Probability Prelim
August 24, 2007

Problem 1.

Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. random variables.

a. Prove that the following are equivalent:

(i) $n \cdot P(|X_1| > \varepsilon \cdot \sqrt{n}) \xrightarrow[n\to\infty]{} 0, \ \forall \varepsilon > 0$

(ii) $[1 - P(|X_1| > \varepsilon \cdot \sqrt{n})]^n \xrightarrow[n\to\infty]{} 1, \ \forall \varepsilon > 0$

(iii) $\dfrac{\max\limits_{1\leq k\leq n}\{|X_k|\}}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$ in probability.

b. Assume that $E(X_1^2) < \infty$. Do (i), (ii) and (iii) from part a. hold? Prove or give a counter example.

Problem 2.

Let $X \geq 0$ be a random variable. Assume $\sum_{n=1}^{\infty} P(X > a_n) < \infty$ where $(a_n)_{n\geq 0}$ denote a sequence of numbers so that $a_0 = 0$, $a_{n+1} > a_n$ and $\dfrac{a_n}{n} \uparrow \infty$. Let $Y_n = X \cdot 1_{\{X < a_n\}}$, $n \geq 1$. Prove the following

a. $\sum_{m=1}^{\infty} m \cdot P(a_{m-1} \leq X < a_m) < \infty$

b. For every $N < n$ we have

$$\frac{\sum_{m=1}^{n} E(Y_m)}{a_n} < \frac{n \cdot E(Y_N)}{a_n} + \sum_{m=N+1}^{n} \frac{m}{a_m} \cdot E(X; a_{m-1} \leq X < a_m)$$

Hint: Observe that $\sum_{m=1}^{n} E(Y_m) < n \cdot E(Y_n)$. Also use: $\dfrac{n}{a_n} \leq \dfrac{m}{a_m}$ if $m \leq n$.

c. $\dfrac{\sum_{m=1}^{n} E(Y_m)}{a_n} \xrightarrow[n\to\infty]{} 0$

Problem 3.

Let $\{X_n\}_{1\le n}$ be a sequence of independent random variables. The distribution of X_n, $n \ge 1$ is given by :

$$X_n = \begin{cases} \pm 1 \ \text{ with probability } \frac{1}{2} - \frac{c}{2 \cdot n^2} \\ \pm n \cdot k^3 \ \text{ with probability } \frac{1}{2 \cdot n^2 k^3}, \ k \ge 2 \end{cases}$$

with $c = \sum_{k=2}^{\infty} 1/k^3 < 1$. Let $S_n = \sum_{k=1}^{n} X_k$, $n \ge 1$.

a. Prove that $\frac{S_n}{n} \to 0$, a.s. (**Hint**: think about random series)

b. Let $\{Y_n\}_{n\ge1}$ be i.i.d. random variables with $Y_1 = X_1$ in distribution, namely

$$Y_1 = \begin{cases} \pm 1 \ \text{ with probability } \frac{1}{2} - \frac{c}{2} \\ \pm k^3 \ \text{ with probability } \frac{1}{2 \cdot k^3}, \ k \ge 2 \end{cases}$$

Let $T_n = \sum_{k=1}^{n} Y_k$, $n \ge 1$. Prove $\frac{T_n}{n^3} \to 0$, a.s.

Problem 4.

Let $\{X, X_k\}_{k\ge1}$ be a sequence of i.i.d random variables. Denote by φ the c.f. of X.

Let $S_n = \sum_{k=1}^{n} X_k$. Prove the following:

a. If $\varphi'(0) \equiv \lim_{h\to0} \frac{\varphi(h)-1}{h} = 0$ then $\frac{S_n}{n} \to 0$ in distribution.

b. Use the well known fact $\frac{\log(1+z)}{z} \xrightarrow[z\to0]{} 1$ (z denote a complex number) to get the converse of part a.: If $\frac{S_n}{n} \to 0$ in distribution then $\varphi'(0) = 0$.

c. Can the results of parts a. and b. be extended to the case $\varphi'(0) = i \cdot a$ where a is any real-valued number? Either provide a proof or provide a counter example.

Problem 5.

Let $\{X_k\}_{k\geq 1}$ be a sequence of independent random variables and let

$S_n = \sum_{k=1}^{n} X_k$, $\mu_n = E(S_n)$ and $\sigma_n = s.d.(S_n)$. In what follows you are asked to

prove that $\dfrac{S_n - \mu_n}{\sigma_n}$ converges in distribution as $n \to \infty$ and identify the limit distribution.

a. $X_k = Z_k \cdot 1_{\{Z_k \leq 1\}}$ where $Z_k \sim Poisson(1/k),\ k \geq 1$

b. $X_k \sim Poisson(1/k),\ k \geq 1$

c. $X_k \sim Poisson(1/k^2),\ k \geq 1$

Problem 6.

