THE OPERATOR ν-STABLE LAWS

TOMASZ J. KOZUBOWSKI, MARK M. MEERSCHAERT, AND HANS-PETER SCHEFFLER

Abstract. Operator stable laws are the limits of operator normed and centered sums of independent, identically distributed random vectors. The operator ν-stable laws are the analogous limit distributions for randomized sums. In this paper we characterize operator ν-stable laws and their domains of attraction. We also discuss several applications, including the scaling limits of continuous time random walks and the special case of operator geometric stable laws, which are useful in finance.

1. Introduction

We introduce a new class of multivariate distributions, which includes stable and operator stable, geometric operator stable, ν-stable and geometric stable laws as special cases. Let (X_i) be a sequence of independent and identically distributed (i.i.d.) random vectors (r.v.'s) in $\mathbb{R}^d$. Consider a random sum

$$S_{N_n} = X_1 + \cdots + X_{N_n}, \tag{1.1}$$

where N_n is an integer-valued r.v., independent of the X_i's, such that $N_n \xrightarrow{p} \infty$ (in probability) while $N_n/n \Rightarrow Z$ (in distribution) as $n \to \infty$, where $Z > 0$ has distribution ν. If there exists a weak limit of

$$A_n \sum_{i=1}^{N_n} (X_i - b_n), \tag{1.2}$$

where A_n is a linear operator on $\mathbb{R}^d$ and $b_n \in \mathbb{R}^d$, then we call it an *operator ν-stable law*. This generalizes the class of ν-stable laws, which arise as limits in (1.2) under scalar

Date: 3 April 2002.

1991 *Mathematics Subject Classification.* 60E07, 60F05.

Key words and phrases. operator stable laws, randomized sums, domains of attraction, regular variation.
Kozubowski was partially supported by NSF grant DMS-0139927.

Meerschaert was partially supported by NSF grants DES-9980484 and DMS-0139927.

normalization $A_n = a_n \in \mathbb{R}$ (see, e.g., [15]). If the sum in (1.2) is deterministic ($N_n = n$), then the limiting distributions are operator stable laws (see, e.g., [22]) and stable laws (see, e.g., [30]) under scalar normalization. If N_n has a geometric distribution with mean n then the limits are operator geometric stable (OGS) laws (see [14]), and reduce to geometric stable (GS) laws under scalar normalization (see, e.g., [8, 20]). In the latter case, we obtain the class of skew Laplace distributions as the limiting laws in (1.2) when the components have finite second moment (see [13, 17]).

Random summation arises naturally in various fields, including biology, economics, insurance mathematics, physics, reliability and queuing theories among others (see, e.g., [5]). Thus, being limiting distributions of random sums, the operator ν-stable laws should have numerous applications in stochastic modeling. Univariate and multivariate geometric stable distributions, and their special cases of skew Laplace laws, compete successfully with stable and other laws in modeling financial asset returns (see, e.g., [13, 16, 17, 19, 21, 27]). Compared with ν-stable distributions, we obtain more flexibility when normalizing the sum in (1.1) by linear operators, as here different components of the limiting random vector are allowed to have different tail behavior.

In this paper we derive basic properties of operator ν-stable laws. After briefly recounting in Section 2 some essential ideas from the theory of operator stable laws and their (generalized) domains of attraction, we formally define the class of operator ν-stable laws in Section 3. Here, we prove some basic results about these distributions, and characterize their domains of attraction. We show that (under certain conditions on ν) there is a one-to-one correspondence between operator stable and operator ν-stable laws, and they have the same domains of attraction. In Section 4 we study the tail behavior of operator ν-stable laws, generalizing results for ν-stable laws of [15]. In particular, we show that if ν has a finite mean, then the operator ν-stable law belongs to the generalized domain of attraction of the underlying operator stable law. In Section 5 we briefly discuss the connection with continuous time random walks. We conclude with Section 6, where we summarize the special case of operator geometric stable laws.

2. Operator stable laws

Suppose $X, X_1, X_2, \ldots$ are independent and identically distributed random vectors on $\mathbb{R}^d$ with common distribution μ and that Y_0 is a random vector whose distribution ω is full, i.e., not supported on any lower dimensional hyperplane. We say that ω is *operator stable* if there exist linear operators A_n on $\mathbb{R}^d$ and nonrandom vectors $b_n \in \mathbb{R}^d$ such that

$$A_n \sum_{i=1}^{n} (X_i - b_n) \Rightarrow Y_0, \tag{2.1}$$

where $\Rightarrow$ denotes weak convergence. In terms of measures, we can rewrite (2.1) as

$$A_n \mu^n * \varepsilon_{s_n} \Rightarrow \omega \tag{2.2}$$

where $A_n\mu(dx) = \mu(A_n^{-1}dx)$ is the probability distribution of A_nX, μ^n is the nth convolution power, and ε_{s_n} is the unit mass at the point $s_n = -nA_nb_n$. In this case, we say that μ (or X) belongs to the *generalized domain of attraction* of ω (or Y_0), and we write $\mu \in \mathrm{GDOA}(\omega)$, or $X \in \mathrm{GDOA}(Y_0)$. Theorem 7.2.1 in [22] shows that the operator stable law ω is infinitely divisible and

$$\omega^t = t^E \omega * \varepsilon_{a_t} \quad \text{for all } t > 0, \tag{2.3}$$

where $a_t \in \mathbb{R}^d$, E is a linear operator called an exponent of ω, $t^E = \exp(E \log t)$, and $\exp(A) = I + A + A^2/2! + A^3/3! + \cdots$ is the usual exponential operator. If (2.1) holds with all $b_n = 0$ we say that μ belongs to the *strict generalized domain of attraction* of ω. If (2.3) holds with all $a_t = 0$ we say that ω is *strictly operator stable.*

The following two results are basic and well known, but we include them here for completeness, since we could not locate a suitable reference.

