STT 871-872 Preliminary Examination, August 2006
Wednesday, August 23, 2006, 12:30 - 5:30 pm

NOTE: This examination is closed book. Every statement you make must be substantiated. You may do this either by quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of a problem to solve the other part, even if you are unable to solve the part being used. A complete solution of a problem will get a more favorable review than a partial solution.

You must start solution of each problem on the given page. Be sure to put the number assigned to you on the right corner top of every page of your solution.

Throughout, n is a known positive integer denoting the sample size, $\mathbb{R}$ is the real line and $\mathcal{B}$ is the Borel sigma field.

1. For each $\theta = 1, 2, \cdots, \infty$, let the r.v. X have uniform distribution on the set of integers $\{1, 2, \cdots, \theta\}$. Construct an unbiased estimator of θ based on X and show that it is the only such estimator. **(11)**

2. Let $X_1, \cdots, X_n$ be independent random variables having common density

$$f_\theta(x) := \frac{1+\theta x}{2} I(|x| \le 1), \qquad \text{for some } |\theta| \le 1.$$

Let $Y_i := I(X_i > 0)$, $i = 1, \cdots, n$.

(a). Based on $Y_1, \cdots, Y_n$, obtain the maximum likelihood estimator of θ. **(5)**

(b). Prove or disprove: The estimator obtained in (a) is unbiased. **(2)**

(c). Prove or disprove: The estimator obtained in (a) is consistent? **(5)**

3. For a $\theta > 0$, let X have uniform distribution on $[0, \theta]$. Define the loss function $L(\vartheta, \delta) := |\vartheta - \delta|/\vartheta$, $\delta > 0$, $\vartheta > 0$.

(a). Find the estimator $T(X)$ of θ that has uniformly smallest risk among the class of estimators $\mathcal{M}_X := \{\delta_c(X) = cX;\ c \ge 1\}$. **(7)**

(b). Prove or disprove: The estimator $\delta_1(X)$ is minimax. **(5)**

4. Let P_0, P_1 be two probability measures dominated by a sigma finite measure μ on $(\mathbb{R}, \mathcal{B})$ and let f_0, f_1 denote their respective densities with respect to μ such that $f_0(x) = 0$ implies $f_1(x) = 0$, $x \in \mathbb{R}$. Let G_0 denote the distribution function of $T(X) := f_1(X)/f_0(X)$ when X has distribution P_0. Suppose G_0 is continuous and strictly increasing on its support. For an $0 < \alpha < 1$, let $\varphi_\alpha(X) := I(T(X) \geq k(\alpha))$ denote the Neyman-Pearson test of size α for testing P_0 against P_1 and $\beta(\alpha)$ denote its type II error probability. Show that **(12)**

$$\frac{d\beta(\alpha)}{d\alpha} = -k(\alpha).$$

5. Prove or give counter examples to the following statements.

(a). Suppose P_0 and P_1 are two distinct probability measures on the Borel line $(\mathbb{R}, \mathcal{B})$ and $X_1, X_2, \cdots, X_n$ is a random sample from either P_0 or P_1. Then, there exists a sequence of test functions $\phi_n(X_1, X_2, \ldots, X_n)$ such that, **(6)**

$$\lim_{n\to\infty} E_0(\phi_n(X_1, X_2, \ldots, X_n)) = 0, \quad \text{and} \quad \lim_{n\to\infty} E_1(\phi_n(X_1, X_2, \ldots, X_n)) = 1.$$

(b). Suppose P_0, P_1 and P_2 are distinct probability measures on the Borel line $(\mathbb{R}, \mathcal{B})$ and $X_1, X_2, \cdots, X_n$ are i.i.d. under any of these measures. Then, there exists a sequence of test functions $\phi_n(X_1, X_2, \ldots, X_n)$ such that, **(4)**

$$\lim_{n\to\infty} E_0(\phi_n(X_1, X_2, \ldots, X_n)) = 0, \text{ and } \lim_{n\to\infty} \min_{i=1,2} E_i(\phi_n(X_1, X_2, \ldots, X_n)) = 1.$$

