PRELIMINARY EXAMINATION: STT 871-872,
WEDNESDAY, AUGUST 25, 2004, 1:00PM - 6:00 PM,
A506 WELLS HALL

NOTE

1. **This examination contains 8 problems. Every statement you make must be substantiated. You may do this either by quoting a theorem or result and verifying its applicability or by proving things directly. In a problem consisting of multiple parts you may use one part to solve the other part even if you are unable to solve the part being used.**

2. **Please start solution of each problem on the page containing the statement of the problem. Be sure to put your assigned number on the right hand top corner of each page of your solutions.**

GOOD LUCK!!!

1. Let $X = (X_1, \cdots, X_n)$ be a random sample from the exponential distribution $E(0, \theta)$ (p.d.f.: $\frac{1}{\theta} e^{-x/\theta} \mathbf{I}(x > 0)$) with $\theta > 0$ unknown. Let the prior be such that $\omega = \theta^{-1}$ has the gamma distribution $\Gamma(\alpha, \beta)$ (p.d.f.: $\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \mathbf{I}(x > 0)$) with α and β both known and positive. Consider now the estimation of $\eta_t(\omega) = e^{-t\omega}$ $(t \neq 0)$.

   (a) Find the posterior distribution of ω. **(4 points)**

   (b) Find the Bayes estimator $\delta_t(X)$ of $\eta_t(\omega)$ under the squared error loss. **(4 points)**

   (c) Show that $\delta_t(X)$ is consistent. **(2 points)**

2. Let $X_1, \cdots, X_n$ be i.i.d. having the $N(\mu, \sigma^2)$ distribution with an unknown $\mu \in \mathbb{R}$, a known $\sigma^2 > 0$ and $n > 1$.

   (a) Derive the UMVUE (uniformly minimum variance unbiased estimator) T_n of $e^{t\mu}$ with a fixed $t \neq 0$. **(4 points)**

   (b) Determine whether the variance of T_n attains the Cramér-Rao bound. **(3 points)**

   (c) Show that T_n is asymptotically efficient. **(3 points)**

3. Let $X_1, \cdots, X_n$ be i.i.d. from binomial distribution $\text{BIN}(2, \frac{\theta}{1+\theta})$ with $\theta > 0$ (the p.d.f. of $\text{BIN}(n,p)$ is $\binom{n}{x} p^x (1-p)^{n-x}$).

   (a) Show that the solution of the log-likelihood equation for θ, based on $X_1, \cdots, X_n$, is given by $\hat{\theta}_n = \bar{X}_n/(2-\bar{X}_n)$, where $\bar{X}_n = \sum_{i=1}^n X_i/n$. **(2 points)**

   (b) Show that $\sqrt{n}\,(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \theta(1+\theta)^2/2)$. **(5 points)**

   (c) Find a large sample $(1-\alpha)100\%$ two-side confidence interval for θ based on your answer to part (b) above. **(3 points)**

4. Suppose that $y_i = \beta x_i + \epsilon_i$, where $\beta \in \mathbb{R}$ is unknown, $x_i \in (1, 2)$ is given, and ϵ_i's are independent random variables with $E(\epsilon_i) = 0$, $\sup_i E|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$ and $\text{Var}(\epsilon_i) = \sigma^2 x_i$ with an unknown $\sigma^2 > 0$, $i = 1, \cdots, n$.

   (a) Find the LSE (least squares estimator) of β. **(3 points)**

   (b) Derive the BLUE (best linear unbiased estimator) of β. **(4 points)**

   (c) Show that both the LSE and the BLUE are asymptotically normal with the latter being at least as efficient (asymptotically) as the former. **(5 points)**

5. For each $\theta \in \mathbb{R}$, let $(\Omega, \mathcal{A}, P_\theta)$ be a probability space. Let L_i and U_i, $i = 1, 2$, be random variables defined on the given probability space such that $L_i \leq U_i$ almost surely for any $\theta \in \mathbb{R}$ and $i = 1, 2$. Suppose that $P_\theta(L_1 \leq x \leq U_1) \leq P_\theta(L_2 \leq x \leq U_2)$, $\forall \theta, x \in \mathbb{R}$. Show that $E_\theta(U_1 - L_1) \leq E_\theta(U_2 - L_2)$, $\forall \theta \in \mathbb{R}$. **(8 points)**

6. Let $X = (X_1, \cdots, X_n)$ be i.i.d. random variables from the uniform distribution $U(\theta, \theta+1)$, $\theta \in \mathbb{R}$. Suppose that $n > 1$. Find a minimal sufficient statistic for θ. **(10 points)**

7. Let $X \sim E(0, \theta)$ (see problem 1 for the p.d.f.). Show that the UMVUE estimator X of $\theta > 0$ is not admissible w.r.t. the squared error loss when we know $2 \leq \theta \leq 3$. **(10 points)**

8. Let $X_1, \cdots, X_n$ be i.i.d. from the exponential distribution $E(a, b)$ with $a \in \mathbb{R}$ and $b > 0$ (i.e. the p.d.f. is $\frac{1}{b} e^{-(x-a)/b} \mathbf{I}(a < x)$).

   (a) For testing $H_0 : a = a_0$ versus $a = a_1 < a_0$, show that any UMP test T_* of size α has power $\beta_{T_*}(a_1) = 1 - (1-\alpha) e^{-n(a_0 - a_1)/b}$. **(4 points)**

   (b) Derive the UMP test of size α for testing $H_0 : a = a_0$ versus $a \neq a_0$, when b is known. **(6 points)**