Lemma 2.1. *If the distribution ω of Y is operator stable and* (2.3) *holds, then the characteristic function $\hat{\omega}(x) = \mathbb{E}[e^{i\langle x,Y\rangle}]$ satisfies*

$$\hat{\omega}(x)^t = \hat{\omega}(t^{E^*}x)e^{i\langle a_t,x\rangle} \quad \textit{for all } t > 0. \tag{2.4}$$

Proof. Given a probability measure μ_1 on $\mathbb{R}^d$, let $\mu_2 = A\mu_1 * \varepsilon_a$ where A is a linear operator on $\mathbb{R}^d$ and $a \in \mathbb{R}^d$. It is well known that $\hat{\mu}_2(x) = \hat{\mu}_1(A^*x)e^{i\langle a,x\rangle}$, see for example

Proposition 1.3.8 in [22]. If ω is operator stable and (2.3) holds, then (2.4) follows easily, using the fact that $(t^E)^* = t^{E^*}$. □

Lemma 2.2. *If μ belongs to the strict generalized domain of attraction of ω, then ω is strictly operator stable.*

Proof. Assume (2.1) holds with all $b_n = 0$ for some full ω. Fix $t > 0$ and let $\mu_n = A_{[tn]}\mu^{[tn]}$ and $B_n = A_n A_{[tn]}^{-1}$. Then (2.1) implies that $\mu_n \Rightarrow \omega$ and since ω is infinitely divisible, Proposition 3.3.7 of [22] yields $B_n\mu_n = A_n\mu^{[tn]} \Rightarrow \omega^t$. Since ω^t is full, Lemma 2.3.7 in [22] shows that (B_n) is relatively compact. Theorem 2.1.8 in [22] shows that if $B_{n'} \to B_t$ along a subsequence then $\omega^t = B_t\omega$. Then it follows along the same lines as the proof of Theorem 5.2.11 of [22] that (2.3) holds with all $a_t = 0$ for some exponent E, so ω is strictly operator stable. □

3. The operator ν-stable laws

In this section we develop basic properties of operator ν-stable laws. Suppose $X, X_1, X_2, \ldots$ are independent and identically distributed random vectors on $\mathbb{R}^d$ with common distribution μ and that Y is a random vector whose distribution λ is full. Let $Z > 0$ be a random variable with probability distribution ν, and suppose that N_n are integer-valued random variables independent of the X_i's and such that $N_n/n \Rightarrow Z$. We say that λ is *operator ν-stable* if there exist linear operators A_n on $\mathbb{R}^d$ and nonrandom vectors $b_n \in \mathbb{R}^d$ such that

$$A_n \sum_{i=1}^{N_n} (X_i - b_n) \Rightarrow Y. \tag{3.1}$$

In this case, we say that μ (or X) belongs to the *generalized domain of ν-attraction* of λ (or Y) and we write $\mu \in \mathrm{GDOA}_\nu(\lambda)$, or $X \in \mathrm{GDOA}_\nu(Y)$. If (3.1) holds with all $b_n = 0$ we say that μ belongs to the *strict generalized domain of ν-attraction* of λ.

Lemma 3.1. *Suppose that $X \in \mathrm{GDOA}(Y_0)$ and (2.1) holds. If N_n are positive integer-valued random variables independent of (X_i) with $N_n \to \infty$ in probability, and if $N_n/k_n \Rightarrow Z$ for some random variable $Z > 0$ with distribution ν and some sequence of positive* *integers (k_n) tending to infinity, then*

$$A_{k_n} \sum_{i=1}^{N_n} (X_i - b_{k_n}) \Rightarrow Y \tag{3.2}$$

where Y has distribution

$$\lambda(dx) = \int_0^\infty \omega(dx)^t \nu(dt) \tag{3.3}$$

and ω is the distribution of Y_0.