(c). Let P_0, P_1, P_2 and $X_1, X_2, \ldots, X_n$ be as in Part (b) above. Let $\mathcal{P} := \{\lambda P_1 + (1-\lambda)P_2 : 0 \leq \lambda \leq 1\}$ be the convex set generated by P_1 and P_2. Then, for each $n \geq 1$, there exists a test function $\phi_n(X_1, X_2, \ldots, X_n)$ such that, **(3)**

$$\lim_{n\to\infty} E_0(\phi_n(X_1, X_2, \ldots, X_n)) = 0, \text{ and } \inf_{P\in\mathcal{P}} E_P(\phi_n(X_1, X_2, \ldots, X_n)) \to 1.$$

6. Let $\hat{\theta}_n := \bar{X}_n(1 - \bar{X}_n)$ denote an estimator of $\theta := \mu(1-\mu)$ based on a random sample of size n from a population with mean $\mu \in \mathbb{R}$ and finite and positive variance σ^2.

(a). Show that, when $\mu \neq 1/2$, $n^{1/2}(\hat{\theta}_n - \theta)$ is asymptotically normal with appropriate mean and variance. **(7)**

(b). For the case $\mu = 1/2$, determine a sequence $0 < a_n \to \infty$, so that $a_n(\theta - \hat{\theta})$ converges in distribution to a non-degenerate limit and determine this limiting distribution. **(5)**

7. Let (X, Y) be a pair of r.v.'s satisfying the linear regression model

$$Y = \beta_0 + \beta_1 X + \varepsilon, \quad \text{for some } (\beta_0, \beta_1) \in \mathbb{R}^2,$$

where $E\varepsilon = 0$, $E\varepsilon^2 = \sigma^2$, $0 < \sigma < \infty$. Suppose (X_i, Y_i), $1 \leq i \leq n$ are n i.i.d. observations from this model. Set $\bar{X} := \sum_{i=1}^n X_i$ and $S := \sqrt{\sum_{i=1}^n (X_i - \bar{X})^2/n}$.

(a). Obtain the least square estimator $(\tilde{\beta}_0, \tilde{\beta}_1)$ of (β_0, β_1), subject to the condition $\beta_0+\beta_1 S \leq 2$. **(8)**

(b). Assume that $0 < \tau := \sqrt{\mathrm{Var}(X)} < \infty$. Find the two functions $h_j \equiv h_j(\beta_0, \beta_1, \tau)$, $j = 0, 1$, such that for all values of (β_0, β_1), $n^{1/2}(\tilde{\beta}_0 - h_0,\ \tilde{\beta}_1 - h_1)'$ converges in distribution to a non-constant random vector and determine this joint limiting distribution. **(6)**

8. Let X_i, $i = 1, 2, \cdots, n$, be independent r.v.'s with X_i having Lebesgue density $f(\cdot - \mu_i)$, where $\mu_i \in \mathbb{R}$ are unknown such that $\sum_{i=1}^{n} \mu_i = 0$, and f is a Lebesgue density on $\mathbb{R}$. Let f_0 and f_1 be $N(0, 1)$ and standard Cauchy densities on $\mathbb{R}$, respectively. Consider the problem of testing the hypothesis $H_0 : f = f_0$ against the alternative $H_1 : f = f_1$.

(a). Show that this testing problem remains invariant under the transformation **(4)**

$$g(x_1, \cdots, x_n) = (x_1 + b_1, \cdots, x_n + b_n),\ (x_1, \cdots, x_n) \in \mathbb{R}^n,\ (b_1, \cdots, b_n) \in \mathbb{R}^n,\ \sum_{i=1}^{n} b_i = 0.$$

(b). For an $0 < \alpha < 1$, determine uniformly most powerful size α invariant test of H_0 against H_1. **(10)**