Please write your number here: ________________ and on every page

Probability Preliminary Exam
Friday, January 9, 2009

The exam lasts from **9:00 am until 2:00 pm**. Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete. The exam consists of **6** problems, each with several steps designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step.

On each page you turn in, write your assigned code number instead of your name. Separate and staple each problem and return it to its designated folder

Problem 1. Let $X \geq 0$ be a r.v. and let $\{A_k\}$ be a sequence of events.

a. Prove: $\sqrt{E(X^2)} \cdot \sqrt{P(X > 0)} \geq E(X)$

b. Prove $$P(\bigcup_{k\geq 1} A_k) \geq \limsup_{n\to\infty} \frac{[\sum_{1\leq k\leq n} P(A_k)]^2}{\sum_{1\leq i,j\leq n} P(A_i \cap A_j)}$$

Hint: $\bigcup_{k=1}^{n} A_k = \{ \sum_{1\leq k\leq n} 1_{A_k} > 0\}$

c. Assume that $\sum_{k=1}^{\infty} P(A_k) = \infty$. Prove that for each $m \geq 1$

$$P(\bigcup_{k\geq m} A_k) \geq \limsup_{n\to\infty} \frac{[\sum_{1\leq k\leq n} P(A_k)]^2}{\sum_{1\leq i,j\leq n} P(A_i \cap A_j)}$$

d. Assume that $\sum_{k=1}^{\infty} P(A_k) = \infty$. Use part c to conclude that $P(A_k \text{ i.o.}) = 1$ if $\{A_k\}$ is pair-wise independent.

Problem 2. Let $\{X, X_n, n = 1,2,...\}$ be i.i.d r.v.s and assume that $\frac{X_n}{\sqrt{n}} \to 0$, a.s.

a. Prove that $E(X^2) < \infty$

b. Assume also that $E(X) = 0$. Let $S_n = \sum_{k=1}^{n} X_k$. Prove that for each $\varepsilon > 0$

$$\frac{S_n}{n^{1/2+\varepsilon}} \to 0 \text{ a.s.}$$

c. Will the result of part b. still holds if we will replace ε by $1/n$? Justify.

Problem 3. Let $X_n \sim N(0, \sigma_n^2), n = 1,2,...$ and let $\{B(t), t \geq 0\}$ be standard Brownian motion. Prove:

a. If $X_n \to X$ in distribution then $X \sim N(0, \sigma^2)$ with $\sigma_n^2 \to \sigma^2$

b. Prove using part a. that $\int_0^1 B(t)dt$ has normal distribution. Then find its mean and variance. (Hint: $[\int_0^1 B(t)dt]^2 = \int_0^1 B(s)ds \cdot \int_0^1 B(t)dt$)

c. Find $E(B^2(s)B^2(t)), 0 \leq s \leq t$. Use this result to calculate the 2nd moment of $\int_0^1 B^2(t)dt$.

Problem 4. Let (Ω, F, P) be a probability space and assume that the following relations between the σ- algebras F_1, F_2, F_3 hold: $F_1 \subset F_2 \subset F$ and $F_3 \subset F$. In what follows all random variables are assumed to be bounded. Prove:

a. If $E_{F_1}(Z) = E_{F_2}(Z)$ for all $Z \in F_3$ then $E_{F_1}(Y \cdot Z) = E_{F_1}(Z) \cdot E_{F_1}(Z)$ for all $Y \in F_2, Z \in F_3$.

b. If $E(Y \cdot Z) = E(Y \cdot E_{F_1}(Z))$ for all $Y \in F_2, Z \in F_3$ then $E_{F_1}(Z) = E_{F_2}(Z)$ for all $Z \in F_3$.

c. If $E_{F_1}(Y \cdot Z) = E_{F_1}(Z) \cdot E_{F_1}(Z)$ for all $Y \in F_2, Z \in F_3$ then $E_{F_1}(Z) = E_{F_2}(Z)$ for all $Z \in F_3$

Problem 5. Let $\{X_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables and assume that $E(X_1) = 0$ and $E(X_1^2) = 1$.

a. Prove that $\dfrac{X_n}{\sqrt{n}} \xrightarrow[n \to \infty]{} 0$, a.s. Then show that in fact

$$\frac{\max\limits_{1 \leq k \leq n}\{|X_k|\}}{\sqrt{n}} \xrightarrow[n \to \infty]{} 0 \text{ a.s.}$$

Hint for the second part: Show that for any sequence of numbers $\{a_n\}_{n \geq 1}$:

$$\frac{a_n}{\sqrt{n}} \xrightarrow[n \to \infty]{} 0 \text{ implies } \frac{\max\limits_{1 \leq k \leq n}\{|a_k|\}}{\sqrt{n}} \xrightarrow[n \to \infty]{} 0$$

b. Let $X_{k,n} = X_k \cdot 1_{\{|X_k| < \sqrt{n}/2\}}$, $1 \leq k \leq n$. Prove

(i) $\sum_{k=1}^{n}\frac{X_{k,n}}{\sqrt{n}}-\sum_{k=1}^{n}\frac{X_{k,n}^2}{2\cdot n}$ converges in distribution to $Y\sim N(-\frac{1}{2},1)$.

Hint: You may use the CLT for $\{X_n\}_{n\geq 1}$ and part a.

(ii) $\sum_{k=1}^{n}\frac{|X_{k,n}|^3}{n^{3/2}}\xrightarrow[n\to\infty]{}0$, a.s.

c. Prove that $\prod_{k=1}^{n}(1+\frac{X_k}{\sqrt{n}})$ converges in distribution to e^Y where $Y\sim N(-\frac{1}{2},1)$.

Hint: Use the following inequality(follows from Taylor's expansion):

$$|\log(1+y)-y+\frac{y^2}{2}|\leq|y|^3,\quad |y|<1/2$$

Observe that we may have $\frac{X_k}{\sqrt{n}}<-1$.

Problem 6. Let X be a non-degenerate random variable. Define $\varphi_X(\lambda)=E(\lambda^2X^2\wedge 1),\ \lambda\in R$.

a. Prove: (i) $\varphi_X(\lambda)$ is continuous

(ii) $E(X^2)=\lim_{\lambda\to 0}\{\frac{\varphi_X(\lambda)}{\lambda^2}\}=\sup_{|\lambda|>0}\{\frac{\varphi_X(\lambda)}{\lambda^2}\}$

b. Prove: If $\sum_{k=1}^{\infty}\varphi_X(a_k)<\infty$ then $\sum_{k=1}^{\infty}a_k^2<\infty$.

Hint: Prove first that $a_k\to 0$ and then use part a.

c. Prove: If $E(X^2)<\infty$ and $\sum_{k=1}^{\infty}a_k^2<\infty$ then $\sum_{k=1}^{\infty}\varphi_X(a_k)<\infty$.

d. $\sum_{k=1}^{\infty} a_k X_k$ converges a.s. if and only if $\sum_{k=1}^{\infty} \varphi_{X_1}(a_k) < \infty$.