Preliminary Exam: Probability
9:00 am to 2:00 pm, August **25**, 2006

Problem 1. Assume that triangular array $\{X_{n,k} : k = 1,\dots,n\}$ is independent for each $n = 1,\dots$

(a) Prove: $\max_{1\le k\le n}\{|X_{n,k}|\} \xrightarrow[n\to\infty]{} 0$ in probability **if and only if**

$\sum_{k=1}^{n} P(|X_{n,k}| \ge \varepsilon) \xrightarrow[n\to\infty]{} 0,\ \forall \varepsilon > 0.$

Hint: $1 > e^{-x} > 1 - x,\ x > 0.$

(b) Let $\{X_k, k = 1,\dots\}$ be independent and identically distributed with $\frac{P(|X| > x)}{x^{-\alpha}} \xrightarrow[x\to\infty]{} C$ with $0 < C < \infty$ for some $\alpha > 0$. Let $\{a_n\}$ be a positive sequence. Prove: $\frac{\max_{1\le k\le n}\{|X_k|\}}{n^{1/\alpha} a_n} \xrightarrow[n\to\infty]{} 0$ in probability **if and only if** $a_n \to \infty$.

(c) Let $\{X_k, k = 1,\dots\}$ be independent and identically distributed with $X_k \sim N(0,1)$. Find the minimal $\beta > 0$ so that $\frac{\max_{1\le k\le n}\{|\frac{1}{X_k}|\}}{n^{\beta+\varepsilon}} \xrightarrow[n\to\infty]{} 0$ in probability, for each $\varepsilon > 0$.

Problem 2. Let $X$ be a random variable with $P(X \ge x) = x^{-\alpha},\ x \ge 1$ for some $\alpha > 0$ ( so $X \ge 1$ a.s. ).

(a) Calculate the function $f(y,\alpha) \equiv E(X^2;\ X \le y),\ \alpha > 0,\ y > 0.$

(b) For each $\alpha > 0$ find $\lim_{y\to\infty} \frac{y^2 P(X \ge y)}{f(y,\alpha)}$

(c) Find an example of a sequence of **positive** random variables, $\{X_k, k = 1,\dots\}$ , so that the following 2 requirements both hold.

(i) For each $\alpha > 0$ there exist an integer $K_\alpha > 0$ so that

$E[(X_k)^\alpha] = \infty,\ k > K_\alpha$

(ii) $\{X_k\}$ converges in probability to 0.

Problem 3. In what follows a characteristic function (c.f.) of a random variable $X$ is denoted by $\varphi_X(t)$.

(a) Prove that for every $T > 0$:

(i) $\frac{1}{T}\int_{-T}^{T}\varphi_X(t)\,dt = 2\cdot E(\frac{\sin(TX)}{TX})$

(ii) $E(1-\frac{\sin(TX)}{TX}) \geq \frac{1}{2}P(|X| \geq \frac{2}{T})$

Let $\{X_k, k=1,\ldots\}$ be a sequence of random variables. For parts (b) and (c) assume that $\lim_{k\to\infty}\varphi_{X_k}(t)$ exists for $|t|<1$. Denote the limit by $g(t)$ (observe that $g(t)$ is defined only on $|t|<1$.)

(b) Prove that for every $M > 2$ we have

$$M\cdot\int_{-2/M}^{2/M}[1-g(t)]\,dt \geq \limsup_{k\to\infty}\{P(|X_k| \geq M)\}$$

(c) Assume also that: $\lim_{t\to 0} g(t) = 1$.

(i) Use part (b) to show that $\forall\varepsilon > 0\ \exists M > 0$ so that $\sup_k\{P(|X_k| \geq M)\} \leq \varepsilon$.

(i.e.: $\{X_k\}$ is *tight*).

(ii) Use (c)(i) to show that there exists a random variable $Y$ whose c.f. $\varphi_Y(t)$ is an extension of $g(t)$ (i.e. $\varphi_Y(t) = g(t),\ |t|<1$.)

Problem 4. Let $\{D_n, F_n;\ n=1,...\}$ be a sequence of $L^2$ martingale differences, namely $\{D_n\}$ are random variables and $\{F_n\}$ are $\sigma-$algebras, $D_n \in F_n$, $F_n \subset F_{n+1}$, $\mathrm{E}_{F_n}(D_{n+1})=0$ and $\mathrm{E}(D_n^2)<\infty$. We denote $X_n \equiv \sum_{k=1}^{n} D_k$, $A_n \equiv \sum_{k=1}^{n} \mathrm{E}_{F_{k-1}}(D_k^2)$ ($F_0$ is a trivial $\sigma-$algebra) and $A_\infty \equiv \lim_n A_n$ (can be $\infty$ ).

(a) Is $N_C \equiv \inf\{n:\ A_{n+1} > C\}$, $C>0$ a stopping time? Prove or give a counter example.

(b) Does $\lim_n X_{n\wedge N_C}$ exists a.s. for each $C>0$ ? Prove or disprove.