Proof. Let $S_r^n = X_{n1} + \cdots + X_{nr}$ where $X_{ni} = A_{k_n}(X_i - b_{k_n})$. Then (2.1) implies

$$S_{k_n}^n = \sum_{i=1}^{k_n} X_{ni} = A_{k_n} \sum_{i=1}^{k_n} (X_i - b_{k_n}) \Rightarrow Y_0,$$

and a transfer theorem (Theorem 1 in Rosiński [29]) yields

$$S_{N_n}^n = \sum_{i=1}^{N_n} X_{ni} = A_{k_n} \sum_{i=1}^{N_n} (X_i - b_{k_n}) \Rightarrow Y,$$

so that (3.2) holds, where Y is a random vector on $\mathbb{R}^d$ whose distribution has characteristic function

$$\hat{\lambda}(x) = \int_0^\infty \hat{\omega}(x)^t \nu(dt) \tag{3.4}$$

so that (3.3) also holds. □

Lemma 3.2. *Suppose that (X_i) are independent, identically distributed random vectors on $\mathbb{R}^d$, M_n are positive integer-valued random variables independent of (X_i) with $M_n \to \infty$ in probability, and*

$$B_n \sum_{i=1}^{M_n} (X_i - a_n) \Rightarrow Y \tag{3.5}$$

for some random vector Y with distribution λ and some linear operators B_n on $\mathbb{R}^d$ and centering constants $a_n \in \mathbb{R}^d$. Then there exists a sequence of positive integers (k_n) tending to infinity such that for any subsequence (n') there exists a further subsequence (n''), a *random variable $Z > 0$ with distribution ν, and a random vector Y_0 with distribution ω such that $M_{n''}/k_{n''} \Rightarrow Z$,*

$$B_{n''} \sum_{i=1}^{k_{n''}} (X_i - a_{n''}) \Rightarrow Y_0, \tag{3.6}$$

and (3.3) *holds.*

Proof. Let $S_r^n = X_{n1} + \cdots + X_{nr}$ where $X_{ni} = B_n(X_i - a_n)$. Then $S_{N_n}^n \Rightarrow Y$ and the result follows immediately from Theorem 3 in Rosiński [29]. □

Under a certain technical condition, the generalized domain of attraction is the same for randomized and non-randomized sums. The condition on the probability measure ν, supported on the positive real numbers, is that

$$\int_0^\infty \omega_1(dx)^t \nu(dt) = \int_0^\infty \omega_2(dx)^t \nu(dt) \quad \text{implies} \quad \omega_1 = \omega_2 \tag{3.7}$$

for any two infinitely divisible probability measures ω_1, ω_2. Following [15], we shall call the operator ν-stable law *regular* if ν satisfies condition (3.7). As remarked by several authors (see [33, 34, 4]) this condition may be true in general, but as Gnedenko and Korolev [4] have put it, "no one has managed either to prove this or to refute it yet" (see [4], p. 108). However, there are cases where the condition holds. As noted by Szasz and Freyer [34], it holds when ν is standard exponential, or more generally when the function $a(z) = \int_0^\infty z^t \nu(dt)$ admits the inverse function on the unit disk. It is also true when ν has finite mean and the ch.f's $\hat{\omega}_1$ and $\hat{\omega}_2$ are analytic or when ν is arithmetic and has finite mean.

Theorem 3.3. *Suppose that (X_i) are independent, identically distributed random vectors on $\mathbb{R}^d$, let $Z > 0$ be a random variable with probability distribution ν, and suppose that N_n are integer-valued random variables independent of (X_i) such that $N_n/n \Rightarrow Z$. If $X \in \mathrm{GDOA}(Y_0)$ and* (2.1) *holds then $X \in \mathrm{GDOA}_\nu(Y)$ and* (3.1) *holds. Conversely, if $X \in \mathrm{GDOA}_\nu(Y)$ and* (3.1) *holds, and if condition* (3.7) *also holds, then $X \in \mathrm{GDOA}(Y_0)$ and* (2.1) *holds. In either case, the distribution ω of Y_0 and the distribution λ of Y are related by* (3.3).

Proof. If $X \in \mathrm{GDOA}(Y_0)$ and (2.1) holds then Lemma 3.1 with $k_n = n$ shows that $X \in \mathrm{GDOA}_\nu(Y)$, (3.1) holds, and (3.3) holds. Conversely, if $X \in \mathrm{GDOA}_\nu(Y)$ and (3.1) holds then for any subsequence (k_n) of the positive integers, Lemma 3.2 with $B_n = A_{k_n}$, $a_n = b_{k_n}$ and $M_n = N_{k_n}$ shows that there is a subsequence (n'') and a random variable $Z_1 > 0$ with distribution ν_1 such that $N_{k_{n''}}/k_{n''} \Rightarrow Z_1$ and

$$A_{k_{n''}} \sum_{i=1}^{k_{n''}} (X_i - b_{k_{n''}}) \Rightarrow Y_0$$

where the distribution ω of Y_0 satisfies (3.3) with ν_1 in place of ν. Since we also have $N_{k_{n''}}/k_{n''} \Rightarrow Z$ by assumption, $\nu = \nu_1$ and so condition (3.7) implies that the limit distribution ω is the same for any sequence (k_n). Since every sequence (k_n) has a further subsequence with this property, it follows that (2.1) holds. □

Remark 3.4. Note that if the operator and operator ν-stable laws related via (3.3) are symmetric, then (3.7) holds since the relevant characteristic functions are real (see [4]); thus in this case the two distributions have the same domains of attraction.

Corollary 3.5. *If Y is operator ν-stable law then under the assumptions of Theorem 3.3 we also have*

$$Y \stackrel{d}{=} Z^E Y_0 + a_Z. \tag{3.8}$$

where $Z > 0$ has distribution ν, the limit Y_0 in (2.1) *is operator stable with exponent E and independent of Z, and (a_t) is from* (2.3).

