OLDER

(iii). Find the probability $\mathbb{P}\{T_a < T_b\}$.

Question 4. (20 points) Let $Y_n\,(n \geq 1)$ be i.i.d. random variables and assume that $\phi = \mathbb{E}(e^{Y_1}) < \infty$. Define $S_n = Y_1 + \cdots + Y_n$ and $X_n = \exp(S_n - n \ln \phi)$.

(i). Show that $\{X_n\}$ is a martingale. What is $\mathbb{E}(X_n)$?

(ii). Show that $\ln \phi > 2 \ln \tilde{\phi}$, where $\tilde{\phi} = \mathbb{E}(e^{Y_1/2})$.

(iii). Show that $\mathbb{E}(\sqrt{X_n}) = e^{-cn}$ for some constant $c > 0$.

(iv). Show that $\sum_{n=1}^{\infty} \mathbb{E}(\sqrt{X_n}) < \infty$ and $X_n \to 0$ almost surely. Is the sequence $\{X_n\}$ uniformly integrable ?

Question 5. (10 points) Let $\{B(t), t \geq 0\}$ be a real-valued standard Brownian motion. Let $a < 0 < b$ be given constants. Define $\tau = \inf\{t > 0 : B(t) \notin (a, b)\}$ and let T_a and T_b be the first hitting times of a and b, respectively.

(i). Show that the event $\{T_b < T_a\}$ is in $\mathcal{F}_\tau$.

(ii). Use the strong Markov property to show that for any $x \in (a, b)$,

$$\mathbb{E}^x(e^{-\lambda T_a}) = \mathbb{E}^x(e^{-\lambda \tau}; T_a < T_b) + \mathbb{E}^x(e^{-\lambda \tau}; T_b < T_a) \times \mathbb{E}^b(e^{-\lambda T_a}).$$

Question 6. (10 points) Let $\{B(t), t \geq 0\}$ be a real-valued standard Brownian motion.

(i). Use the reflection principle to show that for every $t > 0$,

$$\mathbb{P}\{\max_{0 \leq s \leq t} B(s) \geq u\} \sim \sqrt{\frac{2}{\pi}} \frac{\sqrt{t}}{u} e^{-\frac{u^2}{2t}} \quad \text{as } u \to \infty.$$

(ii). Show that almost surely

$$\limsup_{t \to \infty} \frac{\max_{0 \leq s \leq t} B(s)}{\sqrt{2t \ln \ln t}} \leq 1.$$

[This is the easy half of the law of iterated logarithm. First consider the limsup on the sequence $t_n = \alpha^n$, where $\alpha > 1$ is an arbitrary constant.]

P2003

PROBABILITY PRELIM 8/24/01

- The exam lasts from 9:00 until 2:00.
- Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
- The exam consists of seven main problems, each with several steps designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step.
- There are a total of 17 steps, each worth 5 points. On your work, label the steps this way: 1a, 1b,...
- On each page you turn in, write your assigned code number instead of your name. Separate and staple each main part and return each in its designated folder.

1. Fix $\lambda > 0$, let $X_1, X_2, \ldots$ be iid Poisson(λ), and define, for real x,

$$f_n(x) = e^{-n\lambda} \sum_{0 \le k \le nx} (n\lambda)^k / k! \ .$$

   (a) Express f_n as the probability of some event involving the random variables $X_1, \ldots, X_n$.

   (b) Evaluate

$$\lim_{n\to\infty} f_n(x) \ .$$

2. Let $X_1, X_2, \ldots$ be iid X random variables. Prove the following.

   (a) If $\sum_{n\ge1} X_n/n$ converges a.s. then $E|X| < \infty$.

   (b) If $E|X| < \infty$ and X is symmetrically distributed about 0, then $\sum_{n\ge1} X_n/n$ converges a.s.

   (c) Give a simple example to show that the hypothesis $E|X| < \infty$ alone is not sufficient to imply convergence of the series.

P 2001

PROBABILITY PRELIM 8/25/00

- The exam lasts from 9:00 until 2:00, with a walking break every hour.
- Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
- The exam consists of six main problems, each with several steps designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step.
- There are a total of 22 steps, each worth 5 points. On your work, label the steps this way: 1a, 1b,...
- On each page you turn in, write your assigned code number instead of your name. Separate and staple each main part and return each in its designated folder.

1. Let $X_1, X_2, \ldots$ be iid X, where X has a continuous distribution.

   (a) Let $R_k := \sum_{1 \le j \le k} [X_j \ge X_k]$ denote the relative rank of X_k in $X_1, \ldots, X_k$.
   Claim: The rv's R_k are independent, and the distribution of R_k is uniform on the integers $1, \ldots, k$. Prove this for R_1, R_2, R_3, but use the entire claim below.

   (b) Let $I_k := [R_k = 1]$, so that I_k indicates when X_k is a "record" new high value. Let $S_n := \sum_1^n I_k$ denote the number of records in the first n observations. Prove

   $$ES_n/\log n \to 1, \quad Var(S_n)/\log n \to 1.$$

   (c) Denote the "record times" by

   $$T_n := \min\{k : S_k = n\}, \quad n \ge 1,$$

   so $T_n > m$ iff $S_m < n$. Fix ϵ, $0 < \epsilon < 1$. Prove

   $$P(\log(T_{n^2})/n^2 > 1 + \epsilon \;\; i.o.) = 0, \quad P(\log(T_{n^2})/n^2 \le 1 - \epsilon \;\; i.o.) = 0.$$

   (d) Use the the last result to prove $\log(T_n)/n \to 1$ a.s.

