sim 2 \log\log n$.

(iv) Let $S'_n = \sum_{m=1}^{n} Y_{n,m}$. Then

$$\frac{S'_n}{\sqrt{2n \log\log n}} \Rightarrow \chi \quad \text{as } n \to \infty,$$

where χ is a standard normal random variable.

(v) Let $S_n = \sum_{m=1}^{n} X_m$. Then

$$\frac{S_n}{\sqrt{2n \log\log n}} \Rightarrow \chi \quad \text{as } n \to \infty.$$

Question 5. (15 points) Let $X_1, \ldots, X_n, \ldots$ be i.i.d. r.v.'s with

$$\mathbb{P}(X_1 = 1) = p > 1/2 \quad \text{and} \quad \mathbb{P}(X_1 = -1) = 1 - p.$$

Consider the asymmetric simple random walk $\{S_n, n \ge 0\}$ on $\mathbb{Z}$ defined by $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \ge 1$. Given integers $a < 0 < b$, let $T_a = \inf\{n > 0 : S_n = a\}$ and $T_b = \inf\{n > 0 : S_n = b\}$. It is known that

$$\mathbb{P}\{T_a < T_b\} = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)},$$

where $\varphi(x) = [(1-p)/p]^x$. Prove the following statements:

(i) If $b > 0$, then $\mathbb{P}\{T_b < \infty\} = 1$.

(ii) For every integer $a < 0$,

$$\mathbb{P}\{\min_n S_n \le a\} = \mathbb{P}\{T_a < \infty\} = \left(\frac{p}{1-p}\right)^a.$$

Show that $\mathbb{E}(\min_n S_n) > -\infty$.

(iii) For every integer $b > 0$, $\mathbb{E}(T_b) = b/(2p-1)$.

[Hint: Use the fact that $\{S_n - (2p-1)n\}_{n\ge 0}$ is a martingale].

Question 6. (15 points) Let Y_n $(n \ge 1)$ be i.i.d. normal random variables with mean 0 and variance σ^2, and let $S_n = Y_1 + \cdots + Y_n$. For each $u \in \mathbb{R}$, define

$$X_n^u = \exp\left(uS_n - \frac{1}{2}nu^2\sigma^2\right).$$

(i) Show that for every $u \in \mathbb{R}$, $\{X_n^u\}$ is a martingale. What is $\mathbb{E}(X_n^u)$?

(ii) Show that for every $u \in \mathbb{R}$, $\{X_n^u\}$ converges a.s. to a random variable X_∞^u and $X_\infty^u < \infty$ a.s.

(iii) Show that $\sum_{n=1}^\infty \mathbb{E}\left(\sqrt{X_n^u}\right) < \infty$.

(iv) For each $u \in \mathbb{R}$, what is the distribution of X_∞^u?

(v) For each $u \ne 0$, is the sequence $\{X_n^u\}$ uniformly integrable?

Question 7. (20 points) Let $W = \{W(t), t \ge 0\}$ be a standard Brownian motion in $\mathbb{R}$. For any integer $n \ge 1$, let $I_{n,k} = [k2^{-n}, (k+1)2^{-n}]$ $(k = 0, 1, \ldots, 2^n - 1)$ be dyadic intervals of order n in $[0, 1]$. Define

$$\Delta_{n,k} = \max_{t \in I_{n,k}} \left|W(t) - W(k2^{-n})\right|.$$

(i) Use the reflection principle $\mathbb{P}\{\max_{[0,t]} |W(s)| \ge a\} = 2\mathbb{P}\{|W(t)| \ge a\}$ to show that for any $a > 1$,

$$\mathbb{P}\{\Delta_{n,k} \ge a\, n^{-n/2}\} \le 4 \exp\left(-a^2/2\right).$$

(ii) For any $\varepsilon > 0$, let $b = 2(1+\varepsilon)\log 2$ and $a_n = \sqrt{bn}$. Then

$$\mathbb{P}\{\Delta_{n,k} \ge a_n\, n^{-n/2} \text{ for some } 0 \le k \le 2^n - 1\} \le 4 \cdot 2^{-n\varepsilon}.$$

(iii) Let $\operatorname{osc}(\delta) = \sup\{|W(s) - W(t)| : s, t, \in [0,1], |s-t| \le \delta\}$ be the modulus of continuity of W on $[0,1]$. Prove that

$$\limsup_{\delta \to 0} \frac{\operatorname{osc}(\delta)}{\sqrt{\delta \log(1/\delta)}} \le 6 \quad \text{a.s.}$$

[Hint: Use the Borel-Cantelli Lemma and triangle's inequality.]

Question 8. (Optional) Let $W_i = \{W_i(t), t \ge 0\}$ $(i = 1, 2, \ldots, d)$ be d independent standard Brownian motions in $\mathbb{R}$ $(d \ge 3)$. For $t \ge 0$, let $W(t) = (W_1(t), \ldots, W_d(t))$. Then $W = \{W(t), t \ge 0\}$ is called a Brownian motion in $\mathbb{R}^d$. For any $T > 0$ and $\varepsilon > 0$, define

$$\mathcal{S}_T = \int_T^\infty \mathbb{1}_{\{\|W(s)\| \le \varepsilon\}} ds,$$

which is the total time after T spent by W in the ball $B(0, \varepsilon)$. Assume $\varepsilon < \sqrt{T}$. Prove the following statements:

(i) There exists some constant $c_1 > 0$ such that $\mathbb{E}(\mathcal{S}_T) \ge c_1\, \varepsilon^d T^{1-\frac{d}{2}}$.

(ii) For some finite constant $c_2 > 0$,

$$\mathbb{E}(\mathcal{S}_T^2) \le c_2\, \varepsilon^{d+2} T^{1-\frac{d}{2}}.$$

(iii) Apply the Paley-Zygmund inequality: for all non-negative random variable Y and $0 \le \lambda < 1$,

$$\mathbb{P}\{Y \ge \lambda \mathbb{E}(Y)\} \ge (1-\lambda)^2 \frac{[\mathbb{E}(Y)]^2}{\mathbb{E}(Y^2)}$$

to show that

$$\mathbb{P}\Big\{\exists\, t > T \text{ such that } \|W(t)\| \le \varepsilon\Big\} \ge \Big(\frac{\varepsilon}{\sqrt{T}}\Big)^{d-2}.$$