Proof. Since ω is operator stable, (2.4) implies that the characteristic function of $Z^E Y + a_Z$ is given by

$$\begin{aligned}
\mathbb{E}[e^{i\langle x, Z^E Y_0 + a_Z\rangle}] &= \mathbb{E}[\mathbb{E}(e^{i\langle x, Z^E Y_0 + a_Z\rangle}|Z)] \\
&= \mathbb{E}[\mathbb{E}(e^{i\langle Z^{E^*} x, Y_0\rangle} e^{i\langle x, a_Z\rangle}|Z)] \\
&= \mathbb{E}[\hat{\omega}(Z^{E^*} x) e^{i\langle x, a_Z\rangle}] \\
&= \int_0^\infty \hat{\omega}(t^{E^*} x) e^{i\langle x, a_t\rangle} \nu(dt) \\
&= \int_0^\infty \hat{\omega}(x)^t \nu(dt).
\end{aligned} \tag{3.9}$$

which agrees with (3.3), so that λ is the distribution of $Z^E Y_0 + a_Z$. □

Remark 3.6. If $d = 1$ and Y_0 in (3.8) is strictly stable with some index $0 < \alpha \leq 2$, then $Y \stackrel{d}{=} Z^{1/\alpha} Y_0$ is called a scale mixture of Y_0. Then if $EZ^{\varepsilon/\alpha}$ exists for some $0 < \varepsilon < \alpha$, Proposition 1.3 in [6] implies that Y_0 is uniquely determined by Y, so that condition (3.7) holds. Then Theorem 3.3 shows that $\mathrm{GDOA}_\nu(Y) = \mathrm{GDOA}(Y_0)$ in this case.

Remark 3.7. Lemma 3.1 assumes that the summands X_i are independent of the random variables N_n. There is also a corresponding transfer theorem for dependent sums. Theorem 2.4 in [1] shows that if $\mu \in \mathrm{GDOA}(\omega)$ and (2.2) holds, if N_n are positive integer-valued random variables (not necessarily independent of the X_i) with $N_n \to \infty$ in probability, and if $N_n/k_n \to Z > 0$ *in probability* for some random variable $Z > 0$ with distribution ν and some sequence of positive integers (k_n) tending to infinity, then (3.2) still holds, where the limit distribution λ is given by (3.3).

Remark 3.8. Note that if the random variable ν is infinitely divisible, then the operator ν-stable measure (3.3) is infinitely divisible as well (since ω is infinitely divisible). In this case one can obtain the characteristic triplet of the Lévy - Khintchine representation of $\hat{\lambda}$ from that of $\hat{\omega}$, see [31], Theorem 30.1.

Remark 3.9. Note that in case of classical summation (with deterministic number of terms) the limiting distributions of (1.2) have the stability property: the appropriately normalized sum has the same distribution as each one of the terms. This is usually no longer so in the random summation scheme: The (normalized) random sum may not have the same distribution as each X_i. There are cases, however, when this holds (for example when N_n is geometric). Some authors define N_n-stable laws as distributions admitting this stability property (see, e.g., [3, 4, 8, 9, 10, 11, 25, 26]). In this setting it is usually assumed that $\{N_\theta, \Theta \subset (0,1)\}$ is a family of integer-valued random variables with finite mean $\mathbb{E}N_\theta = 1/\theta$, and the semigroup of generating functions generated by $\{N_\theta\}$ is commutative, in which case one obtains the representation $P_\theta(z) = \phi(\frac{1}{\theta}\phi^{-1}(z))$, where P_θ is the generating function corresponding to N_θ and ϕ is the Laplace transform of ν. Then the N_θ-stable (or operator stable) laws are power mixtures (3.3) where ω is stable (operator stable). Our class of distributions is much larger, as the variable ν is not assumed to have a finite mean and the semigroup of generating functions does not have to be commutative. In fact, in the above setting one always has $\theta N_\theta \Rightarrow \nu$ (see [4]), so that these classes of distributions are special cases of ν-stable laws as defined in this paper. Thus, our Theorem 3.3 extends the results of [11] who showed that the domains of attraction are the same in the above setting under certain additional technical conditions.

4. Regular variation and tail behavior

For a full operator stable law ω with no normal component, Corollary 8.2.12 in [22] shows that (2.2) holds for some $b_n \in \mathbb{R}^d$ if and only if

$$n \cdot A_n \mu \to \phi \tag{4.1}$$

where ϕ is the Lévy measure of the infinitely divisible law ω. The convergence $\mu_n \to \phi$ means that $\mu_n(B) \to \phi(B)$ for any Borel set $B \subset \mathbb{R}^d$ with $\operatorname{dist}(B, \{0\}) > 0$ and $\phi(\partial B) = 0$. Theorem 8.1.5 in [22] shows that we can always choose A_n in (2.2) to vary regularly with index $-E$, meaning that

$$A_{[tn]} A_n^{-1} \to t^{-E} \quad \text{as } n \to \infty$$

for all $t > 0$. Proposition 6.1.10 in [22] shows that the convergence (4.1) is equivalent to regular variation of the probability measure μ (at infinity) with exponent E.

