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Ergodic Properties of Sum– and Max– Stable Stationary Random Fields via Null and Positive Group Actions

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Abstract

We establish characterization results for the ergodicity of symmetric α–stable (SαS) and α–Fréchet max–stable stationary random fields. We first show that the result of Samorodnitsky [35] remains valid in the multiparameter setting, i.e., a stationary SαS ($0 < \alpha < 2$) random field is ergodic (or equivalently, weakly mixing) if and only if it is generated by a null group action. The similarity of the spectral representations for sum– and max–stable random fields yields parallel characterization results in the max–stable setting. By establishing multiparameter versions of Stochastic and Birkhoff Ergodic Theorems, we give a criterion for ergodicity of these random fields which is valid for all dimensions and new even in the one–dimensional case. We also prove the equivalence of ergodicity and weak mixing for the general class of positively dependent random fields.

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1 Introduction

A process is called sum–stable (max–stable, respectively) if so are its finite dimensional distributions and it arises as a limit, under suitable affine transformations, of sums (maxima, respectively) of independent processes. Convenient stochastic integral representations have been developed and actively used to study the structure and properties of sum–stable processes and fields (see, e.g., [36], [22], [24], [23], [19], [33], [34], [35], [31]), [29] and [30]. On the other hand, the seminal works of de Haan [7] and de Haan and Pickands

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[8] as well as the recent developments in [38], [43], and [10] have developed similar tools to represent and handle general classes of max–stable processes.

The ergodic properties of stationary stochastic processes and fields are of fundamental importance and hence well-studied. See, e.g., Maruyama [16], Rosiński and Żak [25, 26], Roy [27, 28] for results on infinite divisible processes and Cambanis *et al.* [2], Podgórski [20], Gross and Robertson [6], Gross [5] for results on stable processes. These culminated in the characterization of Samorodnitsky [35], which shows that the ergodicity of a stationary symmetric stable process is equivalent to the null–recurrence of the underlying nonsingular flow. On the other hand, necessary and sufficient conditions for the ergodicity of max–stable processes have only recently been obtained in [37]. Kabluchko [10] has shown that as in the sum–stable case, one can associate a nonsingular flow to the process and that the characterization of Samorodnitsky [35] remains valid in the max–stable setting, as well. The question whether this is the case for sum–stable and max–stable random fields remained open.

We resolve the above open question by characterizing the ergodicity for both classes of sum– and max–stable stationary random fields. For simplicity of exposition as well as mathematical tractability, we work with symmetric α–stable (SαS), ($0 < \alpha < 2$) sum–stable fields and α–Fréchet max–stable fields ($\alpha > 0$). As in the case of processes, we use minimal representations to relate the random fields to the underlying nonsingular actions following [23] and then show, by establishing multiparameter versions of the Stochastic and Birkhoff Ergodic Theorems for actions (of both $\mathbb{Z}^d$ and $\mathbb{R}^d$), that the sum– or max–stable random fields are ergodic if and only if they are generated by null group actions. These ergodic theorems can be of independent interest in both probability and ergodic theory.

The main obstacle to this work is unavailability of higher–dimensional analogue of the work of Krengel [13], which helps to characterize stationary SαS processes generated by positive and null flows; see [35] for details. In contrast to the case of processes, we use the work of Takahashi [40], to develop tractable and dimension–free criterion for verifying whether a given spectral representation corresponds to a random field generated by a null (or positive) action. These results offer alternative characterizations of ergodicity even in the one–dimensional case.

In this work, we also establish random fields analogues of some of the classical results on stable and max–stable processes. In particular, we show, following closely the work of Podgórski [20], that ergodicity and weak mixing are equivalent for stationary SαS random fields. We also extend a well–known result of Gross [5] and give necessary and sufficient condition for a stationary SαS random field to be weakly mixing and in the process fill in a gap in [5] (see Remark A.3 below). Similarly, in the α–Fréchet case, we obtain a multiparameter version of a characterization of ergodicity given by Stoev [37].

Rosiński and Żak [26] have shown that weak mixing and ergodicity are equivalent for (sum–)infinitely divisible processes. Recently, Kabluchko and Schlather [11] established the equivalence of weak mixing (of all orders) and ergodicity for max–infinitely divisible processes. In Section 5, we obtain a result showing the equivalence of weak mixing and ergodicity for the general class of positively dependent stationary random fields, which includes as particular cases max–infinitely divisible and max–stable random fields.

The paper is organized as follows. In Section 2, we start with some auxiliary results from ergodic theory. In particular, we establish multiparameter versions of the *Stochastic* and *Birkhoff* Ergodic Theorems. In Section 3, we establish the positive–null decomposition for measurable stationary SαS random fields. Section 4 contains the main results on the ergodicity of SαS random fields as well as the extensions of some other classical results on stable processes. The max–stable setting is discussed in Section 5. We conclude with a couple of examples in Section 6. Some technical proofs and auxiliary results are given in the Appendix.

2 Preliminaries on Ergodic Theory

In this section, we start with some preliminaries on ergodic theory used in the rest of the paper. We then establish multiparameter ergodic theorems that can be of independent interest. Throughout this paper, we let $(S, \mathcal{B}, \mu)$ denote a standard Lebesgue space (see Appendix A in [19]). Let ϕ denote a bi–measurable and invertible transformation on S. We say that ϕ is *non–singular*, if $\mu \circ \phi^{-1} \sim \mu$. In this case, one can define the *dual operator* $\widehat{\phi}$, as a mapping from $L^1(S, \mu)$ to $L^1(S, \mu)$ such that:

$$\widehat{\phi} f(s) \equiv \left[\widehat{\phi} f\right](s) := \left(\frac{\mathrm{d}(\mu \circ \phi^{-1})}{\mathrm{d}\mu}\right)(s)\ f \circ \phi^{-1}(s)\,. \tag{2.1}$$

Note that $\widehat{\phi}$ is a positive linear isometry (hence a contraction) on $L^1(S, \mu)$.

Dual operators facilitate the study of the corresponding point mappings. In particular, the existence of a finite ϕ–invariant positive measure $\nu \ll \mu$, $\nu \circ \phi^{-1} = \nu$ is equivalent to $\widehat{\phi}(\mathrm{d}\nu/\mathrm{d}\mu) = \mathrm{d}\nu/\mathrm{d}\mu$, i.e. the existence of a fixed point of the dual operator $\widehat{\phi}$ (see, e.g.,Proposition 1.4.1 in [1]). The characterization results in the next section are in terms of dual operators.

2.1 Group Actions

Let now $G \equiv (G, +)$ be a locally compact, topological abelian group with identity element 0. Equip G with the Borel σ–algebra $\mathcal{A}$.

