STT 871-872 Preliminary Examination, January 2010
Wednesday, January 6, 2010, 12:30 p.m. - 5:30 p.m.

NOTE: This exam is closed book. You must substantiate each statement you make by either quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of the problem to solve the other part, regardless if you are able to solve the part you use. Clearly written and detailed partial solutions earn more partial credits than sketchy ones. You must start solution of each problem on the given page. Be sure to put the number assigned to you on the upper right corner of each page of your solution. In all problems, n denotes the sample size, a positive integer, and $R^1 = (-\infty, \infty)$ the set of real numbers.

1. Let X denote a random variable taking values in $\{1, 2, ..., n\}$ where n is a fixed and known integer, $n > 2$. The pdf of X is given by

$$f_\theta(x) = q^{I(x \neq \theta)} c(\theta, q), \theta \in \{1, 2, ..., m\}, m > 2$$

where $q \in (0, 1)$ is a known constant and $c(\theta, q) > 0$ the normalizing constant.

(a) Find the expression for $c(\theta, q)$. **(5)**

(b) Is the statistic X complete when $m = 2$? **(4)**

(c) Find the smallest m such that the statistic X is complete. **(5)**

2. Let $X_1, \ldots, X_n$ be iid sample with Lebesgue density

$$f_\theta(x) = \frac{\alpha\theta^\alpha I(x \geq \theta)}{x^{\alpha+1}}, \ x \in R^1$$

where $\theta \in (0, \infty)$ is the unknown parameter, $\alpha > 0$ is a known constant.

(a) Find the MLE for θ. **(3)**

(b) Show that the MLE is inadmissible under squared error loss. **(7)**

3. Let $X_1, \ldots, X_n$ be iid sample of Bernoulli(p) for unknown parameter $p \in (0, 1)$.

(a) Find the LRT statistic LRT_n for testing $H_0 : p \geq 1/2$ vs. $H_1 : p < 1/2$. **(5)**

(b) Let the true value of p be some $p_0 \neq 1/2$, find a function $T_n(\mathrm{LRT}_n, p_0)$ such that it converges to a non-degenerate distribution as $n \to \infty$ and identify the distribution explicitly. **(7)**

4. Let $X_1, \ldots, X_n$ be a random sample from $f_\theta(x) = f(x - \theta)$ for parameter $\theta \in R^1$ where

$$f(x) = \frac{2}{\Gamma(1/4)} \exp[-x^4], \ x \in R^1,$$

is a density function on R^1. Denote by $\overline{X}_n$ the sample mean. What is the asymptotic distribution of $\sqrt{n}(\overline{X}_n - \theta)$? Prove your assertion and identify the limit explicitly. **(10)**

5. Let $X_1, \ldots, X_n$ be iid sample with Lebesgue density

$$f_\theta(x) = \frac{1}{2} \frac{\theta^7 |x|^6 \exp(-\theta|x|)}{\Gamma(7)}, \ x \in R^1$$

where $\theta \in (0, \infty)$.

(a) Find a UMVUE for θ. **(6)**

(b) Find the Bayes estimator for θ w.r.t. prior $\frac{1}{2}\theta^2 \exp(-\theta) I_{(0.+\infty)}(\theta)$. **(8)**

6. Consider the linear regression model with data $X_i, i = 1, \ldots, n$, which are independent, and

$$X_i \text{ is } N(\alpha + \beta Z_i, \sigma^2),$$

where $\alpha \in R^1$, $\beta \in R^1$ and σ^2 are unknown parameters. For convenience, is is assumed that the known constants $Z_1, \ldots, Z_n$ satisfy $\sum_{i=1}^n Z_i = 0$ and $\sum_{i=1}^n Z_i^2 = 1$.

(a) Show that the MLE of α and β are $\hat{\alpha} = n^{-1} \sum_{i=1}^n X_i$, $\hat{\beta} = \sum_{i=1}^n Z_i X_i$. **(4)**

(b) Consider testing $H_0 : \beta \leq 0$ versus $H_1 : \beta > 0$ with statistic

$$t = \frac{\sqrt{n-2}\hat{\beta}}{\sqrt{\sum_{i=1}^n (X_i - \hat{\alpha} - \hat{\beta} Z_i)^2}}$$

rejecting H_0 if t is larger than the α^{th} upper percentile of the t_{n-2} distribution. Show that this test is unbiased. **(6)**

7. Let F be an absolutely continuous distribution function with density f, G be a σ-finite measure on the Borel line R^1, $0 < a_n \to \infty$, as $n \to \infty$ and Y be a random variable, all satisfying

$$\int\int f(x+s) P(|Y| \geq a_n |s|)\, dG(x)\, ds \to 0.$$

Show that as $n \to \infty$, $E \int \left| F(x + a_n^{-1} Y) - F(x) \right| dG(x) \to 0$. **(16)**

8. Let $\mathcal{F}$ denote the class of Lebesgue density f on R^1 such that $f(x) = f(-x)$. Let $X_1, \cdots, X_n$ be a random sample from $f_\theta(x) = f(x - \theta)$, for some $\theta \in R^1$, let $\overline{X}_n$ denote the sample mean and let $A_n := [\overline{X}_n - 2n^{-1/2}, \overline{X}_n + 2n^{-1/2}]$. Let $P_{f,\theta}$ denote the probability measure when underlying density is $f_\theta(x)$. Show that $\inf_{f \in \mathcal{F}, \theta \in R^1} \lim_{n \to \infty} P_{f,\theta}(\theta \in A_n) = 0$. **(14)**