Theorem 4.1. *Suppose that $Z > 0$ is a random variable with distribution ν independent of a strictly operator stable law ω with exponent E and no normal component. If $m = \mathbb{E}Z < \infty$ and λ is given by* (3.3) *then*

$$n \cdot n^{-E} \lambda \to m \cdot \phi \quad \text{as } n \to \infty, \tag{4.2}$$

so that λ varies regularly with exponent E.

Proof. We first show that for any Borel set $B \subset \mathbb{R}^d$ with $\operatorname{dist}(B, \{0\}) > 0$ and $\phi(\partial B) = 0$ we have

$$\sup_{t>0} t \cdot (t^{-E}\omega)(B) = C(B) < \infty. \tag{4.3}$$

In fact if $0 < t \leq 1$, then $t \cdot (t^{-E}\omega)(B) \leq 1 \cdot 1 = 1$. On the other hand if $\sup_{t \geq 1} t \cdot (t^{-E}\omega)(B) = \infty$, then there exists a sequence $t_n \to \infty$ such that $t_n \cdot (t_n^{-E}\omega)(B) \to \infty$ as $n \to \infty$. But since $\omega \in \mathrm{GDOA}(\omega)$ with norming sequence $A_n = n^{-E}$ we have $n \cdot (n^{-E}\omega) \to \phi$ as $n \to \infty$ or equivalently $t \cdot (t^{-E}\omega) \to \phi$ as $t \to \infty$. Since $\phi(B) < \infty$ we get a contradiction.

Since ω is strictly operator stable we know that $\omega^r = (r^E\omega)$ for all $r > 0$. Now for Borel sets B as above write

$$\begin{aligned} n \cdot (n^{-E}\lambda)(B) &= n \int_0^\infty (n^{-E}\omega^r)(B) d\nu(r) \\ &= \int_0^\infty n \cdot \big((n/r)^{-E}\omega\big)(B) d\nu(r) \\ &= \int_0^\infty f_n(r) d\nu(r), \end{aligned}$$

where $f_n(r) = n \cdot ((n/r)^{-E}\omega)(B)$.

Then $\lim_{n\to\infty} f_n(r) = r \cdot \phi(B)$ and $f_n(r) \leq M \cdot r$ for all $n \geq 1$ and $r > 0$. In fact note that

$$f_n(r) = r \cdot (\frac{n}{r}) \cdot \big((\frac{n}{r})^{-E}\omega\big)(B) \to r \cdot \phi(B) \quad \text{as } n \to \infty$$

since $t \cdot (t^{-E}\omega) \to \phi$ as $t \to \infty$. Moreover, by (4.3) we have

$$\begin{aligned} f_n(r) &= r \cdot (\frac{n}{r}) \big((\frac{n}{r})^{-E}\omega\big)(B) \\ &\leq r \cdot \sup_{t>0} t \cdot (t^{-E}\omega)(B) \\ &= C(B) \cdot r. \end{aligned}$$

Since $\int_0^\infty r d\nu(r) < \infty$ we get by dominated convergence that

$$\begin{aligned} \lim_{n\to\infty} n \cdot (n^{-E}\lambda)(B) &= \lim_{n\to\infty} \int_0^\infty f_n(r) d\nu(r) \\ &= \phi(B) \int_0^\infty r d\nu(r) \\ &= m \cdot \phi(B). \end{aligned}$$

Then the Portmanteau Theorem, Proposition 1.2.19 in [22], yields (4.2). □

Corollary 4.2. *Under the assumptions of Theorem 4.1 we have:*

$$n^{-E}\lambda^n * \varepsilon_{a_n} \Rightarrow \omega^m = m^E\omega \quad \textit{as } n \to \infty \tag{4.4}$$

for some sequence (a_n) of shifts, or equivalently

$$(mn)^{-E}\lambda^n * \varepsilon_{b_n} \Rightarrow \omega \quad \text{as } n \to \infty$$

for some sequence (b_n) of shifts.

Proof. Note that $m \cdot \phi$ is the Lévy measure of ω^m. Then it follows from Corollary 8.2.11 of [22] that (4.2) implies (4.4). □

The next results provides a converse to Theorem 4.1. Together these two results show that the operator ν-stable law λ given by (3.3), where ω is strictly operator stable with no normal component, belongs to the generalized domain of normal attraction of ω if and only if ν has a finite mean. Here normal refers to the special form n^{-E} of norming and not to a normal limit.

Theorem 4.3. *Suppose that ω is a full strictly operator stable law with no normal component, $Z > 0$ has distribution ν independent of ω, and λ is the operator ν-stable law given by* (3.3)*. If*

$$n \cdot (n^{-E}\lambda) \to \Phi \quad \text{as } n \to \infty$$

for some Borel measure Φ on $\mathbb{R}^d \setminus \{0\}$, which is finite on sets bounded away from the origin and is not supported on any lower dimensional subspace of $\mathbb{R}^d$, then $\mathbb{E}(Z) < \infty$.