Definition 2.1. A collection of measurable transformations $\phi_t : S \to S, t \in G$ is called a *group action* of G on S (or a *G–action*), if

(i) $\phi_0(s) = s$ for all $s \in S$,

(ii) $\phi_{v+u}(s) = \phi_u \circ \phi_v(s)$ for all $s \in S, u, v \in G$,

(iii) $(s, u) \mapsto \phi_u(s)$ is measurable w.r.t. the product σ–algebra $\mathcal{A} \otimes \mathcal{B}$.

A G–action $\mathcal{G} = \{\phi_t\}_{t \in G}$ on (S, μ) is called non–singular, if ϕ_t is non–singular for all $t \in G$.

In the sequel, let $\mathcal{G} = \{\phi_u\}_{u \in G}$ denote a non–singular G–action on (S, μ). The existence of a $\mathcal{G}$-invariant finite measure ν, equivalent to μ, is an important problem in ergodic theory. The investigation of this problem was initiated by Neveu [17] and further explored by Krengel [13] and Takahashi [40] among others. In the rest of this section, we present results due essentially to Takahashi [40]. We will see that the invariant finite measures induce a modulo μ unique decomposition of S. This decomposition will play an important role in the characterization of ergodicity for sum and max–stable random fields. The proofs of the results mentioned in this section are given in the Appendix.

Consider the class of finite (positive) $\mathcal{G}$-invariant measures on S, absolutely continuous with respect to μ:

$$\Lambda(\mathcal{G}) := \{\nu \ll \mu : \nu \text{ finite measure on } S, \nu \circ \phi^{-1} = \nu \text{ for all } \phi \in \mathcal{G}\} .$$

For all $\nu \in \Lambda(\mathcal{G})$, let $S_\nu \equiv \mathrm{supp}(\nu) := \{\mathrm{d}\nu/\mathrm{d}\mu > 0\}$ denote the support of ν (mod μ) and set $I(\mathcal{G}) := \{S_\nu : \nu \in \Lambda(\mathcal{G})\}$.

Lemma 2.2. *$I(\mathcal{G})$ has a modulo μ unique maximal element $P_\mathcal{G}$. That is,*

(i) For all $S_\nu \in I(\mathcal{G})$, $\mu(S_\nu \setminus P_\mathcal{G}) = 0$.

(ii) If there exists another $Q_\mathcal{G}$ such that (i) holds, then $P_\mathcal{G} = Q_\mathcal{G} \mod \mu$.

This result suggests the decomposition:

$$S = P_\mathcal{G} \cup N_\mathcal{G}, \tag{2.2}$$

where $N_\mathcal{G} := S \setminus P_\mathcal{G}$. The set $P_\mathcal{G} \equiv S_{\nu_0},\ \nu_0 \in \Lambda(\mathcal{G})$ is the largest (mod μ) set, where one can have a finite $\mathcal{G}$-invariant measure ν_0, equivalent to $\mu|_{P_\mathcal{G}}$.

Consequently, there are no finite measures supported on $N_{\mathcal{G}}$, invariant w.r.t. $\mathcal{G}$ and absolutely continuous w.r.t. μ.

The next theorem provides a convenient characterization of the decomposition (2.2).

Theorem 2.3. *Consider any* $f \in L^1(S,\mu)$, $f > 0$. *Let* $P_{\mathcal{G}}$ *denote the unique maximal element of* $I(\mathcal{G})$ *and set* $N_{\mathcal{G}} := S \setminus P_{\mathcal{G}}$. *We have:*

(i) The sets $P_{\mathcal{G}}$ *and* $N_{\mathcal{G}}$ *are invariant w.r.t.* $\mathcal{G}$, *i.e., for all* $\phi \in \mathcal{G}$, *we have*

$$\mu(\phi^{-1}(P_{\mathcal{G}}) \triangle P_{\mathcal{G}}) = 0 \quad \textit{and} \quad \mu(\phi^{-1}(N_{\mathcal{G}}) \triangle N_{\mathcal{G}}) = 0\,.$$

(ii) Restricted to $P_{\mathcal{G}}$,

$$\sum_{n=1}^{\infty} \widehat{\phi}_{u_n} f(s) = \infty\,, \mu\text{-a.e. for all } \{\phi_{u_n}\}_{n\in\mathbb{N}} \subset \mathcal{G}\,. \tag{2.3}$$

(iii) Restricted to $N_{\mathcal{G}}$,

$$\sum_{n=1}^{\infty} \widehat{\phi}_{u_n} f(s) < \infty\,, \mu\text{-a.e. for some } \{\phi_{u_n}\}_{n\in\mathbb{N}} \subset \mathcal{G}\,. \tag{2.4}$$

The decomposition in (2.2) is unique (mod μ). It is referred to as the *positive–null* decomposition w.r.t. $\mathcal{G}$. The sets $P_{\mathcal{G}}$ and $N_{\mathcal{G}}$ are referred to as the *positive* and *null* parts of S w.r.t. $\mathcal{G}$, respectively. If $\mu(N_{\mathcal{G}}) = 0$ ($\mu(P_{\mathcal{G}}) = 0$, resp.), then $\mathcal{G}$ is said to be *positive* (*null*, resp.) G–action.

The next result provides an equivalent characterization of (2.2), based on the notion of weakly wandering set. Recall that a measurable set $W \subset S$ is *weakly wandering*, w.r.t. $\mathcal{G}$, if there exists $\{\phi_{t_n}\}_{n\in\mathbb{N}} \subset \mathcal{G}$ such that $\mu(\phi_{t_n}^{-1}(W) \cap \phi_{t_m}^{-1}(W)) = 0$ for all $n \neq m$.

Theorem 2.4. *Under the assumptions of Theorem 2.3, we have:*

(i) The positive part $P_{\mathcal{G}}$ *has no weakly wandering set of positive measure.*

(ii) The null part $N_{\mathcal{G}}$ *is a union of weakly wandering sets w.r.t.* $\mathcal{G}$.

We conclude this section with some remarks as follows.

Remark 2.5. Theorems 2.3 and 2.4 follow from Theorems 1 and 2 in [40], which are valid for the general case when G is an *amenable* semigroup. See e.g. [32] for more on amenable groups.

Remark 2.6. In the case when $G = \mathbb{Z}$, one can further express $N_{\mathcal{G}}$ via an *exhaustively weakly wandering set* W. Namely, $N_{\mathcal{G}} = \bigcup_{n=1}^{\infty} \phi_{t_n}^{-1}(W)$, where $\mu\left(\phi_{t_n}^{-1}(W) \cap \phi_{t_m}^{-1}(W)\right) = 0$, $\forall n \neq m$ (see, e.g., [9]). We don't know whether this is the case for general G.

