STT 871-872 Preliminary Examination
Wednesday, August 22, 2007, 1 - 6 pm

Instructions

• This examination is closed book. Every statement you make must be substantiated. You may do this either by quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of a problem to solve the other part, even if you are unable to solve the part being used.

• Solution of each problem should be started on the page of its statement. A complete solution of a problem will get a more favorable review than a partial solution.

• Be sure to put the number assigned to you on the right corner top of every page of your solution.

• Throughout, n is a known positive integer denoting the sample size, $\mathbb{R}$ is the real line and $\mathcal{B}$ is the Borel sigma field.

GOOD LUCK !!!

1. Let $\theta \in \{0, 1, 2, \cdots\}$. Let X and Y be two independent r.v.'s with the following distributions:

$$P(X = \theta) = 1/2 = P(X = \theta + 1), \quad P(Y = \theta) = 1/2 = P(Y = \theta - 1).$$

(a) Let $T_1 := X - 1/2$, $T_2 = Y + 1/2$. Obtain a minimum variance unbiased estimator of θ among the class of estimators $aT_1 + bT_2$, $a \in \mathbb{R}, b \in \mathbb{R}$. **(5)**

(b) Prove or disprove: The family of the joint distributions of (X, Y) is complete. **(8)**

2. Let $X_1, \cdots, X_n$, be a random sample of size $n \geq 1$ from the density

$$f_\theta(x) := \frac{\theta}{x^2} I(x > \theta), \quad \theta = 1, 2, 3, \cdots.$$

(a) Prove that the MLE $\hat{\theta}$ of θ is a.s. consistent for θ. **(6)**

(b) Let the loss function be $L(\theta, \delta) = I(\delta \neq \theta)$, for any estimator δ.
Prove or disprove: $\hat{\theta}$ is minimax under this loss function. **(8)**

3. Let X_1 and X_2 be two independent r.v.'s with X_i having $N(\theta_i, 1)$ distribution for some $\theta_i \in \mathbb{R}$, $i = 1, 2$. Let $\theta = (\theta_1, \theta_2)'$ and consider the problem of estimating θ_1 under the loss function $L(\theta, T) := (\theta_1 - T)^2$, for any estimator T of θ_1. Show that among the class of estimators T for which $E_\theta L(\theta, T) < \infty$, $\forall\, \theta \in \mathbb{R}^2$, $\delta := \text{sign}(X_2)$ is an admissible estimator of θ_1. **(10)**

[Hint: Show that the estimator δ can not be beaten by any other estimator at both $\theta = (1, \theta_2)'$ and $\theta = (-1, \theta_2)'$ and for all $\theta_2 \in \mathbb{R}$.]

4. Let X and Y be independent r.v.'s, with

$$P(X = x) = p_1(1-p_1)^x, \quad x = 0, 1, 2, \cdots,\ 0 < p_1 < 1,$$
$$P(Y = y) = p_2(1-p_2)^y, \quad y = 0, 1, 2, \cdots,\ 0 < p_2 < 1.$$

Derive uniformly most powerful unbiased test of the hypothesis $H_0 : p_1 = p_2$ against the alternative $H_1 : p_2 > p_1$, in as much detail as possible. **(12)**

5. Let $X_1, \cdots, X_n$ be i.i.d. $N(\mu, 1)$ r.v.'s, $n\bar{X} := \sum_{i=1}^n X_i$, $n \geq 1$, and let Φ denote distribution function of $N(0,1)$ r.v. Consider the problem of testing $H_0 : 0 \leq \mu \leq 3$ vs. the alternative $\mu > 3$.

