CLOSED BOOK
NO NOTES
NO ELECTRONICS (INCL NO CALCULATORS)

STT351 Final Exam Fall 2008

Tests of hypotheses.

1. For the of score x = skin thickness, a 95% confidence interval [0.822, 0.847] has been obtained for the population mean μ. It is desired to harness this CI to test

$$H_0: \mu = 0.85 \text{ versus } H_a: \mu < 0.85.$$

1a. Which action, reject the null hypothesis or fail to reject the null hypothesis, is taken based on the test employing this CI?

95% CI = [0.822, 0.847] $\boxed{\text{Reject } H_0}$

$H_0: \mu = 0.85$

The 95% CI falls entirely to the left of $\mu = 0.85$ therefore, reject H_0

1b. Is this test one-sided or two-sided?

One sided because $H_a: \mu < 0.85$

1c. What is the probability α, of type one error, for this test?

$\alpha = P(\text{type I}_{\text{error}}) = \frac{1-.95}{2} = \frac{0.05}{2} = 0.025$

1d. Ideally, what would be the desired probability of rejecting H_0 if μ is 0.84?

desired probability = 1

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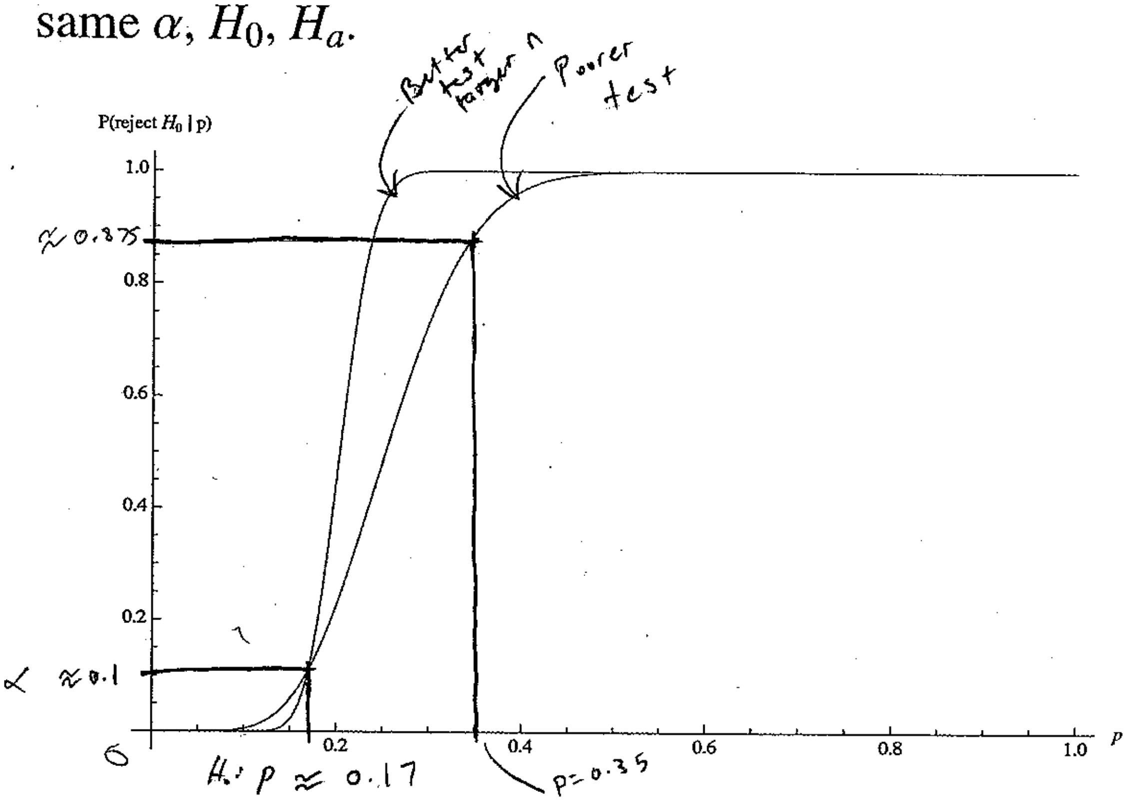
2. Plots of P(reject H_0 | p) are shown for two tests having the same α, H_0, H_a.



2a. Determine H_0. $H_0 : p = 0.17$

2b. Determine H_a. $H_a : p > 0.17$

2c. Determine α. $\alpha \approx 0.1$

2d. For the poorer test, determine $\beta(0.35)$.

