Preliminary Exam: Probability, January 12, 2008

The exam lasts from **10:00 am until 3:00 pm**. Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete. The exam consists of **8** problems, each with several steps designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step.

On each page you turn in, write your assigned code number instead of your name. Separate and staple each problem and return each it in its designated folder.

Problem 1. For a random variable X we define $\varphi_X(t) = E(e^{t\cdot X}), t \in R$ ("moment generating function")

a. Prove: $P(X \geq a) \leq \inf_{t>0}\{e^{-t\cdot a}\varphi_X(t)\}, \; a > 0$

b. Let $X \sim Poisson(\lambda), \; \lambda > 0$ and let $k > \lambda$. Prove:

$$P(X \geq k) \leq e^{-\lambda}\left(\frac{e\lambda}{k}\right)^k$$

Is the condition $k > \lambda$ really needed?

c. Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d random variables with the following distribution :

$P(X_1 = \pm 1) = 1/2$. Prove: $P(\bar{x} \geq a) \leq e^{-n\cdot a^2/2}, \; a > 0$, where $\bar{x} \equiv \frac{\sum_{k=1}^{n} X_k}{n}$

Hint: You may want to use the inequality: $e^t + e^{-t} \leq 2e^{t^2/2}, t \in R$, (follows from Taylor expansion.)

Problem 2. Let $0 \leq X_k \leq 1, k \geq 1$ be a sequence of **uncorrelated** random variables defined on (Ω, F, P). Assume that $\sum_k E(X_k) = \infty$.

a. Prove that for each $n > 0$ we have $\sum_{k=1}^{n} E(X_k) \geq Var(\sum_{k=1}^{n} X_k)$. Prove also that

$$\frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} E(X_k)} \xrightarrow[n\to\infty]{} 1 \text{ in probability.}$$

b. For the rest of the problem (a_n) denotes a sequence of numbers with $a_n \uparrow \infty$ and $\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty$. Let $n_m = \inf\{L : \sum_{k=1}^{L} E(X_k) \geq a_m\},\ m \geq 1$. Prove that for each $\delta > 0$:

$$\sum_{m=1}^{\infty} P(|\sum_{k=1}^{n_m} [X_k - E(X_k)]| > \delta \cdot a_m) < \infty.$$

What can you say about the convergence of $\frac{\sum_{k=1}^{n_m} X_k}{\sum_{k=1}^{n_m} E(X_k)}$ as $m \to \infty$?

c. Assume also that (a_n) satisfy $\frac{a_{n+1}}{a_n} \to 1$. Prove that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} E(X_k)} = \limsup_{m\to\infty} \frac{\sum_{k=1}^{n_m} X_k}{\sum_{k=1}^{n_m} E(X_k)}, \quad \text{a.s.}$$

Problem 3. Let $\{X_{N,k}\}_{k\geq 1}$ be a sequence of i.i.d random variables for each $N \geq 1$. The distribution is given by : $P(X_{N,k} = j) = p_{N,j},\ 1 \leq j \leq N$, where $\sum_{j=1}^{N} p_{N,j} = 1,\ p_{N,j} \geq 0$. Let $T_N \sim Poisson(\lambda),\ \lambda > 0$ and assume that T_N and $\{X_{N,k}\}_{k\geq 1}$ are independent.

Define: (i) $M_{N,j} = \sum_{k=1}^{T_N} 1_{\{X_{N,k}=j\}}$, $1 \leq j \leq N$, and (ii) $U_N = \sum_{j=1}^{N} 1_{\{M_{N,j}\geq 1\}}$

a. Find $P(M_{N,1}=m_1,\dots,M_{N,N}=m_N)$. Name the distribution of $M_{N,j}$. What is the relationship between the random variables $\{M_{N,j}:1\le j\le N\}$?

Hint: Condition on T_N.

b. What is the distribution of U_N in the special case $p_{N,j}=1/N$?

c. Back to the general case. Assume: $\max_{1\le j\le N}\{p_{N,j}\}\xrightarrow[N\to\infty]{}0$. Prove that U_N converges in distribution as $N\to\infty$ and identify the limit distribution.

Hint: You may want to use the inequality: $x-x^2/2\le 1-e^{-x}\le x,\ 0<x$.

Remark. If it helps you to have a concrete interpretation of the problem, then think about N as the # of floors in a building, $X_{N,k}$ as the destination floor of the k'th person entering the elevator on ground floor, $p_{N,j}$ as the proportion of those who go to the j'th floor and T_N as the # of people entering the elevator on ground floor. In this setup $M_{N,j}$ **is the** # people going to the j'th floor and U_N **is the** # of stops the elevator is making on its way up.

Problem 4. Let $\{X_k\}_{k\ge1}$ be a sequence of i.i.d random variables with $P(X_k=j)=1/N,\ 1\le j\le N$, where $N\ge1$. We are removing numbers from the set $\{1,\dots,N\}$ by using the following rule: At the *k*'th step, the number X_k is being removed (if it wasn't removed before) together with the smallest 2 remaining numbers. Let $H_N=\min\{n\ge2:\ X_n \text{ has been removed before the n'th step}\}$.

