STT 872, 867-868 Fall Preliminary Examination
Wednesday, August 20, 2014
12:30 - 5:30 pm

1. Let $X_1, ..., X_n$ be iid from a Geometric distribution with a parameter $p \in (0,1)$, i.e. $P(X_i = x) = p(1-p)^x,\ x = 0, 1, 2, ...$

(a) Find a complete and sufficient statistics for p based on $X_1, ..., X_n$.

(b) Find the moment generating function of X_1 and the first two moments of X_1.

(c) Let $g(p) = p^{-2}$. Formulate the two methods of finding UMVUE and use each to find it for $g(p)$ when $n = 2$.

(d) Show that $\frac{n}{n+\sum_{i=1}^n X_i}$ is the MLE of p. Compute its asymptotical variance, $n > 1$.

(e) Find the MLE of $1/p$. Using delta-method find its asymptotical variance, $n > 1$.

(f) Let $g(p) = 1/p$ and let the loss function be $\mathbb{E}(p\delta(X_1, ..., X_n) - 1)^2$. Using a $\beta(a, b)$ prior find a Bayes estimator of $g(p)$, $n > 1$.

(g) Calculate the loss of the Bayes estimator in (f). Find conditions on a, b that would make the estimator in (f) minimax. Are they satisfied? $n > 1$.

(h) Calculate the asymptotical efficiency of the MLE of $1/p$ and the Bayes estimator of $1/p$ with the prior $\beta(\sqrt{n}, n - \sqrt{n})$, $n > 1$.

(i) Under the loss $\mathbb{E}(p\delta(X_1, ..., X_n) - 1)^2$, show that the MLE of $1/p$ is inadmissible when n is large enough.

(j) Derive a UMP unbiased test of size $\alpha \in (0,1)$ for testing $H : 1/3 \le p \le 2/3$ vs $K : p < 1/3$ or $p > 2/3$ in the fullest possible detail, $n > 1$.

2. Let $X_1, ..., X_n$ be iid from the density $\lambda x^{-2} I(x > \lambda), \lambda > 0$.

(a) Construct the MRE estimator of λ under the loss $\mathbb{E}(\delta(X_1, ..., X_n)/\lambda - 1)^2$, $n > 2$.

(b) Is the estimator in (a) consistent in probability for λ?

(c) Derive a UMP test of size $\alpha \in (0,1)$ for testing $H : \lambda = 1$ vs $K : \lambda > 1$ in the fullest possible detail.

3. An experiment is designed to compare moisture content in three types of pigment pastes. For each type of pigment paste, 12 observations are obtained. Let Y_{ij} be the moisture level of the j-th observation in the i-th type of pigment paste. The main interest is to compare the average moisture level among three types of pigment pastes. A fixed-effect model may be used to fit the data as follows:

$$Y_{ij} = \mu + \theta_i + e_{ij}, e_{ij} \overset{iid}{\sim} N(0, \sigma^2)$$

for $i = 1, 2, 3$ and $j = 1, \cdots, 12$. To avoid identifiability issues, we set $\sum_{i=1}^3 \theta_i = 0$ and remove θ_3 from the above formulation, that is, the parameters in our model are μ, θ_1, and θ_2.

(a) Let $\boldsymbol{\beta} = (\mu, \theta_1, \theta_2)^T$. Specify the design matrix $\mathbf{X}$ in the linear model of matrix form $\mathbf{Y} = \mathbf{X}\beta + \mathbf{e}$ and compute the correlation between the least squares estimates (LSE) for θ_1 and θ_2.

(b) Construct $(1-\alpha)100\%$ confidence intervals for $\theta_1 - \theta_2$ based on the LSEs $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\sigma}^2$ for θ_1, θ_2 and σ^2 respectively, and t distribution.

(c) Show that a $(1-\alpha)$ joint confidence region for θ_1 and θ_2 can be specified by

$$(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \le \frac{2\hat{\sigma}^2}{12} F_{2,33;\alpha},$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$, $\hat{\boldsymbol{\theta}}$ is the LSE and $F_{2,33;\alpha}$ is upper α quantile of $F_{2,33}$.

4. In the previous problem, the samples were considered to be independent and identically distributed. In fact, the data set was actually obtained through the following nested study design: (a) sample 3 barrels of pigment paste; (b) take 2 samples from the content of each barrel; (c) each sample is mixed evenly and divided into 2 parts. Then the measurement of the moisture content is obtained through each part. Let Y_{ijkl} be the moisture content for the l-th part of the k-th sample from the j-th barrel of the i-th type.

Consider the following mixed effects model

$$Y_{ijkl} = \mu + \theta_i + \beta_{ij} + \delta_{ijk} + e_{ijkl} \tag{1}$$

for $i, j = 1, \cdots, 3$, $k = 1, 2$ and $l = 1, 2$, where μ is the fixed effect part, θ_i is the fixed type effect, β_{ij} is the random barrel effect and δ_{ijk} is the random sample effect and e_{ijkl} is the measurement error. Assume that β_{ij} are iid $N(0, \sigma_\beta^2)$, δ_{ijk} are iid $N(0, \sigma_\delta^2)$ and e_{ijkl} are iid $N(0, \sigma^2)$. In addition, β_{ij}, δ_{ijk} and e_{ijkl} are independent.

(a) Complete the ANOVA table for the above nested random effects model. Provide the formula for sum of squares in terms of Y_{ijkl}.

source	df	sum of squares	mean squares
Type			MST
Barrel			MSA
Sample			MSB
error			MSE

Given the model (1), obtain the expectation of the mean squares.

$$E(MSE) = \qquad\qquad\qquad , E(MSB) = \qquad\qquad\qquad , E(MSA) = \qquad\qquad\qquad .$$

(b) Assume that MSE, MSA, MAB are known. Using the ANOVA table in part (a), obtain the unbiased moment estimates for σ^2, σ_β^2 and σ_δ^2.

(c) Suppose we would like to obtain a new observation from the type 1 pigment paste through the same procedure described in the problem. Please give the best linear unbiased prediction for the moisture content of the new sample $Y_1^* = \mu + \theta_1 + \beta_{11} + \delta_{111}$.

5. Let g be a function with $\int g(x)dx = 1$, and let f be a density with support S. Define $g^* = gI_S / \int_S g(x)dx$. Prove that $\int |g^*(x) - f(x)|dx \leq \int |g(x) - f(x)|dx$.