Preliminary Examination: STT 871-872,
Wednesday, August 24, 2005, 1:00pm - 6:00 pm,
A506 Wells Hall

1. Let $X_1, \cdots, X_n$ be i.i.d. from a distribution with Lebesgue p.d.f. $f(x,\theta) = e^{-(x-\theta)}$ for $\theta < x < \infty$ and $f(x,\theta) = 0$ otherwise, where $\theta \in \mathbb{R}$.

   (a) Show that $T_n = X_{(1)} - \frac{1}{n}$ is the UMVUE (uniformly minimum variance unbiased estimator) of θ, where $X_{(1)}$ is the minimum order statistic. **(7 points)**

   (b) Show that the inequality $\mathrm{var}(T_n) \geq 1/\left(n E[\frac{\partial}{\partial\theta} \log e^{-(X-\theta)}]^2\right)$ does not hold for any $\theta \in \mathbb{R}$ and $n > 1$. **(4 points)**

   (c) Explain if the result in (b) conflicts with the Cramé-Rao lower bound. **(4 points)**

2. Suppose that one observes 6 independent normal r.v.'s Y_i, $i = 1, \cdots, 6$, with a common variance and the means given as follows:

$$E(Y_1) = E(Y_2) = \mu_1, E(Y_3) = E(Y_4) = \mu_2, E(Y_5) = E(Y_6) = \mu_1 + \mu_2.$$

   (a) What is the best linear unbiased estimator of $\mu_1 - \mu_2$. **(5 points)**

   (b) Describe a $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ in as much detail as possible for an $0 < \alpha < 1$. **(5 points)**

3. Let X have $N(\theta, 1)$ distribution for some $\theta \in \mathbb{R}$ and let $0 < \alpha < 1$. Suppose one observes only $Y = |X|$. Let f_θ denote the Lebesgue density of Y.

   (a) Show that $f_\theta(y)/f_0(y)$ is strictly increasing in y (> 0) for all $\theta \neq 0$. **(5 points)**

   (b) Show that the UMP size α test of $H_0 : \theta = 0$ v.s. $H_1 : \theta \neq 0$ is $T(Y) = I(Y > y_\alpha)$, where y_α is the $(1-\alpha)$th quantile of the null distribution of Y. **(5 points)**

4. Let X be a random sample of size 1 from the uniform $(\theta - 1/2, \theta + 1/2)$ distribution for some $\theta \in \mathbb{R}$.

   (a) Show that X is minimal sufficient but not complete. **(6 points)**

   (b) Show that X is an unbiased estimator but not a UMVUE of θ. **(4 points)**

5. Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be two independent samples from $E(0,\mu)$ (Lebesgue p.d.f. $f(t,\mu) = \frac{1}{\mu} e^{-t/\mu}$, $t > 0$, $\mu > 0$) and $E(0,\lambda)$ respectively. Suppose that we observe only $Z_i = \min\{X_i, Y_i\}$ and $\Delta_i = I(X_i \geq Y_i)$, $i = 1, \cdots, n$.

   (a) Show that Z_i and Δ_i are independent with distributions $E(0, \frac{\mu\lambda}{\mu+\lambda})$ and Binomial$(1, \frac{\mu}{\mu+\lambda})$ respectively, for every $i = 1, \cdots, n$. **(5 points)**

   (b) Show that $(\hat{\mu}, \hat{\lambda}) = \left(\frac{\sum_{i=1}^n Z_i}{n - \sum_{i=1}^n \Delta_i}, \frac{\sum_{i=1}^n Z_i}{\sum_{i=1}^n \Delta_i}\right)$ is the MLE (maximum likelihood estimator) of (μ, λ). **(5 points)**

   (c) Show that $\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} N(0, \sigma^2)$ and determine σ^2. **(5 points)**

6. Let X be a random sample of size 1 from a distribution with Lebesgue density f. Consider the problem of testing $H_0 : f = f_0$ v.s. $H_1 : f = f_1$, where $f_0(x) = e^{-x} I(x > 0)$, $f_1(x) = \frac{1}{2} e^{-(x-1)/2} I(x > 1)$.