Proof. Let ϕ denote the Lévy measure of ω. Choose a Borel set $B \subset \mathbb{R}^d$ with $\operatorname{dist}(B, \{0\}) > 0$ such that $\phi(B) > 0$ and that B is a Φ-continuity set. Let $f_n(t) = n \cdot ((n/t)^{-E}\omega)(B)$ and note that $\lim_{n\to\infty} f_n(t) = t \cdot \phi(B)$ as in the proof of Theorem 4.1. Then, by Fatou's lemma we obtain

$$\begin{aligned}
\int_0^\infty t d\nu(t) &= \frac{1}{\phi(B)} \int_0^\infty t \cdot \phi(B) d\nu(t) \\
&= \frac{1}{\phi(B)} \int_0^\infty \liminf_{n\to\infty} f_n(t) d\nu(t) \\
&\leq \frac{1}{\phi(B)} \liminf_{n\to\infty} \int_0^\infty f_n(t) d\nu(t) \\
&= \frac{1}{\phi(B)} \liminf_{n\to\infty} n \cdot (n^{-E}\lambda)(B) \\
&= \frac{\Phi(B)}{\phi(B)} < \infty
\end{aligned}$$

showing that $\mathbb{E}(Z) < \infty$. □

Various properties of operator ν-stable limits follow from Theorem 4.1 along with known results about regular variation and generalized domains of attraction. Let $a_1 < \cdots < a_p$ denote the real parts of the eigenvalues of E. Then Theorem 8.2.14 in [22] implies that there exists a function $\rho : \Gamma \to \{a_p^{-1}, \dots, a_1^{-1}\}$ such that for all $\theta \in \Gamma = \mathbb{R}^d \setminus \{0\}$ the radial moments

$$\int |\langle y, \theta\rangle|^\gamma \lambda(dy) \tag{4.5}$$

exist for $0 \le \gamma < \rho(\theta)$ and diverge for $\gamma > \rho(\theta)$. Corollary 8.2.15 in [22] implies that

$$\int \|y\|^\gamma \lambda(dy) \tag{4.6}$$

exists if $\gamma < 1/a_p$ and is infinite if $\gamma > 1/a_p$. Also, Theorem 6.4.15 in [22] gives the power law tail behavior of the truncated moments and tail moments

$$\int_{|\langle x,y\rangle| \le r} |\langle x, y\rangle|^\zeta \lambda(dy) \quad \text{and} \quad \int_{|\langle x,y\rangle| > r} |\langle x, y\rangle|^\eta \lambda(dy) \tag{4.7}$$

in terms of multivariable R-O variation. Roughly speaking, this result says that the tail $P(|\langle Y, \theta\rangle| > r)$ falls off like $r^{-\rho(\theta)}$ as $r \to \infty$.

Remark 4.4. These moments results can be sharpened using the fact that λ belongs to the generalized domain of normal attraction of the operator stable limit in (4.4). Then the results in Meerschaert [24] show that the integral (4.5) also diverges when $\rho = \rho(\theta)$.

5. Applications to continuous time random walks

Continuous time random walks were introduced in [28], and are now used in physics to model a wide variety of phenomena connected with anomalous diffusion [7, 32, 35] and relaxation [12]. Given a sequence of nonnegative i.i.d random variables (J_i), set $T_0 = 0$ and $T_n = \sum_{j=1}^n J_i$. The random variable T_n represents the time of the nth jump. Given a sequence of i.i.d. random vectors (X_j) on $\mathbb{R}^d$, let $S(0) = 0$ and $S(t) = \sum_{i=1}^{[t]} X_j$. The random vector $S(n)$ represents the position of the particle after the nth jump. Finally let $N_t = \max\{n \ge 0 : T_n \le t\}$ denote the number of jumps by time $t \ge 0$ so that

$$W(t) = S(N_t) = X_1 + \cdots + X_{N_t} \tag{5.1}$$

is the position of the particle at time t.

The following results are proven in [23]. Let $\stackrel{f.d.}{\Longrightarrow}$ denote convergence of all finite dimensional distributions of a stochastic process. If J_1 belongs to the strict domain of attraction of some stable law with index $0 < \beta < 1$, then

$$\left\{b(c)^{-1}N_{ct}\right\} \stackrel{f.d.}{\Longrightarrow} \{V_t\} \quad \text{as } c \to \infty \tag{5.2}$$

where $b(tc)/b(c) \to t^{\beta}$ as $c \to \infty$ for all $t > 0$, and $\{V_t\}$ is the inverse process [2]

$$V_\tau = \inf\{t : D(t) > \tau\} \tag{5.3}$$

for the stable subordinator $\{D(t)\}$ of index β, a stationary independent increment process such that $D = D(1)$ has Laplace transform $\mathbb{E}[e^{-sD}] = e^{-s^\beta}$. The random variable $V_t \stackrel{d}{=} (D/t)^{-\beta}$ has moments of all orders. If (X_j) are independent of (J_i) and their common distribution μ belongs to the strict generalized domain of attraction of some full operator stable law ω with exponent E, then

$$\{A_c S(ct)\} \stackrel{f.d.}{\Longrightarrow} \{Y(t)\} \quad \text{as } c \to \infty \tag{5.4}$$

where $A_{tc}A_c^{-1} \to t^{-E}$ as $c \to \infty$ for any $t > 0$, $\{Y(t)\}$ has stationary independent increments, and $Y(1)$ has distribution ω. Finally

$$\{A_{b(c)}W(ct)\} \stackrel{f.d.}{\Longrightarrow} \{Y(V_t)\} \quad \text{as } c \to \infty. \tag{5.5}$$