$\beta(0.35) = 1 - 0.875 = 0.125$

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(x, y) Data.

$$S_y = \sqrt{\frac{(0-4)^2+(6-4)^2+(6-4)^2}{3-1}} = \sqrt{\frac{16+4+4}{2}} = \sqrt{12}$$

3. Given data

	x	y	x^2	y^2	xy
	0	0	0	0	0
	0	6	0	36	0
	3	6	9	36	18
total	3	12	9	72	18
avg	1	4	3	24	6

$n = 3$

To receive credit for #3 you must CALCULATE your answers from the GENERAL FORMULAS for obtaining slope and then intercept. Pretend that you cannot see the picture, only the averages above.

3a. Sample standard deviation s_x. $= \sqrt{\frac{n}{n-1}}\sqrt{\overline{x^2} - \overline{x}^2}$

$$S_x = \sqrt{\frac{3}{3-1}}\sqrt{3-1^2} = \sqrt{\frac{3}{2}}\sqrt{2}$$

3b. Estimated slope $\hat{\beta}_1$ of regression of y on x.

$$\hat{\beta}_1 = R\frac{\hat{\sigma}_y}{\hat{\sigma}_x} = \frac{\overline{xy} - \overline{x}\,\overline{y}}{\overline{x^2} - \overline{x}^2} = \frac{6-(1)(4)}{3-1^2} = \frac{6-4}{3-1} = \frac{2}{2} = 1$$

3c. Estimated y-intercept $\hat{\beta}_0$ of regression of y on x.

$$\hat{y} = 1x + \hat{\beta}_0 \longrightarrow 6 = 1(3) + \hat{\beta}_0$$

$$\hat{\beta}_0 = 6-3 = 3$$

3d. Fraction of $\sqrt{\overline{y^2} - \overline{y}^2}$ explained by regression of y on x.

$= R^2$

$= \left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{4}}$

$$R = \frac{\overline{xy} - \overline{x}\,\overline{y}}{\sqrt{\overline{x^2}-\overline{x}^2}\sqrt{\overline{y^2}-\overline{y}^2}} = \frac{6-(1)(4)}{\sqrt{3-1^2}\sqrt{24-4^2}} = \frac{6-4}{\sqrt{2}\sqrt{8}} = \frac{2}{\sqrt{2}\,2\sqrt{2}} = \frac{1}{2}$$

3e. 95% t-based CI for μ_y.

$\alpha = 0.05$

$df = 3-1 = 2$

$t_{crit} = 4.303$

95% CI: $\mu_y \pm 4.303\frac{S_y}{\sqrt{3}} \longrightarrow \mu_y \pm 4.303\frac{\sqrt{12}}{\sqrt{3}}$

3f. 95% regression-based t-based CI for μ_y if $\mu_x = 1.5$.

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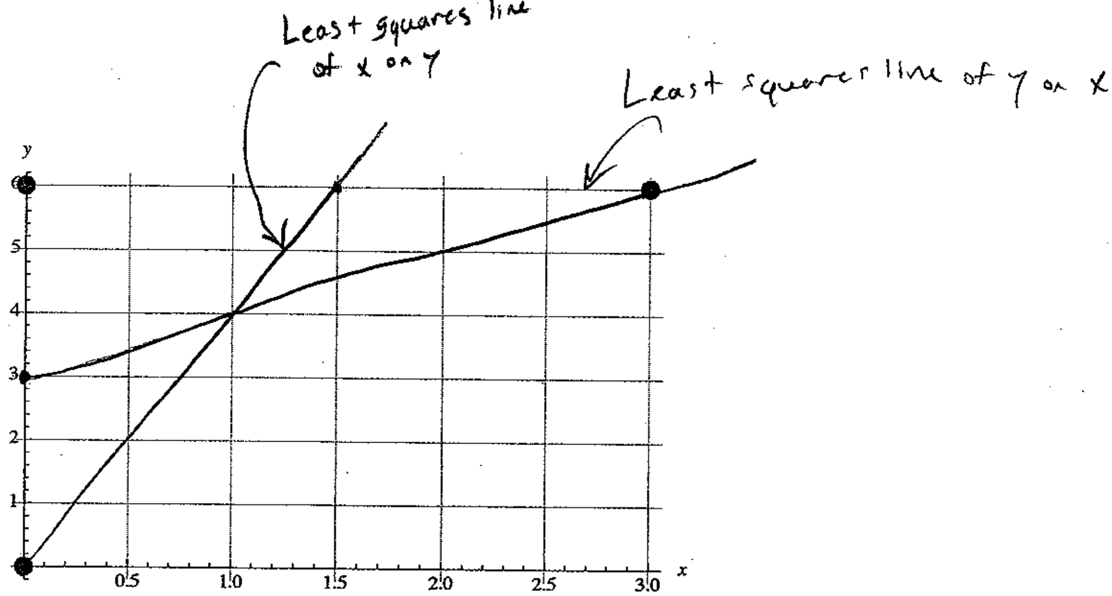
MLR.



