Preliminary Exam: Statistics

Wednesday, August 22, 2001
1:00 PM – 6:00 PM

This examination consists of 7 problems. Each part of a problem is worth 5 points, for a total of 80 possible points.

You are allowed to solve a subproblem based on the conclusion of an earlier subproblem, even if you did not solve the earlier subproblem.

Your solutions should be detailed and clearly formulated. Make sure you give complete justification. If you want to use a theorem, clearly specify the theorem and show that it is applicable.

Please use separate sheets of paper for separate problems. *Make sure that your identification number and the problem number are written on every page.*

Good Luck!

1. Let X and Y be random variables. Suppose that $g(X)$ and $h(Y)$ are independent for some non-degenerate functions g and h. Is it true that X and Y are independent?

2. Let $X_1, \ldots, X_n$ be i.i.d. random variables with common distribution P. Find the most powerful α level test for testing the null hypothesis that P is a Poisson distribution versus the alternative hypothesis that P is a normal distribution.

3. Let $\Theta = (-\infty, \infty)$ and for each $\theta \in \Theta$ let X have density f_θ given by

$$f_\theta(x) = c(\theta)\, e^{\theta T(x)},$$

where T is real-valued. Consider estimation of $\psi(\theta) = E_\theta[T(X)]$ under squared error loss. For an estimator δ of ψ, let $R_\delta(\theta)$ be the risk of δ at θ and let $b_\delta(\theta)$ be the bias of δ at θ.

   (a) Show that $R_T(\theta) = I(\theta)$ and $\psi'(\theta) = I(\theta)$. Here $I(\theta)$ is the information.

   (b) Suppose that

$$R_\delta(\theta) \le R_T(\theta) \quad \text{for all} \quad \theta \in \Theta.$$

   Show that for all $\theta \in \Theta$,

$$2b'_\delta(\theta) + [b_\delta(\theta)]^2 \le 0.$$

   (c) Use (b) to show that T is admissible for ψ.

4. Let $X_1, X_2, \ldots$ be i.i.d. normally distributed with mean θ and variance 1. The mean θ lies in the parameter space $\Theta = \{0, \pm 1, \pm 2, \ldots\}$.

   (a) Find the mle $T_n = T(X_1, \ldots, X_n)$ of θ.

   (b) Show that the mle is minimax with respect to squared error loss by obtaining it as a limit of Bayes estimates. State the result that you use.

   (c) Find the limiting distribution of $\sqrt{n}\,(T_n - \theta)$ for $\theta \in \Theta$.

5. Let $X_1, X_2, \ldots$ be i.i.d. random variables for which

$$P_\theta(X_1 = 0) = \theta/2; \quad P_\theta(X_1 = \theta) = 1 - \theta; \quad P_\theta(X_1 = 2\theta) = \theta/2.$$

   Here $\theta \in (0, 1)$.

   (a) Find a function f such that for each $\theta \in (0,1)$, $\sqrt{n}\,[f(\overline{X}_n) - f(\theta)]$ converges in distribution under P_θ to a standard normal random variable.

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(b) Let $\hat{\psi}_n = (1/n)\sum_{j=1}^n X_j\,[X_j^2 - 3(\overline{X}_j)^3]$. Show that $\hat{\psi}_n$ is weakly consistent for $\psi = \mathrm{Var}(X_1)$.

(c) Let $Y_j = \sqrt{j}\,(X_j - \theta)$ and let $s_n^2 = \sum_{j=1}^n \mathrm{Var}(Y_j)$. Show that for each $\theta \in (0,1)$, $\sum_{j=1}^n Y_j/s_n$ converges in distribution to a standard normal distribution.

6. Let $(\Omega, \mathcal{A})$ be a measurable space, and for each $\theta \in \mathbb{R}$, let P_θ be a probability measure on $(\Omega, \mathcal{A})$. Let X be a random variable on $(\Omega, \mathcal{A})$, and let $[L(X), U(X)]$ be a confidence interval for θ based on X.

(a) Show that for each $\theta \in \mathbb{R}$,

$$E_\theta[U(X) - L(X)] = \int_{\theta' \neq \theta} P_\theta(L(X) \leq \theta' \leq U(X))\, d\theta'.$$

(b) Show that if $[L(X), U(X)]$ is obtained by inverting the acceptance regions $A(\theta)$ of nonrandomized tests, then for each $\theta \in \mathbb{R}$,

$$E_\theta[U(X) - L(X)] = \int_{\theta' \neq \theta} P_\theta(X \in A(\theta'))\, d\theta'.$$

(c) Now suppose that P_θ is the normal distribution with mean θ and variance 1, and find the confidence interval with minimum expected length when $\theta = 0$.

7. Let $\{Y_n\}$ be a sequence of random variables; let $\{a_n\}$ be a sequence of constants; and let μ be a constant. The following result is well-known:

Well-known result: *Suppose that $a_n \to \infty$ and $a_n(Y_n - \mu) \xrightarrow{\mathcal{L}} X$ as $n \to \infty$. If the function g is differentiable at μ, then $a_n[g(Y_n) - g(\mu)] \xrightarrow{\mathcal{L}} g'(\mu)X$ as $n \to \infty$.*

(a) Prove the well-known result using Skorohod's representation of a weakly converging sequence by an almost-surely converging sequence.

(b) Prove the well-known result without using Skorohod's representation. You may require more conditions on $\{Y_n\}$ or on the function g. If so, make the conditions explicit.

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