Since a continuous time random walk is a randomized sum, the results of this paper apply. Fix $t > 0$ and let $c_n = n/t$, $k_n = [b(c_n)]$. Since $b(c) \to \infty$ as $c \to \infty$ we have $k_n \sim b(c_n)$, and then the uniform convergence theorem, Theorem 4.2.1 in [22], implies that $A_{k_n} \sim A_{b(c_n)}$. Then multivariable convergence of types, Theorem 2.3.17 in [22], along with (5.5) yields

$$A_{k_n}(X_1 + \cdots + X_{N_n}) = A_{k_n}A^{-1}_{b(c_n)} \cdot A_{b(c_n)}W(c_n t) \Rightarrow Y(V_t). \tag{5.6}$$

On the other hand, (5.2) implies that $N_n/k_n \Rightarrow V_t$, and hence Corollary 3.5 yields

$$A_{k_n}(X_1 + \cdots + X_{N_n}) \Rightarrow V_t^E Y(1). \tag{5.7}$$

Since $Y(t) \stackrel{d}{=} t^E Y(1)$ and V_t is independent of $\{Y(t)\}$, $Y(V_t) \stackrel{d}{=} V_t^E Y(1)$, so that the limits in (5.6) and (5.7) are identically distributed. In particular, the distribution of $Y(V_t)$

is operator ν-stable. If ω has no normal component, then Theorem 4.1 shows that the distribution of $Y(V_t)$ varies regularly with exponent E, and Corollary 4.2 shows that $Y(V_t)$ belongs to the generalized domain of normal attraction of $Y(t)$. Theorem 4.7 of [23] shows that $Y(V_t)$ is not operator stable, but since $Y(V_t)$ belongs to the generalized domain of normal attraction of the operator stable law ω, the moment and tail behavior of $Y(V_t)$ are quite similar to $Y(t)$.

6. Operator geometric stable laws

Operator geometric stable laws arise as limiting distributions in (3.2) with a geometrically distributed number of terms N_n with mean n in the sum (see [11, 14]). Random sums with a geometric distribution of summands arise quite naturally (see, e.g., [5]), so that their limiting distributions have many potential applications. Several results on operator geometric stable laws appear in [14], along with an application to finance. We give a brief summary here. A geometric operator stable law λ has the representation (3.3) where ν is standard exponential. Then (3.8) also holds with Z standard exponential. A geometric operator stable law λ is infinitely divisible, with characteristic function of the form

$$\hat{\lambda}(t) = (1 - \log \hat{\omega}(t))^{-1}, \tag{6.1}$$

where $\hat{\omega}$ is the characteristic function of an operator stable law ω in (3.3). If ω has no normal component then λ has Lévy measure

$$\int_0^\infty \omega^t(dx) t^{-1} e^{-t} dt.$$

If ω is strictly operator stable with exponent E and (3.3) holds with ν standard exponential and independent of ω, then we say that the λ is strictly operator ν-stable. In this case we also have

$$n^E Y \stackrel{d}{=} Y_1 + \cdots + Y_{N_n} \stackrel{d}{=} Z^E Y_0$$

where $Y, Y_1, Y_2, \ldots$ are i.i.d. with distribution λ, N_n is geometric with mean n, Z is standard exponential, and Y_0 has the strictly operator stable distribution ω corresponding to λ via (6.1).

References

[1] Becker-Kern, P. (2002) Limit theorems with random sample size for generalized domains of semistable attraction, *J. Math. Science* **111**(6), 3820–3829.

[2] Bingham, N. (1971) Limit theorems for occupation times of Markov processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **17**, 1–22.

[3] Bunge, J. (1996). Composition semigroups and random stability, *Ann. Probab.* **24**(3), 1476-1489.

[4] Gnedenko, B.V. and Korolev, V. Yu. (1996). *Random Summation: Limit Theorems and Applications*, CRC Press, Boca Raton.

[5] Kalashnikov, V. (1997). *Geometric Sums: Bounds for Rare Events with Applications*, Kluwer Acad. Publ., Dordrecht.

[6] Keilson, J. and Steutel, F.W. (1974). Mixtures of distributions, moment inequalities and measures of exponentiality and normality, *Ann. Probab.* **2**(1), 112-130.

[7] Klafter, J., Blumen, A. and Shlesinger, M.F. (1987). Stochastic pathways to anomalous diffusion, *Phys. Rev. A* **35** 3081–3085.

[8] Klebanov, L.B., Maniya, G.M., and Melamed, I.A. (1984). A problem of Zolotarev and analogs of infinitely divisible and stable laws in a scheme for summing a random number of random variables, *Theory Probab. Appl.* **29**, 791-794.

[9] Klebanov, L.B., Maniya, G.M., and Melamed, I.A. (1985). Analogs of infinitely divisible and stable laws for sums of a random number of random variables, in: *Proc. Fourth Internat. Conf. Probab. Statist.* **2**, pp. 40-41, Lit. Akad. Nauk, Vilnius (in Russian).