Remark 2.7. In the one–dimensional case, Krengel [13] (for $G = \mathbb{Z}$) and Samorodnitsky [35] (for $G = \mathbb{R}$) establish alternative characterizations of the decomposition (2.2). These results involve certain integral tests, which we were unable to extend to multiple dimensions. Takahashi's characterizations, employed in Theorem 2.3, are valid for all dimensions.

2.2 Multiparameter Ergodic Theorems

In the rest of the paper, we focus on $\mathbb{T}^d$–actions, where $\mathbb{T}$ stands for either the integers $\mathbb{Z}$ or the reals $\mathbb{R}$. We equip $\mathbb{T}^d$ with the measure $\lambda \equiv \lambda_{\mathbb{T}^d}$, which is either the counting (if $\mathbb{T} = \mathbb{Z}$) or the Lebesgue (if $\mathbb{T} = \mathbb{R}$) measure.

In this section, we establish multiparameter versions of the *stochastic ergodic theorem* and *Birkhoff theorem* for the case of $\mathbb{T}^d$–actions. These two results provide important tools for studying the ergodicity of sum– and max–stable random fields.

Introduce the *average functional* A_T, defined for all locally integrable $h : \mathbb{T}^d \to \mathbb{R}$:

$$A_T h \equiv A_{\mathbb{T}^d,T} h := \frac{1}{C(T)} \int_{B(T)} h(t)\lambda(\mathrm{d}t)\,,$$

with $B(T) \equiv B_{\mathbb{T}^d}(T) := (-T, T]^d \cap \mathbb{T}^d$ and $C(T) \equiv C_{\mathbb{T}^d}(T) := (2T)^d$.

Consider now a collection of functions $\{f_t\}_{t\in\mathbb{T}^d} \subset L^1(S, \mu)$ such that $(t, s) \mapsto f(t, s) \equiv f_t(s)$ is jointly measurable when $\mathbb{T} \equiv \mathbb{R}$. Then, one can define the *average operator*:

$$(A_T f)(s) := \frac{1}{C(T)} \int_{B(T)} f_t(s)\lambda(\mathrm{d}t)\,. \tag{2.5}$$

Let $\|\cdot\|$ denote the L^1 norm. If $t \mapsto \|f_t\|$ is locally integrable (i.e. integrable on finite intervals) then Fubini's theorem implies that $A_T f \in L^1(S, \mu)$, for all $T > 0$. Recall also that a sequence of measurable functions $\{f_n\}_{n\in\mathbb{N}} \subset L^\alpha(S, \mu)$ converges *stochastically* (or locally in measure) to $g \in L^\alpha(S, \mu)$, in short $f_n \stackrel{\mu}{\to} g$, as $n \to \infty$, if

$$\lim_{n\to\infty} \mu\Big(\{s : |f_n(s)-g(s)\big| > \epsilon\} \cap B\Big) = 0 \text{ for all } \epsilon > 0, B \in \mathcal{B} \text{ with } \mu(B) < \infty\,. \tag{2.6}$$

Remark 2.8. By Theorem A.1 in [12], there exists a strictly positive measurable function $(t,s) \mapsto w(t,s)$, such that for all $t \in \mathbb{T}^d$,

$$w(t,s) = \frac{\mathrm{d}\mu \circ \phi_t}{d\mu}(s)$$

for μ–almost all s, and for all $t, h \in \mathbb{T}^d$ and for all $s \in S$

$$w(t+h,s) = w(h,s)w(t,\phi_h(s)). \tag{2.7}$$

From now on, we shall use $w(t,s)$ as the version of the Radon-Nikodym derivative $\frac{\mathrm{d}\mu\circ\phi_t}{d\mu}(s)$.

Theorem 2.9 (Multiparameter Stochastic Ergodic Theorem for Nonsingular Actions)**.** *Let $\{\phi_t\}_{t\in\mathbb{T}^d}$ be a nonsingular $\mathbb{T}^d$–action on the measure space (S,μ). Let $f_0 \in L^1(S,\mu)$ and define $f(t,s) \equiv (\widehat{\phi}_{-t}f_0)(s) := w(t,s)f_0 \circ \phi_t(s)$. Then, there exists $\widetilde{f} \in L^1(S,\mu)$, such that*

$$A_T f \equiv \frac{1}{C(T)} \int_{B(T)} f(t,\cdot)\lambda(\mathrm{d}t) \overset{\mu}{\to} \widetilde{f} \ \textit{as}\ T \to \infty\,. \tag{2.8}$$

Moreover, $\widetilde{f}$ is invariant w.r.t. $\widehat{\mathcal{G}}$, i.e., $\widehat{\phi}_t\widetilde{f} = \widetilde{f}$ for all $t \in \mathbb{T}^d$.

Proof. Suppose first that $\mathbb{T} = \mathbb{Z}$. The existence of $\widetilde{f}$ follows from Krengel's stochastic ergodic theorem (Theorem 6.3.10 in [14]). To see that $\widetilde{f}$ is L^1–integrable, pick a subsequence T_n such that $A_{T_n}f \to \widetilde{f}$, μ-a.e., as $n \to \infty$. By Fatou's Lemma,

$$\|\widetilde{f}\| = \|\lim_{n\to\infty} A_{T_n}f\| \le \liminf_{n\to\infty} \|A_{T_n}f\| \le \|f_0\| < \infty\,,$$

which implies $\widetilde{f} \in L^1(S,\mu)$. Here we used the fact that

$$\int_S |A_T f| d\mu \le A_T \int_S |\widehat{\phi}_{-t}f_0| d\mu = A_T\|f_0\| = \|f_0\|.$$

We now prove that $\widetilde{f}$ is invariant w.r.t. $\widehat{\mathcal{G}}$. Fix $\tau \in \mathbb{T}^d$ and let $T_n \to \infty$ be such that $g_n := A_{T_n}f \to \widetilde{f}$, μ-a.e., as $n \to \infty$. Then, since ϕ_τ is non–singular,

$$\begin{aligned}(\widehat{\phi}_{-\tau}g_n)(s) &\equiv \frac{\mathrm{d}(\mu\circ\phi_\tau)}{\mathrm{d}\mu}(s) g_n \circ \phi_\tau(s) \\ &\longrightarrow \frac{\mathrm{d}(\mu\circ\phi_\tau)}{\mathrm{d}\mu}(s)\widetilde{f}\circ\phi_\tau(s) \equiv (\widehat{\phi}_{-\tau}\widetilde{f})(s), \quad \mu\text{-a.e.,}\end{aligned} \tag{2.9}$$

as $n \to \infty$. On the other hand, since $f(t, \phi_\tau(s)) = w(t, \phi_\tau(s)) f_0 \circ \phi_{t+\tau}(s)$, we obtain by (2.7) and Fubini's Theorem that

$$(\widehat{\phi}_{-\tau} g_n)(s) = \frac{1}{C(T_n)} \int_{B(T_n)} w(\tau + t, s) f_0(\phi_{\tau+t}(s)) \lambda(\mathrm{d}t)$$
$$= \frac{1}{C(T_n)} \int_{B(T_n)+\tau} f(t, s) \lambda(\mathrm{d}t), \ \mu\text{-a.e.}$$