(c) (i) Does $\lim_n X_n$ exists a.s. on $\bigcup_{C>0}\{N_C=\infty\}$? Explain.

(ii) What is the relationship between the events $\bigcup_{C>0}\{N_C=\infty\}$ and $\{A_\infty=\infty\}$ ?

Problem 5. Here we use the setup and notations of problem 4.

(a) Let $b_n \uparrow \infty$ . Prove that on the event $\{\sum_{k=1}^{\infty} \mathrm{E}_{F_{k-1}}(\frac{D_k^2}{b_k^2})<\infty\}$ we have

$$\frac{X_n}{b_n} \xrightarrow[n\to\infty]{} 0, \text{ a.s.}$$

(b) Prove that $\sum_{k=2}^{\infty} \mathrm{E}_{F_{k-1}}(\frac{D_k^2}{A_k^2}) \le \int_{A_1}^{\infty} \frac{dt}{t^2}$, a.s.

(c) Prove (i) Assume $\mathrm{E}(D_1^2)>0$. Prove that $\sum_{k=1}^{\infty} \frac{D_k}{A_k}$ converges a.s.

(ii) $\frac{X_n}{A_n} \xrightarrow[n\to\infty]{} 0$, a.s. on the event $\{A_\infty=\infty\}$

(iii) $\frac{X_n}{A_n}$ converges a.s.

Problem 6. $\{X_n\}$ is called Uniformly Integrable (UI) if $\sup_n E(|X_n|\cdot 1_{\{|X_n|>M\}}) \xrightarrow[M\to\infty]{} 0$.

(a) Prove: $\{X_n\}$ is UI **if and only if**

(i) $\sup_n E(|X_n|) < \infty$ , and

(ii) $\forall \varepsilon > 0\ \exists \delta > 0$ so that $P(A) < \delta \Rightarrow \sup_n E(|X_n|; A) \le \varepsilon$

(b) Give an example of a sequence $\{X_n\}$ that **is not** UI but at the same time the following 3 requirements hold: (i) $X_n \xrightarrow[n\to\infty]{} 0$, a.s. , (ii) $E(X_n) \xrightarrow[n\to\infty]{} 0$, (iii) $\sup_n E(|X_n|) < \infty$ .

(c) Give an example of a sequence $\{X_n\}$ so that (i) $\{X_n\}$ is UI (ii) $X_n \xrightarrow[n\to\infty]{} 0$, a.s. (iii) There exist a $\sigma$-algebra $F$ so that $\mathrm{E}_F(X_n)$ doesn't converge a.s.

Problem 7. Let $\{X_k, k = 1, ...\}$ be independent and identically distributed with **symmetric** distribution. Let $a_0 = 0$, $a_n \uparrow \infty$ and denote $Y_n \equiv X_n \cdot 1_{\{|X_n| \le a_n\}}$. Assume

(i) There is $C$>0 so that $\sum_{n=m}^{\infty} a_n^{-2} \le C \cdot m \cdot a_m^{-2}$, $\forall m \ge 1$, and

(ii) $\sum_{n=1}^{\infty} P(|X_1| \ge a_n) < \infty$

(a) Prove : $\sum_{n=1}^{\infty} \frac{E(Y_n^2)}{a_n^2} < \infty$. Hint: $E(Y_n^2) = \sum_{m=1}^{n} E(X_m^2; a_{m-1} < |X_m| \le a_m)$, etc.

(b) Prove: $\frac{\sum_{k=1}^{n} X_k}{a_n} \xrightarrow[n\to\infty]{} 0$, a.s.

(c) Assume $E|X_1|^p<\infty$, $1<p<2$. Show by using part (b) that

$$\frac{\sum_{k=1}^{n} X_k}{n^{1/p}} \xrightarrow[n\to\infty]{} 0, \text{ a.s.}$$

Problem 8. Let $\{B(t)\}$ denote standard Brownian motion.

(a) (i) Prove that $\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-z^2/2}dz \le \frac{1}{2}\cdot e^{-x^2/2}$, $x>0$ (Hint: use $z=x+y$).

Remark: in the book the upper bound is $\frac{e^{-x^2/2}}{x}$ which is worse for $x<2$.

(ii) Use the inequality in (i) to present a function $f(t,x)$ so that

$P(\max_{0<u<t}|B_u|>x)\le f(t,x)$, $x>0$.

(b) Prove that $E\max_{0<u<t}B_u^2 \le 2\cdot t$.

(c) Let $\Delta_n=\max\{\Delta_{m,n} : 1\le m\le 2^n\}$ where

$\Delta_{m,n}=\max\{|B(t)-B(\frac{m-1}{2^n})|\} : \frac{m-1}{2^n}\le t<\frac{m}{2^n}\}$. Prove: There is $C<\infty$ so that

$\Delta_n \le C\cdot\sqrt{n\cdot 2^{-n}}$, $n\ge N(\omega)$