[10] Klebanov, L.B., Maniya, G.M., and Melamed, I.A. (1987). ν_p-strictly stable laws and estimation of their parameters, in: *Stability Problems for Stochastic Models* (Varna, 1985), pp. 23-31; *Lecture Notes in Math.* **1233**, Springer-Verlag, Berlin.

[11] Klebanov, L.B. and Rachev, S.T. (1996). Sums of random number of random variables and their approximations with ν-accompanying infinitely divisible laws, *Serdica Math. J.* **22**, 471-496.

[12] Kotulski, M. and Weron, K. (1996). Random walk approach to relaxation in disordered systems, in: *Proceedings of Athens Conference on Applied Probability and Time Series Analysis, Vol. I*, pp. 379-388; *Lecture Notes in Statist.* **114**, Springer, New York.

[13] Kotz, S., Kozubowski, T.J. and Podgórski, K. (2001). *The Laplace distribution and generalizations: A Revisit with Applications to Communications, Economics, Engineering, and Finance*, Birkhäuser, Boston.

[14] Kozubowski, T.J., Meerschaert, M.M., Panorska, A.K. and Scheffler, H.-P. (2002), Operator geometric stable laws, preprint.

[15] Kozubowski, T.J. and Panorska, A.K. (1998). Weak limits for multivariate random sums, *J. Multivariate Anal.* **67**, 398–413.

[16] Kozubowski, T.J. and Panorska, A.K. (1999). Multivariate geometric stable distributions in financial applications, *Math. Comput. Modelling* **29**, 83-92.

[17] Kozubowski, T.J. and Podgórski, K. (2000). A multivariate and asymmetric generalization of Laplace distribution, *Comput. Statist.* **15**, 531-540.

[18] Kozubowski, T.J. and Podgórski, K. (2001). Asymmetric Laplace laws and modeling financial data, *Math. Comput. Modelling* **34**, 1003-1021.

[19] Kozubowski, T.J. and Rachev, S.T. (1994). The theory of geometric stable distributions and its use in modeling financial data, *European J. Oper. Res.* **74**, 310-324.

[20] Kozubowski, T.J. and Rachev, S.T. (1999). Multivariate geometric stable laws, *J. Comput. Anal. Appl.* **1**(4), 349-385.

[21] Madan, D.B., Carr, P.P. and Chang, E.C. (1998). The variance gamma process and option pricing, *European Finance Rev.* **2**, 79-105.

[22] Meerschaert, M.M. and Scheffler, H.-P. (2001). *Limit Theorems for Sums of Independent Random Vectors*, Wiley, New York.

[23] Meerschaert, M.M. and Scheffler, H.-P. (2000). Limit Theorems for continuous time random walks, preprint.

[24] Meerschaert, M.M. (1990) Moments of random vectors which belong to some domain of normal attraction. *Ann. Probab.* **18**, 870–876.

[25] Melamed, I.A. (1987). Limit theorems in the set up of summation of a random number of independent identically distributed random variables, in: *Stability Problems for Stochastic Models*, (Eds., V.V. Kalashnikov and V.M. Zolotarev), pp. 194-228; *Lecture Notes in Math.* **1412**, Springer-Verlag, Berlin.

[26] Melamed, I.A. (1992). Inequalities for the moments for ν-infinitely divisible laws and the characterization of probability distributions, *J. Soviet Math.* **59**(3), 960-970 (in Russian).

[27] Mittnik, S. and Rachev, S.T. (1993). Modeling asset returns with alternative stable distributions, *Econometric Rev.* **12**(3), 261-330.

[28] Montroll, E.W. and Weiss, G.H. (1965). Random walks on lattices II, *J. Mathematical Phys.* **6**, 167–181.

[29] Rosiński, J. (1976). Weak compactness of laws of random sums of identically distributed random vectors in Banach spaces, *Colloq. Math.* **35**(2), 313–325.

[30] Samorodnitsky, G. and Taqqu, M. (1994). *Stable Non-Gaussian Random Processes*, Chapman & Hall, New York.

[31] Sato, K., (1999). *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge.

[32] Scher, H. and Lax, M. (1973). Stochastic transport in a disordered solid I, Theory. *Phys. Rev. B* **7**(10), 4491–4502.

[33] Szasz, D. (1972). On classes of Limit distributions for sums of a random number of identically distributed independent random variables, *Theory Probab. Appl.* **17**(3), 401-415.

[34] Szasz, D. and Freyer, B. (1971). A problem of summation theory with random indices, Litovsk. Mat. Sb. **11**, 181-187 (in Russian).

[35] Uchaikin, V.V. and and Zolotarev, V.M. (1999). *Chance and Stability: Stable Distributions and Their Applications.*, VSP, Utrecht.

Tomasz J. Kozubowski, Department of Mathematics, University of Nevada, Reno, USA
E-mail address: tkozubow@unr.edu

Mark M. Meerschaert, Department of Mathematics, University of Nevada, Reno, USA
E-mail address: mcubed@unr.edu

Hans-Peter Scheffler, Fachbereich Mathematik, University of Dortmund, 44221 Dortmund, Germany
E-mail address: hps@math.uni-dortmund.de