Therefore, by performing cancelations and applying Fubini's theorem, we get:

$$\|\widehat{\phi}_{-\tau} g_n - g_n\| \leq \frac{\lambda((B(T_n) + \tau)\Delta B(T_n))}{C(T_n)} \|f_0\|,$$

where $D\Delta E = (D \setminus E) \cup (E \setminus D)$ is the symmetric difference of sets. The last term vanishes, as $n \to \infty$, since $\tau \in \mathbb{Z}^d$ is fixed. This implies that $\widehat{\phi}_{-\tau} g_n \xrightarrow{\mu} \widetilde{f}$, as $n \to \infty$, which in view of (2.9), yields $\widehat{\phi}_{-\tau}\widetilde{f} = \widetilde{f}$, μ-a.e. This, since $\tau \in \mathbb{Z}^d$ was arbitrary, establishes the desired invariance of the limit $\widetilde{f}$.

Suppose now that $\mathbb{T} = \mathbb{R}$. Since we will use the result proved for $\mathbb{T} = \mathbb{Z}$, we explicitly write $A_{\mathbb{Z}^d,T}$ and $A_{\mathbb{R}^d,T}$ to distinguish between the discrete and integral average operators, respectively. In view of part *(i)*, for all $\delta > 0$, we have

$$A_{\mathbb{R}^d,n\delta} f_0 \equiv \frac{1}{(2n\delta)^d} \int_{(-n\delta,n\delta]^d} \widehat{\phi}_{-\tau} f \mathrm{d}\tau$$
$$= \frac{1}{(2n)^d} \sum_{t \in (-n,n]^d \cap \mathbb{Z}^d} \widehat{\phi}_{-\delta t} g^{(\delta)} \equiv A_{\mathbb{Z}^d,n} g^{(\delta)}, \quad (2.10)$$

where

$$g^{(\delta)}(s) := \frac{1}{\delta^d} \int_{(-\delta,0]^d} (\widehat{\phi}_{-\tau} f_0)(s) \mathrm{d}\tau \ \in L^1(S, \mu) \,.$$

As already shown for the case $\mathbb{T} = \mathbb{Z}$, the right–hand side of (2.10) converges stochastically, as $n \to \infty$, to $\widetilde{g}^{(\delta)} \in L^1(S, \mu)$, where $\widetilde{g}^{(\delta)}$ is $\widehat{\phi}_{-\delta t}$–invariant, for all $t \in \mathbb{Z}^d$.

On the other hand, for all $\delta > 0$, one can show $\|A_{\mathbb{R}^d,T} f - A_{\mathbb{R}^d,\lfloor \frac{T}{\delta} \rfloor \delta} f\| \to 0$ as $T \to \infty$. Therefore, we have that

$$A_{\mathbb{R}^d,T} f \xrightarrow{\mu} \widetilde{g}^{(\delta)} \quad \text{as} \quad T \to \infty \,,$$

which shows in particular that $\widetilde{g}^{(\delta)} = \widetilde{g} \in L^1(S, \mu)$ must be independent of $\delta > 0$. Since $\widetilde{g}$ is invariant w.r.t. $\widehat{\phi}_{\delta t}$ for all $\delta > 0$ and $t \in \mathbb{Z}^d$, it follows that $\widetilde{g}$ is $\widehat{\mathcal{G}}$–invariant. □

Theorem 2.10 (Multiparameter Birkhoff Theorem)**.** *Assume the conditions of Theorem 2.9 hold. Suppose, moreover, that the action* $\{\phi_t\}_{t\in\mathbb{T}^d}$ *is measure preserving on* (S,μ)*, and that* μ *is a probability measure. Then,*

$$A_T f \to \widetilde{f} := \mathbb{E}_\mu(f|\mathcal{I}) \textit{ almost surely and in } L^1\,,$$

where $\mathcal{I}$ *is the* σ*–algebra of all* $\mathcal{G}$*–invariant measurable sets.*

Proof. Suppose first that $\mathbb{T} = \mathbb{Z}$. The almost sure convergence and the structure of the limit $\widetilde{f}$ follow from Tempel'man's Theorem (Theorem 6.2.8 in [14] p.205). The L^1–convergence is clear when f_0 is bounded. Suppose now that $f_0 \in L^1(S,\mu)$. Consider the sequence $A_T f,\ T \in \mathbb{N}$. For all $\epsilon > 0$ there exists a bounded $f_0^{(\epsilon)} \in L^\infty(S,\mu)$ such that $\|f_0 - f_0^{(\epsilon)}\| < \epsilon/3$. Then, by the triangle inequality and the fact that A_T is a linear contraction, we get

$$\begin{aligned}\|A_{T_1}f - A_{T_2}f\| &\le \|A_{T_1}f^{(\epsilon)} - A_{T_2}f^{(\epsilon)}\| + 2\|f_0 - f_0^{(\epsilon)}\|\\ &\le \|A_{T_1}f^{(\epsilon)} - A_{T_2}f^{(\epsilon)}\| + 2\epsilon/3 < \epsilon,\end{aligned}$$

for all sufficiently large T_1 and T_2. This is because $A_T f^{(\epsilon)}$ converges in L^1. We have thus shown that $A_T f,\ T \in \mathbb{N}$ is a Cauchy sequence in the Banach space $L^1(S,\mu)$, and hence it has a limit, which is necessarily $\widetilde{f}$.