4a. Without calculation, plot the least squares line of y on x (per the usual vertical discrepancies). Label it so.

4b. Without calculation, plot the least squares line of x on y (per horizontal discrepancies). Label it so.

4c. Set up the design matrix for a fit of the polynomial model y = $\beta_0 + \beta_1 x + \beta_1 x^2$ to the above data.

$$\text{design matrix xx} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 3 & 9 \end{pmatrix}$$

4d. Determine $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ given

$$\text{Pseudo-Inverse of xx} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{60} & -\frac{1}{60} & \frac{1}{30} \\ -\frac{1}{20} & -\frac{1}{20} & \frac{1}{10} \end{pmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -1/60 & -1/60 & 1/30 \\ -1/20 & -1/20 & 1/10 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix} = \begin{array}{l} \frac{1}{2}(0) + \frac{1}{2}(6) + 0(6) = 0 + 3 + 0 = 3 = \beta_0 \\ \frac{-1}{60}(0) - \frac{1}{60}(6) + \frac{1}{30}(6) = 0 - \frac{1}{10} + \frac{2}{10} = \frac{1}{10} = 0.1 = \beta_1 \\ \frac{-1}{20}(0) - \frac{1}{20}(6) + \frac{1}{10}(6) = 0 - \frac{6}{20} + \frac{6}{10} = \frac{3}{10} = 0.3 = \beta_2 \end{array}$$

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Probability.

5. Box I: {4 R, 3 G, 7 Y}, Box II: {8 R, 2 B, 4Y}. Box I is chosen with probability 0.8, otherwise Box II. Then balls are selected from the chosen box with equal probability and without replacement. $P(I) = 0.8$ $P(II) = 0.2$

5a. P(R1 Y2 | I) = $\left(\frac{4}{14}\right)\left(\frac{7}{13}\right)$

5b. P(R1) = $0.8\left(\frac{4}{14}\right) + 0.2\left(\frac{8}{14}\right)$

5c. P(I | R1) = $\frac{P(I \cap R1)}{P(R1)} = \frac{P(I)P(R1|I)}{P(R1)} = \frac{(0.8)\left(\frac{4}{14}\right)}{0.8\left(\frac{4}{14}\right)+0.2\left(\frac{8}{14}\right)}$

$P(Y1|R1) = 0$ Cannot select yellow 1st if Red is first

5d. P(R1 ⋃ Y1) = $P(R1) + P(Y1) - P(R1 \cap Y1) = P(R1) + P(Y1) - P(R1)P(Y1|R1)$

$= 0.8\left(\frac{4}{14}\right)+0.2\left(\frac{8}{14}\right) + 0.8\left(\frac{7}{14}\right)+0.2\left(\frac{4}{14}\right) - \left[\cancel{0.8\left(\frac{4}{14}\right)+0.2\left(\frac{8}{14}\right)}\right]0$

$= 0.8\left(\frac{4}{14}\right)+0.2\left(\frac{8}{14}\right)+0.8\left(\frac{7}{14}\right)+0.2\left(\frac{4}{14}\right)$

Bonus +1

6. P(OIL) = 0.1, P(+ | OIL) = 0.8, P(+ | OIL^C) = 0.3.

6a. P(OIL −) = $P(OIL)P(-|OIL) = (0.1)(1-0.8) = (0.1)(0.2)$

6b. P(+) = $(0.1)(0.8) + (0.9)(0.3)$

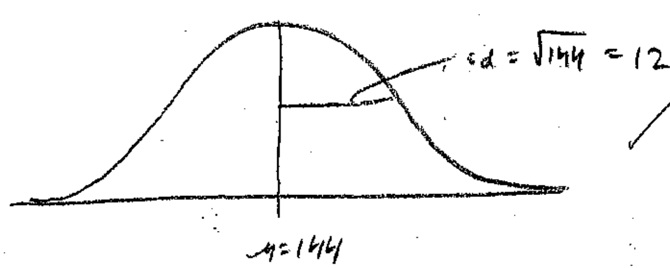
OIL=0.1 — $+|OIL = .8$ — $OIL+ = (.1)(.8)$
OIL=0.1 — $-|OIL = 0.2$ — $OIL- = (.1)(.2)$
$OIL^c = 0.9$ — $+|OIL^c = .3$ — $OIL^c+ = (.9)(.3)$
$OIL^c = 0.9$ — $-|OIL^c = .7$ — $OIL^c- = (.9)(.7)$

6c. P(OIL | +) = $\frac{P(OIL \cap +)}{P(+)} = \frac{(0.1)(0.8)}{(0.1)(0.8)+(0.9)(0.3)}$

6d. Are events OIL, + *independent*? Why?