(a) Find the most power test of size exactly equal to $\alpha = e^{-1}$. **(5 points)**

(b) Show that the test above is admissible under the usual $0-1$ loss (i.e., other size α test is no better than this one in terms of risk). **(5 points)**

7. Suppose that $X_1, \cdots, X_n$ are i.i.d. from $N(0, \sigma^2)$ with $\sigma^2 > 0$ unknown.

   (a) Show that the MLE of σ^2 is inadmissible for the squared error loss: $L(\cdot, \sigma^2) = (\cdot - \sigma^2)^2$. (Hint: $E(\chi^2(m)) = m$, $\text{var}(\chi^2(m)) = 2m$.) **(4 points)**

   (b) Find, under the loss function $L(\cdot, \sigma^2) = (\cdot - \sigma^2)^2/\sigma^4$, an admissible estimator of σ^2 that is the unique minimax estimator. **(6 points)**

   (c) Suppose that $f(x, \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0)$, with α and β both positive and known, is the Lebesgue density of the prior for $\omega = 1/(2\sigma^2)$. Find the Bayes estimator of σ^2 under the squared error loss. **(5 points)**

8. Let $\mathcal{F}$ be a collection of distribution functions on $\mathbb{R}$ and $\delta_x \in \mathcal{F}$ be the point mass distribution function at a point $x \in \mathbb{R}$. Let $T(\cdot)$ be a functional defined on $\mathcal{F}$. The *influence function* (IF) of $T(G)$, $G \in \mathcal{F}$, at x is defined as (if it exists)

$$IF(x; T(G)) = \lim_{t \to 0} \frac{T((1-t)G + t\delta_x) - T(G)}{t},$$

   where the limit is taken over those $t \in (0, 1)$ for which $(1-t)G + t\delta_x \in \mathcal{F}$.

   Let $\mathcal{D} = \{c(G_1 - G_2) : c \in \mathbb{R}, G_n \in \mathcal{F}, n = 1, 2\}$ and ρ be a distance on $\mathcal{F}$. The functional T is said to be *ρ-Fréchet differentiable* at $G \in \mathcal{F}$ if there is a linear functional T_G on $\mathcal{D}$ such that for any sequence $\{G_j\}$, $j \geq 1$, in $\mathcal{F}$ and $\rho(G_j, G) \to 0$ as $j \to \infty$,

$$T(G_j) - T(G) - T_G(G_j - G) = o(\rho(G_j, G)), \quad \text{as } j \to \infty.$$

   For $0 < \alpha < \beta < 1$, define

$$L(H) = \frac{1}{\beta - \alpha} \int y I(\alpha < H(y) \leq \beta) dH(y), \quad H \in \mathcal{F}.$$

   Now fix an $F \in \mathcal{F}$.

   (a) Show that $IF(x; L(F))$ is bounded in $x \in \mathbb{R}$ and $E[IF(X; L(F))] = 0$ for r.v. X with distribution F. **(4 points)**

   (b) Show that L is ρ_∞-Fréchet differentiable at F, where ρ_∞ is defined as: $\rho_\infty(G_1, G_2) = \sup_{x \in \mathbb{R}} |G_1(x) - G_2(x)|$. **(6 points)**

   (c) Now let $F(x) = F_0(x - \theta)$ for a fixed $\theta \in \mathbb{R}$ with F_0 symmetric about 0 and having a density $f_0 > 0$ and let $\beta = 1 - \alpha$ in the above set up. Let F_n denote the empirical distribution function of a random sample of size n from F and $F_n \in \mathcal{F}$ for every $n \geq 1$. Show that $\sqrt{n}(L(F_n) - \theta) \xrightarrow{d} N(0, \sigma_\alpha^2)$ and determine σ_α^2. **(5 points)**