Let now $\mathbb{T} = \mathbb{R}$. First, by a discretization argument as in the proof of Theorem 2.9, we can show $A_T f \to \widetilde{f}$ almost surely, for all $f_0 \in L^1(S,\mu)$. The L^1–convergence can be established as in the proof in the discrete case. ☐

3 Stationary Sum–Stable Random Fields

Here, we investigate the structure of stationary sum–stable random fields $\mathbf{X} = \{X_t\}_{t\in\mathbb{T}^d}$, indexed by $\mathbb{T}^d$. We focus on the general class of measurable symmetric α–stable (SαS) random fields with $0 < \alpha < 2$. These fields have convenient stochastic integral representations:

$$\{X_t\}_{t\in\mathbb{T}^d} \overset{\mathrm{d}}{=} \left\{\int_S f_t(s) M_\alpha(\mathrm{d}s)\right\}_{t\in\mathbb{T}^d}\,, \tag{3.1}$$

where $\{f_t\}_{t\in\mathbb{T}^d} \subset L^\alpha(S,\mu)$, and the integral is with respect to an independently scattered *SαS random measure* M_α on S with control measure μ (see Chapters 3 and 13 in [36], for more details). Without loss of generality, we shall also assume that $\{f_t\}_{t\in\mathbb{T}^d}$ has *full support* in $L^\alpha(S,\mu)$. Namely, there is no $B \in \mathcal{B}$ with $\mu(B) > 0$, such that $\int_B |f_t(s)|^\alpha \mu(\mathrm{d}s) = 0$, for all $t \in \mathbb{T}^d$.

The measurability of $\mathbf{X}$ allows us to choose (S,μ) in (3.1) to be a standard Lebesgue space and the functions $(t,s) \mapsto f_t(s)$ to be jointly measurable (see, e.g., Proposition 11.1.1 and Theorem 13.2.1 in [36]). Relation (3.1) will be also referred to as a *spectral representation* of the random field $\mathbf{X}$.

It is known from Rosiński [22] and [23] that when $\mathbf{X}$ is stationary, there exists a *minimal* spectral representation (3.1) with

$$f_t(s) = c_t(s)\Big(\frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}(s)\Big)^{1/\alpha} f_0\circ\phi_t(s)\,, t\in\mathbb{T}^d\,, \tag{3.2}$$

where $f_0 \in L^\alpha(S,\mu)$, $\{\phi_t\}_{t\in\mathbb{T}^d}$ is a non–singular $\mathbb{T}^d$–action on $(S,\mathcal{B},\mu)$, and $\{c_t\}_{t\in\mathbb{T}^d}$ is a cocycle for $\{\phi_t\}_{t\in\mathbb{T}^d}$ taking values in $\{-1,1\}$. Namely, $(t,s)\mapsto c_t(s)\in\{-1,1\}$ is a measurable map, such that for all $u,v\in\mathbb{T}^d$,

$$c_{u+v}(s) = c_v(s)c_u\left(\phi_v(s)\right)\,, \mu\text{-a.e. } \ s\in S\,.$$

The representation (3.1) is minimal, if the ratio σ–algebra $\sigma(f_t/f_\tau : t,\tau\in\mathbb{T}^d)$ is equivalent to $\mathcal{B}$ (see Definition 2.1 in [22]). We say that a random field $\{X_t\}_{t\in\mathbb{T}^d}$ with the minimal representation (3.1) and (3.2) is *generated by the* $\mathbb{T}^d$*–action* $\{\phi_t\}_{t\in\mathbb{T}^d}$ *and the cocycle* $\{c_t\}_{t\in\mathbb{T}^d}$. In this case, we also say $\{X_t\}_{t\in\mathbb{T}^d}$ has an action representation $(f_0,\mathcal{G}\equiv\{\phi_t\}_{t\in\mathbb{T}^d},\{c_t\}_{t\in\mathbb{T}^d})$.

It turns out, moreover, the action $\{\phi_t\}_{t\in\mathbb{T}^d}$ is determined by the distribution of $\{X_t\}_{t\in\mathbb{T}^d}$, up to the equivalence relationship of $\mathbb{T}^d$–actions (see Theorem 3.6 in [22]). Thus, structural results for the $\mathbb{T}^d$–actions imply important structural results for the corresponding SαS fields. In particular, by using Theorem 2.3, we obtain the following result:

Theorem 3.1. *Let* $\{X_t\}_{t\in\mathbb{T}^d}$ *be a measurable stationary SαS random field with spectral representation* (3.1). *We suppose that* $(S,\mathcal{B},\mu)$ *is a standard Lebesgue space and the spectral representation* $\{f_t(s)\}_{t\in\mathbb{T}^d}$ *is measurable. Assume, in addition that*

$$g(s) := \int_{T_0} a_\tau |f_\tau(s)|^\alpha \lambda(\mathrm{d}\tau) \ \textit{is } L^1\textit{–integrable and } \mathrm{supp}(g) = S\,, \tag{3.3}$$

for some $T_0\in\mathcal{B}_{\mathbb{T}^d}$ *and* $a_\tau>0$, $\forall\tau\in T_0$. *Then,*

(i) $\{X_t\}_{t\in\mathbb{T}^d}$ *is generated by a positive* $\mathbb{T}^d$*–action if and only if*

$$\sum_{n=1}^\infty \int_{T_0} a_\tau |f_{\tau+t_n}(s)|^\alpha \lambda(\mathrm{d}\tau) = \infty\,, \mu\textit{-a.e.}\,, \ \textit{for all } \{t_n\}_{n\in\mathbb{N}}\subset\mathbb{T}^d\,. \tag{3.4}$$

(ii) $\{X_t\}_{t\in\mathbb{Z}^d}$ *is generated by a null* $\mathbb{T}^d$*–action if and only if*

$$\sum_{n=1}^\infty \int_{T_0} a_\tau |f_{\tau+t_n}(s)|^\alpha \lambda(\mathrm{d}\tau) < \infty\,, \mu\textit{-a.e.}\,, \ \textit{for some } \{t_n\}_{n\in\mathbb{N}}\subset\mathbb{T}^d\,. \tag{3.5}$$

In particular, the classes of stationary SαS random fields generated by positive and null $\mathbb{T}^d$–actions are disjoint.

Remark 3.2. One can always choose $\{a_\tau\}_{\tau\in T_0}$ such that (3.3) holds, if the spectral functions $\{f_t\}_{t\in\mathbb{T}^d}$ have full support in $L^\alpha(S,\mu)$.

Proof of Theorem 3.1. Suppose that the spectral functions $\{f_t\}_{t\in\mathbb{T}^d}$ are minimal and have the form (3.2). Observe that, for all $t,\tau\in\mathbb{T}^d$, we have

$$|f_{\tau+t}(s)|^\alpha = \frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}(s)\frac{\mathrm{d}(\mu\circ\phi_\tau)}{\mathrm{d}\mu}\circ\phi_t(s)|f_0\circ\phi_\tau\circ\phi_t(s)|^\alpha, \quad \mu\text{-a.e.}$$

Since both the l.h.s. and the r.h.s. are measurable in (τ,s), by Fubini's theorem,

$$\int_{T_0} a_\tau |f_{\tau+t}(s)|^\alpha \lambda(\mathrm{d}\tau) = \frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}(s)\int_{T_0} a_\tau|f_\tau\circ\phi_t(s)|^\alpha\lambda(\mathrm{d}\tau) = (\widehat{\phi}_{-t}g)(s),\ \mu\text{-a.e.},$$

where the last relation follows from (2.1). Therefore,

$$\sum_{n=1}^\infty \int_{T_0} a_\tau|f_{\tau+t_n}(s)|^\alpha\lambda(\mathrm{d}\tau) = \sum_{n=1}^\infty \widehat{\phi}_{-t_n}g, \quad \mu\text{-a.e.},\ \forall\{t_n\}_{n\in\mathbb{N}}\subset\mathbb{T}^d.$$

Hence Theorem 2.3 *(ii)* and *(iii)*, applied to the strictly positive function $g\in L^1(S,\mu)$, implies the statements of parts *(i)* and *(ii)*, respectively.