(a) Show that the test that rejects H_0 for large values of $n^{1/2}(\bar{X} - 3)$ has p-value $\hat{p} := \Phi(-n^{1/2}(\bar{X} - 3))$. Determine the distribution of $\hat{p}$ when $\mu = 3$, for each n. **(3)**

(b) Let $a_n := n/(n+1)$. Suppose μ has a prior distribution $N(0,1)$. Show that the posterior probability of H_0 is **(3)**

$$\tilde{p} := \Phi\Big(\frac{-n^{1/2}(a_n\bar{X} - 3)}{a_n}\Big) - \Phi\Big(\frac{-n^{1/2}a_n\bar{X}}{a_n}\Big).$$

(c) Determine the limiting distribution of $\tilde{p}$ when $\mu = 3$. **(4)**

(d) Compute the limit of $\tilde{p}/\hat{p}$, in probability, as $n \to \infty$, when $\mu > 3$ is fixed and when $\mu = 3 + (a/\sqrt{n})$, $a > 0$. **(4)**

6. Let (X, Y) be a bivariate random vector with distribution function H on $\mathbb{R}^2$. Let F and G denote marginal distribution functions of X and Y, respectively. Assume $E|X|^2 + E|Y|^2 < \infty$. Let (X_i, Y_i), $i = 1, 2$ be two independent copies of (X, Y).

(a) Show that $\text{Cov}(X, Y)$ exists and **(2)**

$$2\text{Cov}(X, Y) = E\{(X_1 - X_2)(Y_1 - Y_2)\}.$$

(b) Show that $H(x, y) - F(x)G(y) \geq 0$, for all $(x, y) \in \mathbb{R}^2$, implies that $\text{Cov}(X, Y) \geq 0$. **(9)**

7. Let x_i, $1 \leq i \leq n$, be known constants with $\sum_{i=1}^n (x_i - \bar{x})^2 > 0$, $\bar{x} := \sum_{i=1}^n x_i/n$. Consider the observations $(Y_{i1}, Y_{i2}), i = 1, 2, \cdots, n$, obtained from the linear regression model

$$Y_{i1} = \alpha_1 + \beta_1 x_i + \varepsilon_{i1}, \quad Y_{i2} = \alpha_2 + \beta_2 x_i + \varepsilon_{i2},$$

where ε_{ij}, $i = 1, 2, \cdots, n$, $j = 1, 2$, are i.i.d. $N(0, \sigma^2)$ r.v.'s.

(i) Derive the least squares test of the hypothesis that the two regression lines are equal, i.e., $H_0 : \alpha_1 = \alpha_2,\ \beta_1 = \beta_2$, vs. the alternative that H_0 is not true, describing the null distribution of the test statistic obtained in the fullest possible detail. **(6)**

(ii) For an $0 < \alpha < 1$, derive the Scheffe's $1 - \alpha$ level confidence sets for the parameter vector $(\alpha_1 - \alpha_2,\ \beta_1 - \beta_2)$. **(6)**

8. Let m and k be known positive integer, Z_i, $i = 1, 2, \cdots, n$, be independent k-dimensional random vectors, and let $A_1, \cdots, A_n$ be fixed $m \times k$ real matrices. Consider the group transformation $g_i(Z_i) = Z_i + b_i$, $i = 1, \cdots, n$ with b_i's $k \times 1$ vectors such that $\sum_{i=1}^n A_i b_i = 0$.

(a) Find a maximal invariant statistic under the above transformation group. **(4)**

(b) Let f be a Lebesgue density on $\mathbb{R}$ and

$$\mu := \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} := \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} \gamma \\ \theta \end{pmatrix}, \quad \gamma, \theta \in \mathbb{R}.$$

Suppose in part (a), $k = 2$ and $Z_i := (X_i, Y_i)'$ has the bivariate density

$$f(x - \mu_1) f(y - \mu_2), \quad \text{for some } \gamma, \theta \in \mathbb{R}, \text{ and } \forall\, (x, y) \in \mathbb{R}^2.$$

Let $0 < \alpha < 1$. Assume $n > 1$. Derive the most powerful invariant size α test of the hypothesis **(10)**

$$H_0 : f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},\ x \in \mathbb{R}, \quad \text{vs.} \quad H_1 : f(x) = \frac{1}{\pi} \frac{1}{1 + x^2},\ x \in \mathbb{R},$$

under the condition that $A_i \equiv A$, $b_i \equiv b$ in the above transformation group. Also describe the null distribution of the test statistic obtained.