NO because $P(OIL)P(+) \neq P(OIL)P(+|OIL) \rightarrow (0.1)[(.1)(.8)+(.9)(.3)] \neq 0.1[0.8]$

7. The number X of road service calls in one day is approximately Poisson distributed with mean 144. Sketch the normal approximation of the distribution of X. $\mu = 144$



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Expectation, Var, sd.

8. Random variables X, Y have

$E\,X = 5 \qquad \text{Var}\,X = 3$

$E\,Y = 9 \qquad \text{Var}\,Y = 4$

8a. $E(6X - 7Y + 11 - X) =$ $6\,EX - 7\,EY + 11 - EX = 6(5) - 7(9) + 11 - 5$
$= 30 - 63 + 11 - 5 = -33 + 11 - 5 = -22 - 5 = -27$

8b. If X, Y are independent $E(XY) =$ $(EX)(EY) = (5)(9) = 45$

8c. If X, Y are independent $\text{Var}(6X - 7Y + 11 - X) =$
$= \text{Var}(5x - 7y + \cancel{11}) = 5^2\,\text{Var}\,X - 7^2\,\text{Var}\,Y = 25(3) - 49(4)$
$= 75 - 196 = -121$

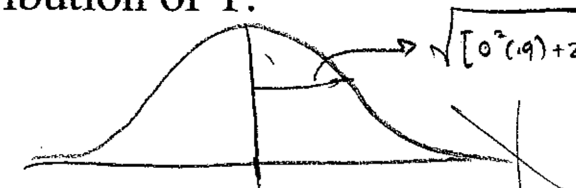
9. The distribution of r.v. W is:

w	p(w)
0	0.9
20	0.1

9a. $E\,W =$ $0(0.9) + 20(0.1)$

9b. $\text{Var}\,W =$ $\left[0^2(0.9) + 20^2(0.1)\right] - \left[0(0.9) + 20(0.1)\right]^2$

9c. Let T denote the ***total*** of 100 independent plays of the lottery whose returns are distributed as W. Sketch the bell-approximation of the distribution of T.



$\sqrt{\left[0^2(.9) + 20^2(.1)\right] - \left[0(.9) + 20(.1)\right]^2}$ $\quad \sqrt{100}$

$100\left[0(.9) + 20(.1)\right]$

9d. $E\left(\frac{1}{1+W}\right) =$ $\frac{1}{1+0}(0.9) + \frac{1}{1+20}(0.1)$
$= 0.9 + \frac{1}{21}(0.1)$

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Continuous models.

10. Lifetime T of an electronic component is exponentially distributed with mean $\mu = \text{E T} = 8$ years. $p(x) = e^{\mu} \frac{\mu^{-x}}{x!}$ $\quad e^{-t/\mu} = e^{-8/8}$

10a. Determine P(T > 8). ~~$= 1 - P(8) = 1 - e^{8}\left(\frac{8^{-8}}{8!}\right)$~~ $= e^{-1}$

?10b. Determine P(T > 20 | T > 12). $= P(T > 8) = e^{-1}$

11. Time T and wear W are jointly distributed with density:

$$f(t, w) = (1/6)(t + w), \quad 0 < t < 1, \quad 0 < w < 3$$
$$= 0 \text{ elsewhere.}$$

11a. Verify that f is a joint probability density.

$\frac{1}{6}\int_0^1\int_0^3 t+w \; dw\,dt = \frac{1}{6}\int_0^1 \left(tw + \frac{w^2}{2}\Big]_0^3 = 3t + \frac{9}{2}\right) dt = \frac{1}{6}\left[\frac{3}{2}t^2 + \frac{9}{2}t\right]_0^1 = \frac{1}{6}\left[\frac{3}{2} + \frac{9}{2}\right]$

$= \frac{1}{6}\left[\frac{12}{2}\right] = 1$

$= 1$ therefore F is a joint probability density

11b. Determine the *marginal* density $f_T(t)$.

$f_T(t) = \int_0^3 f(t,w)\,dw = \frac{1}{6}\int_0^3 t+w \; dw = \frac{1}{6}\left[tw + \frac{1}{2}w^2\right]_0^3 = \frac{1}{6}\left[3t + \frac{9}{2}\right]$

$\boxed{f_T(t) = \frac{1}{2}t + \frac{9}{12}}$

11c. Determine the *conditional* density $f_{W|T}(w \mid t)$. $= \frac{f(t,w)}{f_T(t)}$

$= \frac{f(t,w)}{f_T(t)} = \frac{\frac{1}{6}(t+w)}{\frac{1}{2}t + \frac{9}{12}}$

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