P-2000

2. Let $X_1, X_2, \ldots$ be iid X, where $E(X) = 0$, $E(X^2) = 1$. Build a triangular array, $\{X_{n,k} := a_{n,k}X_k,\ n = 1, 2, \ldots,\ k = 1, \ldots, n\}$, where the constants $a_{n,k}$ are chosen so that $S_n := \sum_{k=1}^n X_{n,k}$ has variance 1. Let $M_n^2 := \max_k a_{n,k}^2$.

   (a) Show that $M_n^2 \to 0$ implies the Lindeberg condition for the array.

   (b) Show that the the Lindeberg condition implies the array is null, that is : $\max_k P(|X_{n,k}| > \epsilon) \to 0$ for all $\epsilon > 0$.

   (c) Show that $M_n^2 \to 0$ if the array is null.

3. Given rv X, for $t > 0$, let $r(t) := EX^2[|X| \le t]/t^2$.

   (a) Prove $\lim_{t\to\infty} r(t) = 0$.

   (b) Let

   $$b_n := 1 \vee \sup\{t > 0 : r(t) \ge 1/n\}, \quad n = 1, 2, \ldots,$$

   where the supremum over an empty set is taken to be zero. Clearly the b_n are finite and increase. Show that

   $$\text{2pt.} \quad b_n > 1 \quad \Longrightarrow \quad nr(b_n) = 1,$$

   and

   $$\text{3pt.} \quad b_n \uparrow b < \infty \quad \Longrightarrow \quad X = 0 \ a.s.$$

   (c) Suppose X is nondegenerate and $\lim_{t\to\infty} P(|X| > t)/r(t) = 0$. Show that for $0 < c < 1$ $\lim_{t\to\infty} r(t)/r(ct) = c^2$.

   (d) Let $X_1, X_2, \ldots$ be iid X, where X is symmetric, nondegenerate, and satisfies the condition of (c). Show that $\sum_1^n X_k/b_n$ converges in distribution to standard normal. [HINT: Truncate at b_n.]

4. Let Z_k, $k = 0, \ldots, n$ be integrable and let $\mathcal{B}_k := \sigma\{Z_j, 0 \le j \le k\}$. Recursively define

   $$X_n := Z_n, \quad X_k := \max\{Z_k, E(X_{k+1}|\mathcal{B}_k)\}, \quad k = n-1, \ldots, 0.$$

   (a) Show that X is the smallest supermartingale dominating Z.

   (b) Define the stopping time $T := \min\{k : X_k = Z_k\}$ and let S be any stopping time such that $S \le T$. Show that $(X_{S\wedge k},\ 0 \le k \le n)$ is a martingale.

   (c) Conclude that $EZ_T = \sup_\tau EZ_\tau$, where the supremum is taken over all stopping times τ with values in $0, \ldots, n$.

P2000

5. Let B be standard Brownian motion. For each positive integer n let $\pi_n = \{t_{n,0} = 0 < t_{n,1} < \cdots < t_{n,k_n} = 1\}$ be a partition with steps $d_{n,k} := t_{n,k} - t_{n,k-1}$ and mesh $\pi_n^* := \max_k d_{n,k}$. Introduce the random variables

$$D_{n,k} := B(t_{n,k}) - B(t_{n,k-1}), \quad V_n := \sum_1^{k_n} |D_{n,k}|, \quad W_n := \sum_1^{k_n} D_{n,k}^2.$$

(a) Suppose $d_{n,k} \equiv 2^{-n}$. Show that with probability one, V_n is eventually greater than $2^{n/2}E|B(1)|/2$.

(b) Show that W_n converges to a constant, in L_2, if $\pi_n^* \to 0$.

(c) Show that W_n converges to a constant $a.s.$ if $\sum \pi_n^* < \infty$.

(d) Now suppose, for all n, that $k_n = n$ and $\pi_n \subset \pi_{n+1}$. Form the σ-algebras

$$\mathcal{B}_n := \sigma\{D_{m,k}^2 : m \geq n, \ k = 1, \ldots, m\},$$

and show that

$$E(W_{n-1} - W_n | \mathcal{B}_n) = 0.$$

[HINT: Suppose $t_{n,j}$ is the unique point in $\pi_n \setminus \pi_{n-1}$; get a simple expression for $W_{n-1} - W_n$ and argue using symmetry.] Conclude that W_n converges $a.s.$ for such nested partitions with mesh going to zero.

6. Let X denote the Markov chain with transition matrix $P = (P_{i,j})$ and states $i, j = 1, \ldots, 7$, :

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let d denote the period of state 2, and let $Y_n := X_{dn}, \ n = 0, 1, \ldots$.

(a) Find d, and the irreducible classes for the X chain.

(b) Find the transition matrix and the irreducible classes for the Y chain.

(c) Find the stationary measure for Y, when Y is restricted to the irreducible class containing state 2. Find the expected return time to state 2 starting from state 2.

(d) Find $\lim_{n\to\infty} P^{nd}$.

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