Using Remark 2.5 in [22] and a standard Fubini argument, it can be shown that a test function (3.3) in the general case corresponds to one in the situation when the integral representation $\{f_t\}_{t\in\mathbb{T}^d}$ of the field is of the form (3.2). Therefore, an argument parallel to the proof of Corollary 4.2 in [22] shows that the tests described in this theorem can be applied to any full support integral representation, not necessarily of the form (3.2). This completes the proof. □

Remark 3.3. Theorem 3.1 provides dimension–free characterizations of the fields generated by positive or null $\mathbb{T}^d$–actions. The seminal work of Samorodnitsky [35] gives alternative characterizations in the case $d = 1$ (see also Remark 2.7).

The above characterization motivates the following decomposition of an arbitrary measurable stationary SαS random field $\mathbf{X} = \{X_t\}_{t\in\mathbb{T}^d}$. Without loss of generality, let $\mathbf{X}$ have a representation $(f_0, \mathcal{G} \equiv \{\phi_t\}_{t\in\mathbb{T}^d}, \{c_t\}_{t\in\mathbb{T}^d})$ as in (3.1) and (3.2). Then, by Lemma 2.2 $S = P_{\mathcal{G}} \cup N_{\mathcal{G}}$ and one can write:

$$\{X_t\}_{t\in\mathbb{T}^d} \stackrel{\mathrm{d}}{=} \left\{X_t^P + X_t^N\right\}_{t\in\mathbb{T}^d} \tag{3.6}$$

with

$$X_t^P = \int_{P_{\mathcal{G}}} f_t(s) M_\alpha(\mathrm{d}s) \ \text{ and } \ X_t^N = \int_{N_{\mathcal{G}}} f_t(s) M_\alpha(\mathrm{d}s) \text{ for all } t \in \mathbb{T}^d\,.$$

Corollary 3.4. *(i) The decomposition* (3.6) *is unique in law. That is, if there is another representation* $(f_0^{(2)}, \mathcal{G}^{(2)} \equiv \{\phi_t^{(2)}\}_{t\in\mathbb{T}^d}, \{c_t^{(2)}\}_{t\in\mathbb{T}^d})$ *satisfying* (3.1) *and* (3.2)*, then*

$$\{X_t^P\} \stackrel{\mathrm{d}}{=} \left\{\int_{P_{\mathcal{G}^{(2)}}} f_t^{(2)} \mathrm{d}M_\alpha\right\} \quad \textit{and} \quad \{X_t^N\} \stackrel{\mathrm{d}}{=} \left\{\int_{N_{\mathcal{G}^{(2)}}} f_t^{(2)} \mathrm{d}M_\alpha\right\}.$$

(ii) The components $\mathbf{X}^P = \{X_t^P\}_{t\in\mathbb{T}^d}$ *and* $\mathbf{X}^N = \{X_t^N\}_{t\in\mathbb{T}^d}$ *are independent,* $\mathbf{X}^P$ *is generated by a positive* $\mathbb{T}^d$*–action and* $\mathbf{X}^N$ *is generated by a null* $\mathbb{T}^d$*–action.*

Proof. Proof of (ii) is trivial. To prove (i), observe that by Remark 2.5 in [22], there exist measurable functions $\Phi : S_2 \to S$ and $h : S_2 \to \mathbb{R} \setminus \{0\}$ such that for all $t \in \mathbb{T}^d$,

$$f_t^{(2)}(s) = h(s) f_t \circ \Phi(s) \quad \mu_2\text{–almost all } s \in S_2\,, \tag{3.7}$$

and $\mathrm{d}\mu = (|h|^\alpha \mathrm{d}\mu_2) \circ \Phi^{-1}$. Using (3.7) and an argument parallel to the proof of (2.18) in [35], it can be shown that $P_{\mathcal{G}^{(2)}} = \Phi^{-1}(P_{\mathcal{G}})$ and $N_{\mathcal{G}^{(2)}} = \Phi^{-1}(N_{\mathcal{G}})$ modulo μ_2, from which the distributional equalities in (i) follows as in the proof of Theorem 4.3 in [22]. □

4 Ergodic Properties of Stationary SαS Fields

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\{\theta_t\}_{t\in\mathbb{T}^d}$ a measure–preserving $\mathbb{T}^d$–action on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the random field $X_t(\omega) = X_0 \circ \theta_t(\omega)$, $t \in \mathbb{T}^d$. The random field $\{X_t\}_{t\in\mathbb{T}^d}$ defined in this way is stationary and conversely, any stationary measurable random field can be expressed in this form.

We start by introducing some notation. For all $t = (t_1, \cdots, t_d) \in \mathbb{T}^d$, let $\|t\|$ denote $\max_{1\le i\le d} |t_i|$. We consider the class $\mathcal{T}$ of all subsequences that converge to infinity in the following sense:

$$\mathcal{T} := \left\{\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{T}^d : \lim_{n\to\infty} \|t_n\| = \infty\right\}.$$

Recall that a set $E \subset \mathbb{T}^d$ is said to have *density zero* in $\mathbb{T}^d$, if

$$\lim_{T\to\infty} \frac{1}{C(T)} \int_{B(T)} \mathbf{1}_E(t)\lambda(\mathrm{d}t) = 0\,. \tag{4.1}$$

A set $D \subset \mathbb{T}^d$ is said to have *density one* in $\mathbb{T}^d$ if $\mathbb{T}^d \setminus D$ has density zero in $\mathbb{T}^d$. The class of all sequences on D that converge to infinity will be denoted by

$$\mathcal{T}_D := \Big\{\{t_n\}_{n\in\mathbb{N}} : t_n \in \mathbb{T}^d \cap D, \lim_{n\to\infty} \|t_n\| = \infty\Big\}\,.$$

