August 22, 2012

STT 871-872 Fall Preliminary Examination
Wednesday, August 22, 2012
12:30 - 5:30 pm

1. Suppose an observation X takes values $-1, 0, 1$, with respective probabilities $\theta/2, 1-\theta, \theta/2$, for some $0 \le \theta \le 1$, i.e., it has the following density:

$$f_\theta(x) \quad := \quad \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1, \; 0 \le \theta \le 1.$$

(a) Determine the class $\mathcal{U}$ of all unbiased estimators of zero based on X. **(5)**

(b) Obtain maximum likelihood estimator $\hat{\theta}(X)$ of θ. **(3)**

(c) Prove or disprove: $\hat{\theta}(X)$ is UMVU estimator for θ. **(2)**

2. Suppose $k \ge 1$ is a known integer, $X \sim P_\theta$, $\theta \in \mathbb{R}^k$, $q : \mathbb{R}^k \mapsto \mathbb{R}$.

(a) Let $T_1(X)$ and $T_2(X)$ be two UMVU estimators of $q(\theta)$. Show that **(4)**

$$P_\theta(T_1(X) = T_2(X)) = 1, \; \forall\, \theta \in \mathbb{R}^k.$$

(b) Let $X \sim \mathcal{N}(\theta, 1)$, $\theta \in \mathbb{R}$. Show that $T(X) := X^2 - 1$ is UMVU estimator of θ^2. **(2)**

(c) Let $S(X) := T(X)I(|X| < 1)$. Show that $E_0 S^2(X) > 0$ and $T(X) = X^2 - 1$ is an inadmissible estimator of θ^2 w.r.t. the square error loss. **(6)**

3. Let X be a $Binomial(n, p)$ random variable, for some $0 \le p \le 1$. Define a class of estimators

$$\begin{aligned} T_\alpha(X) \;&:=\; X/n, && \text{with prob. } 1-\alpha, \\ &:=\; 1/2, && \text{with prob. } \alpha, \quad 0 \le \alpha < 1. \end{aligned}$$

Let $R(p, T_\alpha)$ denote the risk of the estimator T_α under the square error loss. Compare $R(p, T_\alpha)$ for $0 < \alpha < 1$ with $R(p, T_0)$. Find an $\alpha \in (0,1)$ such that $\sup_{0 \le p \le 1} R(p, T_\alpha) < \sup_{0 \le p \le 1} R(p, T_0)$. Is the estimator T_0 minimax? **(12)**

4. Let a and k be unknown positive integers. Let f_θ denote the density of uniform distribution on $(\theta - 1/2, \theta + 1/2)$, $\theta \in \mathbb{R}$. For $\theta_1 = a$, $\theta_2 = a+1, \cdots, \theta_k = a + k - 1$, let

$$f_k(x) := \frac{1}{k} \sum_{j=1}^{k} f_{\theta_j}(x), \quad k = 1, 2, \cdots.$$

Let $X_1, X_2, \cdots, X_m$ be i.i.d. observations from f_k, $k = 1, 2, \cdots, m$.

(a) Find the MLE $\hat{k}$ of k. **(7)**

(b) Show that $\hat{k}$ is consistent for k, as $m \to \infty$. **(5)**

5. Let $X_1, X_2, \cdots, X_n$ be i.i.d. r.v.'s from the parametric density

$$f_\theta(x) := 2\theta x e^{-\theta x^2}, \quad x > 0, \ \theta > 0.$$

(a) Derive UMP unbiased test of size $0 < \alpha < 1$, for testing $H_0 : \theta = 1$, vs. $H_1 : \theta \neq 1$, in the fullest possible detail. **(8)**

(b) Discuss the corresponding confidence interval for θ. What optimality properties does it have, if any. **(4)**

6. Let $Y_1, \cdots, Y_m$ and $Z_1, \cdots, Z_m$ be mutually independent observable r.v.'s with $Y_i \sim N(\xi_i, \sigma^2)$, $Z_i \sim N(\xi_i + c, \sigma^2)$, $i = 1, 2, \cdots, m$. Here, $\xi_1, \xi_2 \cdots, \xi_m$, c and σ^2 are all unknown parameters. Consider the problem of testing

$$H \ : \xi_1 = \xi_2 = \cdots = \xi_m \qquad \text{vs.} \qquad K \ : \text{not } H.$$

(a) Obtain the explicit expression for the residual sum of squares, SSE, under the full model $H \cup K$ and show that it is non-negative, and describe the F-test for the above problem in complete detail. **(8)**

(b) Is the estimator $SSE/(m-1)$ consistent for σ^2, as $m \to \infty$? **(4)**

(c) Consider the alternatives where $m^{-1} \sum_{i=1}^{m} (\xi_i - \bar{\xi})^2 \to \psi > 0$, as $m \to \infty$. Show that the power of the test in (a) for these alternatives converges to 1, as $m \to \infty$ **(4)**

7. Recall that in the regression model $Y_i = c_i \beta + \tau_i \eta_i$, where the r.v.'s η_i's have zero mean and unit variance, and c_i, $\tau_i > 0$ are known numbers, the weighted least square estimator of β is obtained by minimizing the sum $\sum_{i=1}^{n} \tau_i^{-2} (Y_i - c_i b)^2$ w.r.t. b.

Let $0 < x_1 < x_2 < \cdots < x_n$ be known positive numbers. Suppose that the observations Y_i's are generated by the following model:

$$Y_i = x_i \beta + U_i, \quad U_i = e_1 + e_2 + ... + e_i, \ \ i = 1, 2, ..., n,$$

where e_i, $1 \leq i \leq n$ are i.i.d. r.v.'s with mean 0 and $\text{Var}(e_i) = \sigma^2 (x_i - x_{i-1})$, $i = 1, \cdots, n$, where $x_0 = 0$.

(a) Obtain the weighted least square estimator $\hat{\beta}$ of β and show that it depends only on x_n and Y_n. Prove the consistency of $\hat{\beta}$ for β. Explicitly state the assumptions needed for consistency, if any. **(10)**

(b) Derive an expression for the test statistic for testing $H_0 : \beta = 1$ vs $H_1 : \beta > 1$ and describe the distribution of the test statistics under H_0. **(5)**

8. **(a).** Let X, X_i, $i \geq 1$ be i.i.d. r.v.'s with $EX = 0$, $\sigma^2 := EX^2$, $\tau := EX^4 < \infty$. Let

$$T_n := \Big(n^{-1} \sum_{i=1}^{n} X_i^2 \Big)^{1/2}.$$

Derive the asymptotic distribution of $n^{1/2}(T_n - \sigma)$. **(7)**

(b). Suppose a sequence of r.v.'s Y_n is such that for some constants a and $b > 0$, $n^{1/2}(Y_n - a)$ converges in distribution to $\mathcal{N}(0, b^2)$. Then are the following statements true or false, in general. **(4)**

(i) $\lim_{n\to\infty} EY_n = a$. (ii) $\lim_{n\to\infty} \text{Var}(Y_n) = b$.