Example: if $N=10, X_1=2, X_2=7, X_3=4$, then the first 3 steps are $\{1,2,3,4,5,6,7,8,9,10\}\to\{4,5,6,7,8,9,10\}\to\{6,8,9,10\}\to\{9,10\}$, and $H_N=3$

a. Find a formula for $P(H_N>m),\ 1\le m<\dfrac{N}{3}$

Hint: How many numbers are we removing in the *k*'th step if $k<\min\{H_N,\dfrac{N}{3}\}$?

b. Prove that $\dfrac{H_N}{\sqrt{N}}$ converges in distribution as $N\to\infty$ and identify the cumulative distribution function of the limit.

Problem 5. Let $\{X_k\}_{k\geq 1}$ be a sequence of independent random variables and let

$S_n = \sum_{k=1}^{n} X_k$, $\mu_n = E(S_n)$ and $\sigma_n = s.d.(S_n)$.

a. Prove that if (+) $\frac{\sum_{k=1}^{n} E(|X_k - E(X_k)|^3)}{\sigma_n^3} \xrightarrow[n\to\infty]{} 0$ then $\frac{S_n - \mu_n}{\sigma_n}$ converges in distribution as $n \to \infty$ to standard normal.

b. Assume $X_k \sim Poisson(1/k),\ k \geq 1$. Prove that:
$E(|X_k - E(X_k)|^3) \leq E(X_k^3) + E^3(X_k)$.
Does the (+) condition of part a. hold ?

c. The same question as part b. for the case: $X_k \sim Poisson(1/k^2),\ k \geq 1$.

Problem 6. Let $\{X_k\}_{k\geq 0}$ be a sequence of random variables and denote $D_k = X_k - X_{k-1},\ k \geq 1$. Let T be a finite, positive integer-valued random variable.

Assume: $\sum_{k=1}^{\infty} E(|D_k| \cdot 1_{\{T\geq k\}}) < \infty$.

a. Prove the representation: $X_{n\wedge T} = X_0 + \sum_{k=1}^{n} D_k \cdot 1_{\{T\geq k\}},\ n \geq 0$. Then prove that $\{X_{n\wedge T} - X_0\}_{n\geq 0}$ is uniformly integrable.

b. Assume that $\{X_k\}_{k\geq 0}$ is a MG sequence and T is a stopping time relative to an increasing sequence of σ - algebras. Prove that

$$E(D_k \cdot 1_{\{T\geq k\}}) = 0,\ k \geq 1 \quad \text{and} \quad E(X_T) = E(X_0).$$

c. Show how to **use the earlier parts of the problem** (Don't forget to verifying the needed assumptions) in order to get the following result: Let $\{Y_k\}_{k\geq 1}$ be i.i.d with $E(|Y_1|) < \infty$ and let N be a stopping time relative to $\sigma\{Y_1,\ldots,Y_n\}, n \geq 1$, so that $E(N) < \infty$. Then: $E(\sum_{k=1}^{N} Y_k) = E(Y_1) \cdot E(N)$.

Problem 7. Let $W(t)$, $0 \le t < \infty$, $W(0) \equiv 0$, be a standard Brownian motion defined on Ω which consists of all real-valued and continuous functions on $[0, \infty)$.
In what follows $\tau = \inf(t > 0 : W(t) > 0\}$, $R_t = \inf\{u > t : W(u) = 0\}$ and $(\theta_{R_t}\omega)(u) = \omega[R_t(\omega) + u], u \ge 0, \omega \in \Omega$.

a. Prove the following:
(i) $P(\tau = 0) = 1$.
(ii) For each $t \ge 0$ we have: $P(\tau \circ \theta_{R_t} > 0) = 0$.
(iii) $P(\tau \circ \theta_{R_t} > 0 \text{ for any } t \ge 0 \text{ rational}) = 0$. In other words: with probability 1 there is $t_n \downarrow R_t$ with $\mathrm{W}(t_n) > 0$ for any t rational.

b. Let $I_{s,\delta} = (s - \delta, s) \cup (s, s + \delta)$, $s \in R, \delta > 0$. Prove that
$P[\exists s, \delta > 0 \text{ with } W(s) = 0, \text{and } W(t) < 0 \text{ if } t \in I_{s,\delta}] = 0$.
In other words: with probability 1 there is no zero of Brownian motion that is a local maximum.
Hint: Think about $t \in (s - \delta, s)$ that is rational.

Problem 8. Let $W(t)$, $0 \le t < \infty$, $W(0) \equiv 0$, be a standard Brownian motion. Let $a > 0$, $b > 0$ be fixed throughout. In what follows
$S = \inf(t > 0 : W(t) = a - bt\}$,
$T = \inf(t > 0 : W(t) = a + bt\}$, and
$M_\theta(t) = e^{\theta \cdot W(t) - \theta^2 t/2}, t \ge 0,\ \theta \in R$.

a. Show that for each $\theta \ge 0$ the process $M_\theta(t \wedge S), t \ge 0$ is a **uniformly integrable** martingale.

b. Show that for each $\theta \ge 2b$ the process $M_\theta(t \wedge T), t \ge 0$ is a **uniformly integrable** martingale.

c. Find for $\alpha > 0$ the following: (i) $E(e^{-\alpha \cdot S})$, and (ii) $E(e^{-\alpha \cdot T})$. Justify each step of your calculation.
Hint: Select an appropriate θ in parts a. and b.

d. Find $P(T < \infty)$. Hint: observe that $E(e^{-\alpha \cdot T}; T = \infty) = 0$, $\alpha > 0$.