Now we recall some basic definitions. Write $\sigma_{\mathbf{X}} := \sigma(\{X_t : t \in \mathbb{T}^d\})$ for the σ–algebra generated by the field $\{X_t\}_{t\in\mathbb{T}^d}$. We say $\{X_t\}_{t\in\mathbb{T}^d}$ is

(i) **ergodic**, if

$$\lim_{T\to\infty} \frac{1}{C(T)} \int_{B(T)} \mathbb{P}(A \cap \theta_t(B))\lambda(\mathrm{d}t) = \mathbb{P}(A)\,\mathbb{P}(B) \text{ for all } A, B \in \sigma_{\mathbf{X}}. \tag{4.2}$$

(ii) **weakly mixing**, if there exists a density one set D such that

$$\lim_{n\to\infty} \mathbb{P}(A \cap \theta_{t_n}(B)) = \mathbb{P}(A)\,\mathbb{P}(B) \text{ for all } A, B \in \sigma_{\mathbf{X}}\,, \{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}_D\,. \tag{4.3}$$

(iii) **mixing**, if

$$\lim_{n\to\infty} \mathbb{P}(A \cap \theta_{t_n}(B)) = \mathbb{P}(A)\,\mathbb{P}(B) \text{ for all } A, B \in \sigma_{\mathbf{X}}\,, \{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}\,. \tag{4.4}$$

Ergodicity can be equivalently characterized as follows.

Lemma 4.1. *The $\mathbb{T}^d$–action $\{\theta_t\}_{t\in\mathbb{T}^d}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is ergodic if and only if $A \in \sigma_{\mathbf{X}}$ and $\theta_t^{-1}(A) = A$ for all $t \in \mathbb{T}^d$ implies $\mathbb{P}(A) = 0$ or $\mathbb{P}(A^c) = 0$.*

The proof is given in the Appendix.

Remark 4.2. Another equivalent definition of weak mixing is the following: $\{X_t\}_{t\in\mathbb{T}^d}$ is weakly mixing if,

$$\lim_{T\to\infty} \frac{1}{C(T)} \int_{B(T)} |\,\mathbb{P}(A \cap \theta_t(B)) - \mathbb{P}(A)\,\mathbb{P}(B)|\lambda(\mathrm{d}t) = 0 \text{ for all } A, B \in \sigma_{\mathbf{X}}\,. \tag{4.5}$$

The equivalence of (4.3) and (4.5) follows from the following straightforward multivariate extension of Lemma 6.2 in [18] p. 65.

Lemma 4.3 (Koopman–von Neumann)**.** *Let f be a nonnegative function on $\mathbb{T}^d$, which is bounded: $\sup_{t\in\mathbb{T}^d} f(t) \le M < \infty$. Then,*

$$\lim_{T\to\infty} \frac{1}{C(T)} \int_{B(T)} f(t)\lambda(\mathrm{d}t) = 0\,, \tag{4.6}$$

if and only if, there exists a subset $D \subset \mathbb{R}^d$ of density one, such that

$$\lim_{n\to\infty} f(t_n) = 0\,,\ \textit{for all}\ \{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}_D\,. \tag{4.7}$$

In general, we always have that

mixing $\Rightarrow$ weakly mixing $\Rightarrow$ ergodicity.

For stationary SαS random fields, however, we have the following result.

Theorem 4.4. *A real stationary SαS random field is ergodic if and only if it is weakly mixing.*

Proof. Using Theorem 2.10, proceed as in Theorem 2 and Theorem 3 in [20]. □

The main result of this section is the following theorem.

Theorem 4.5. *A stationary SαS random field is weakly mixing (and equivalently ergodic) if and only if the component $\mathbf{X}^P$ in (3.6) corresponding to a positive $\mathbb{T}^d$–action vanishes.*

Remark 4.6. Theorems 2.4 and 4.5 yield that a stationary SαS random field is weakly mixing if and only if S can be expressed as a union of weakly wandering sets w.r.t. the underlying action. Heuristically, weakly wandering sets are those which do not come back to itself too often and so the same values of the random measure M do not contribute to observations X_t far separated in t. Thus the field ends up having a shorter memory which is manifested in its weakly mixing behavior.

To prove Theorem 4.5, we need the following result, which is an extension of Theorem 2.7 in [5]. The proof is given in the Appendix.

Theorem 4.7. *Assume $\alpha \in (0,2)$ and $\{X_t\}_{t\in\mathbb{T}^d}$ is a stationary SαS random field with spectral representation $\{f_t\}_{t\in\mathbb{T}^d} \subset L^\alpha(S,\mathcal{B},\mu)$. Then, the process $\{X_t\}_{t\in\mathbb{T}^d}$ is* *(i) weakly mixing if and only if there exists a density one set $D \subset \mathbb{T}^d$, such that*

$$\lim_{n\to\infty} \mu\Big\{s : |f_0(s)|^\alpha \in K\,, |f_{t_n^*}(s)|^\alpha > \epsilon\Big\} = 0$$
$$\textit{for all compact } K \subset \mathbb{R}\setminus\{0\}, \epsilon > 0 \textit{ and } \{t_n^*\}_{n\in\mathbb{N}} \in \mathcal{T}_D\,. \quad (4.8)$$

(ii) mixing, if and only if (4.8) *holds with $\mathcal{T}_D$ replaced by $\mathcal{T}$.*

Proof of Theorem 4.5. For this proof, we follow very closely the proof of Theorem 3.1 in [35] with the proof of 'only if' part being exactly the same. For the 'if part', however, we treat the discrete and the continuous parameter scenarios together by virtue of Theorem 2.9, which unifies the two cases. More specifically, in view of (4.8) and Lemma 4.3, it is enough to show that for all $\epsilon > 0$ and compact sets $K \subset \mathbb{R}\setminus\{0\}$,

$$\lim_{T\to\infty} A_T \mu\Big\{s : |f_0(s)|^\alpha \in K\,, |f_{(\cdot)}(s)|^\alpha > \epsilon\Big\} = 0\,, \quad (4.9)$$

where A_T is the average operator defined by (2.5). Following verbatim the argument in the proof of (3.1) in [35], we obtain (4.9) for both discrete and continuous parameter cases with the help of Theorem 2.9. □

Remark 4.8. From the structure results in [31] and [30], and Theorem 4.5 above, we obtain a unique in law decomposition of $\mathbf{X}$ into three independent stable processes in parallel to the one-dimensional case [35], i.e.,

$$\mathbf{X} = \mathbf{X}^{(1)} + \mathbf{X}^{(2)} + \mathbf{X}^{(3)},$$

where $\mathbf{X}^{(1)}$ is a mixed moving average in the sense of [39], $\mathbf{X}^{(2)}$ is weakly mixing with no mixed moving average component and $\mathbf{X}^{(3)}$ has no weakly mixing component.

