STT 871-872 Preliminary Examination, August 2008
Wednesday, August 20, 2008, 12:30 pm—5:30 p.m.

NOTE: This exam is closed book. You must substantiate each statement you make by either quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of the problem to solve the other part, regardless if you are able to solve the part you use. Clearly written and detailed partial solutions earn more partial credits than sketchy ones.
You must start solution of each problem on the given page. Be sure to put the number assigned to you on the upper right corner of each page of your solution.
In all that follows, n denotes the sample size, a positive integer, and R the set of real numbers.

Problem 1 Let $\theta > 0$ and $X_1, \cdots, X_n$ be a random sample from the normal distribution $N(\theta, \theta)$. Obtain the M.L.E. $\hat{\theta}_n$ of θ and derive the asymptotic distribution of $n^{1/2}(\hat{\theta}_n - \theta)$. (**12 points**)

Problem 2 Let $f(x)$ be a Lebesgue pdf on $(0, +\infty)$ and X a single observation from f. Let

$$f_0(x) = e^{-x} I_{(0,+\infty)}(x), f_1(x) = 2^{-1} x^2 e^{-x} I_{(0,+\infty)}(x)$$

and consider testing of the hypotheses $H_0 : f = f_0$ vs $H_1 : f = f_1$. Show that any test of size α $(0 < \alpha < 1)$ has power at most $\alpha \left\{ (\ln \alpha)^2 / 2 - \ln \alpha + 1 \right\}$. (**12 points**)

Problem 3 Suppose $X_1, X_2, \ldots, X_n$ are i.i.d $N(\theta, 1)$
(a) Show that a UMP test for $H_0 : \theta = 0$ vs $H_1 : \theta \neq 0$ does not exist; (**4 points**)
(b) Consider the group of transformations $\{g, e\}$ where $g(x) = -x, e(x) = x$. Find the UMPI test under this group of transformations; (**4 points**)
(c) Let $\phi(X_1, X_2, \ldots, X_n)$ be the UMPI test. As $n \to \infty$, what is the behavior of $E_\theta \phi(X_1, X_2, \ldots, X_n)$? (**4 points**)

Problem 4 Suppose one observes $X \in \mathcal{X}$ having density $f_\theta(x)$ w.r.t. a σ-finite measure ν on $\mathcal{X}$, for $\theta \in \Theta \subset R$. One is interested in estimating $h(\theta)$, a real valued function of θ, under the loss function $L(\delta, h(\theta))$ which is strictly increasing in δ for $\delta > h(\theta)$, and strictly decreasing in δ for $\delta < h(\theta)$.
Suppose that $h(\theta)$ is nonconstant and has a global minimum at a point θ^*. Assume that $f_{\theta^*}(x) > 0$, a.e. ν. Let $T(X)$ be an unbiased estimator of $h(\theta)$ with finite risk under L.
(a) Let $A_\varepsilon := \{T(X) < h(\theta^*) - \varepsilon\}$, $\varepsilon > 0$. Show that there exists an $\varepsilon > 0$ such that $P_{\theta^*}(A_\varepsilon) > 0$; (**6 points**)
(b) Prove that $T(X)$ is inadmissible for $h(\theta)$. (**6 points**)

Problem 5 Let $X_1, \cdots, X_n$ be a random sample from the normal distribution $N(\mu, 1)$. Construct an asymptotically pivotal quantity and a $1 - \alpha$ asymptotically correct confidence set for μ^2. (**12 points**)

Problem 6 Let a parameter $\theta \in \Theta = (1,2)$ and $Y_1, \cdots, Y_n$ a random sample from $U(\theta, 2\theta)$. Suppose that instead of $Y_1, \cdots, Y_n$ one observes $X_1, \cdots, X_n$ which are

$$X_i = \begin{cases} 2 & Y_i > 2 \\ Y_i & Y_i \le 2 \end{cases}.$$

(a) Denote the σ-finite measure $\nu = \delta + m$ in which δ is the point mass measure at $\{2\}$ and m the Lebesgue measure. Show that the pdf of X_1 with respect to ν is $f_\theta(x) = \left(\frac{2\theta-2}{\theta}\right) I_{\{2\}}(x) + \left(\frac{1}{\theta}\right) I_{(\theta,2)}(x)$; (**8 points**)
(b) Let $R = \sum_{i=1}^{n} I(X_i = 2)$, show that the UMVUE of $1-\theta^{-1}$ based on $X_1, \cdots, X_n$ is $n^{-1}R/2$. (**6 points**)

Problem 7 Let $(x_i, Y_i), i = 1, 2, ..., 2n$ be bivariate data with $x_i = i/2n, i = 1, 2, ..., 2n$ and

$$\begin{aligned} Y_i &= \beta_{10} + \beta_{11} x_i + \varepsilon_i, i = 1, 2, ..., n, \\ Y_i &= \beta_{20} + \beta_{21} x_i + \varepsilon_i, i = n+1, 2, ..., 2n \end{aligned}$$

where $\varepsilon_i, i = 1, ..., 2n.$ are iid from $N(0,1)$.
(a) Give the explicit formulae of the LSE $\hat{\boldsymbol{\beta}} = \left(\hat{\beta}_{10}, \hat{\beta}_{11}, \hat{\beta}_{20}, \hat{\beta}_{21}\right)$ of the vector of parameters $\boldsymbol{\beta} = (\beta_{10}, \beta_{11}, \beta_{20}, \beta_{21})$; (**6 points**)
(b) Show that $\hat{\beta}_{10}$ and $\hat{\beta}_{21}$ are independent. (**6 points**)

Problem 8 Let X be a single observation from $\Gamma(\alpha, \gamma)$ with $\alpha > 0$ known and one wishes to estimate $\gamma > 0$ under the loss $L\{\gamma, T(X)\} = \{1 - T(X)\gamma^{-1}\}^2$.
(a) Show that the generalized Bayes estimator with the improper prior $\pi(\gamma) = \gamma^{-2}$ is $T_*(X) = X/(\alpha+2)$; (**6 points**)
(b) Show that $T_*(X)$ is inadmissible. (**8 points**)