Let $\{X_k\}_{k\geq 0}$ be a positive supermartingale with respect to the increasing sequence of $\sigma -$algebras $\{F_k\}_{k\geq 0}$.

a. Let $\{Y_k\}_{k\geq 0}$ be another $\{F_k\}_{k\geq 0}$-supermartingale. Let $T \geq 0$ be a stopping time. Assume $X_T \geq Y_T$, a.s. Prove that $\{W_k\}_{k\geq 0}$ is $\{F_k\}_{k\geq 0}$-supermartingale, where

$$W_k = \begin{cases} X_k & \text{if } 0 \leq k < T \\ Y_k & \text{if } k \geq T \end{cases}$$

b. Let $b > a > 0$ and assume that $X_0 > a$. Define

$S = \inf\{k : X_k \leq a\}$

$T = \inf\{k > S : X_k \geq b\}$

(both S and T can get the value ∞). Let

$$Z_k = \begin{cases} 1 & \text{if } 0 \leq k < S \\ X_k/a & \text{if } S \leq k < T \\ b/a & \text{if } T \leq k \end{cases}$$

Prove that $\{Z_k\}_{k\geq 0}$ is a $\{F_k\}_{k\geq 0}$-supermartingale.

c. We continue with the setup of part b. Let U be the number of up-crossings of $[a,b]$ by $\{X_k\}_{k\geq 0}$. Prove that $E(Z_T) \leq 1$ and $P(U \geq 1) \leq a/b$.

Problem 7.

Let $X, X_k, k \geq 0$ be a sequence of L^1 random variables defined on (Ω, G, P) and let $F_k \subset G$ be a decreasing sequence of $\sigma-$algebras, i.e. $F_k \downarrow F$. In what follows we denote $M_k = \sup_{k_1,k_2 \geq k} \{|X_{k_2} - X_{k_1}|\}, \ k \geq 0$. Prove the following

a. If $E|X_k - X| \xrightarrow[k\to\infty]{} 0$ then $E|E_{F_k}(X_k) - E_F(X)| \xrightarrow[k\to\infty]{} 0$

b. If $E(M_1) < \infty$ then there is an integrable random variable M, so that:
$E|E_F(M_k) - E_F(M)| \xrightarrow[k\to\infty]{} 0$ and $E_F(M_k) \xrightarrow[k\to\infty]{} E_F(M)$ almost surely .

c. If $X_k \xrightarrow[k\to\infty]{} X$ almost surely and $E(M_1) < \infty$ then

$$E_{F_k}(|X_k - X|) \xrightarrow[k\to\infty]{} 0 \text{ almost surely.}$$

Also, prove that: $E_{F_k}(X_k) \xrightarrow[k\to\infty]{} E_F(X)$ almost surely.

Remark. The dominated convergence theorem for conditional expectations in the textbook deals with the case $F_k \uparrow F_\infty$.

Problem 8.

Let $W(t),\ 0 \leq t \leq 1$ be a standard Brownian motion. Let $\{t_k\}_{k\geq 1}$ be a sequence of numbers in (0, 1). For each $n \geq 1$ we denote by
$0 = t_{n,0} < t_{n,1} < t_{n,2} < \ldots < t_{n,n} < t_{n,n+1} = 1$ the order statistics of $\{0, 1, t_1, \ldots t_n\}$. We assume that $\lambda_n \equiv \max_{0\leq k\leq n}\{|t_{n,k+1} - t_{n,k}|\} \xrightarrow[n\to\infty]{} 0$

Finally define $Q_n = \sum_{k=0}^{n} [W(t_{n,k+1}) - W(t_{n,k})]^2$.

a. Prove that $Var(Q_n) \xrightarrow[n\to\infty]{} 0$. What can you say about the convergence in probability of $\{Q_n\}$? Explain.

b. Let $0 < s < t < u < 1$. Let F be a $\sigma-$algebra defined by:
$F = \sigma(|W(u) - W(t)|,\ |W(t) - W(s)|)$.
Find the conditional distribution of $(W(u) - W(t)) \cdot (W(t) - W(s))$ given F. Use it to calculate: $E_F(W(u) - W(s))^2$.

c. Define a **decreasing** sequence of $\sigma-$algebras by $F_n = \sigma(H_n), n \geq 1$, where we let $H_n = \{|W(t_{m,k+1}) - W(t_{m,k})|:\ 0 \leq k \leq m,\ m \geq n\}$. How many random variables are in H_n but not in H_{n+1} (i.e. in $H_n \cap (H_{n+1})^c$) ? What is the relationship to t_{n+1} ?

d. Prove that $(Q_n, F_n)_{n\geq 1}$ is a Backwards Martingale. What can you say about the convergence of $\{Q_n\}$ in almost sure sense? Explain.