5 Max–Stable Stationary Random Fields

In this section, we investigate the structure and ergodic properties of stationary max–stable random fields, indexed by $\mathbb{T}^d$. It turns out that the results are similar to the ones in the sum–stable case in Sections 3 and 4.

For simplicity and without loss of generality, we will focus on α–Fréchet random fields. The random field $\mathbf{X} = \{X_t\}_{t\in\mathbb{T}^d}$ is said to be α–Fréchet, if for all $a_j > 0$, $\tau_j \in \mathbb{T}^d$, $1 \le j \le n$, the max–linear combinations $\xi := \max_{1\le j\le n} a_j X_{\tau_j} \equiv \bigvee_{1\le j\le n} a_j X_{\tau_j}$, have α–Fréchet distributions. Namely,

$$\mathbb{P}(\xi \le x) = \exp\{-\sigma^\alpha x^{-\alpha}\} \text{ for all } x \in (0,\infty),$$

where $\sigma > 0$ referred to as the *scale coefficient* and $\alpha > 0$ is the tail index of ξ. The α–Fréchet fields are max–stable. Conversely, all max–stable fields with α–Fréchet marginals are α–Fréchet fields.

De Haan [7] has developed convenient spectral representations for these processes (fields). An intimate connection between the α–Fréchet and SαS processes ($0 < \alpha < 2$) has long been suspected due to their similar extremal properties and analogous Poisson point process representations. Recently, this connection was formally defined through the notion of association (see [10] and [42]). Wang and Stoev [43], moreover, have developed a theory for the spectral representation of max–stable processes, which parallels the existing representation theory for infinite variance SαS processes.

More precisely, any measurable α–Fréchet field $\mathbf{Y} = \{Y_t\}_{t\in\mathbb{T}^d}$ ($\alpha > 0$) can be represented as

$$Y_t := \int_S^{e} f_t(s) M_{\alpha,\vee}(\mathrm{d}s), \quad (t \in \mathbb{T}^d), \tag{5.1}$$

where $\{f_t\}_{t\in\mathbb{T}^d} \subset L^\alpha_+(S,\mu) := \{f \in L^\alpha(S,\mu) : f \geq 0\}$, '${}^{e}\!\!\int$' stands for the *extremal integral*, $M_{\alpha,\vee}$ is an *independently scattered α–Fréchet random sup–measure* with control measure μ and (S,μ) can be chosen to be a standard Lebesgue space (see [38, 43]). The functions $\{f_t\}_{t\in\mathbb{T}^d}$ in (5.1) are called *spectral functions* of the α–Fréchet random field. By using Theorems 4.1 and 4.2 in [43], if the representation in (5.1) is *minimal*, as in the sum–stable case, one obtains

$$f_t(s) = \Big(\frac{d(\mu\circ\phi_t)}{d\mu}\Big)^{1/\alpha} f_0 \circ \phi_t(s) \text{ for all } t \in \mathbb{T}^d\,, \tag{5.2}$$

where $\phi = \{\phi_t\}_{t\in\mathbb{T}^d}$ is a nonsingular group action and $f_0 \in L^\alpha_+(S,\mu)$.

Thus, the α–Fréchet random field $\mathbf{Y}$ is said to be generated by the group action ϕ if (5.1) is a minimal representation such that (5.2) holds. This allows us to extend the available classification results in the sum–stable case to the max–stable setting. Indeed, by following the proof of Theorem 3.1, we obtain:

Theorem 5.1. *Suppose $\{Y_t\}_{t\in\mathbb{T}^d}$ is a measurable stationary α–Fréchet random field with spectral representation $\{f_t\}_{t\in\mathbb{T}^d}$ as in* (5.1)*. Let $T_0 \in \mathcal{B}_{\mathbb{T}^d}$ and $\{a_\tau\}_{\tau\in T_0}$, $a_\tau > 0$, be such that* (3.3) *holds. Then,*

(i) $\{Y_t\}_{t\in\mathbb{T}^d}$ is generated by a positive $\mathbb{T}^d$–action, if and only if (3.4) *holds.*

(ii) $\{Y_t\}_{t\in\mathbb{T}^d}$ is generated by a null $\mathbb{T}^d$–action, if and only if (3.5) *holds.*

In particular, the classes of stationary α–Fréchet random fields generated by positive and null $\mathbb{T}^d$–actions are disjoint.

This yields the following decomposition result.

Corollary 5.2. *Let $\{Y_t\}_{t\in\mathbb{T}^d}$ be a measurable stationary α–Fréchet random field with representation in form of* (5.1) *and* (5.2)*. We have the unique–in–law decomposition*

$$\{Y_t\}_{t\in\mathbb{T}^d} \stackrel{\mathrm{d}}{=} \left\{Y_t^P \vee Y_t^N\right\}_{t\in\mathbb{T}^d}$$

with

$$Y_t^P = {\int_{P_{\mathcal{G}}}}^{\!\!e} f_t(s) M_{\alpha,\vee}(\mathrm{d}s) \text{ and } Y_t^N = {\int_{N_{\mathcal{G}}}}^{\!\!e} f_t(s) M_{\alpha,\vee}(\mathrm{d}s) \text{ for all } t \in \mathbb{T}^d\,,$$

where $\mathcal{G} \equiv \{\phi_t\}_{t\in\mathbb{T}^d}$. The component $\{Y_t^P\}_{t\in\mathbb{T}^d}$ is generated by positive $\mathbb{T}^d$–action and $\{Y_t^N\}_{t\in\mathbb{T}^d}$ is generated by null $\mathbb{T}^d$–action.

The proof is analogous to that of Corollary 3.4.

Remark 5.3. In contrast to the sum–stable case, one does not encounter a co–cycle in (5.2) because the spectral functions in (5.1) are non–negative. One could also arrive at (5.2) by using *association* as in [10] and [42]. Namely, for any $\alpha-$Fréchet field as in (5.1), the $1-$Fréchet field $\mathbf{Y}^\alpha = \{Y_t^\alpha\}_{t\in\mathbb{T}^d}$ is associated with an S1S random field with spectral functions $\{f_t^\alpha\}_{t\in\mathbb{T}^d} \subset L^1(S,\mu)$. One can thus relate the spectral functions to a group action as in (5.2) by using the available theory in the sum–stable case.

In the rest of this section, let $\mathbf{Y} = \{Y_t\}_{t\in\mathbb{T}^d}$ denote a measurable α–Fréchet random field with spectral representation (5.1) and (5.2). We shall study the ergodic properties of $\mathbf{Y}$.

Theorem 5.4. *$\{Y_t\}_{t\in\mathbb{T}^d}$ is ergodic, if and only if*

$$\lim_{T\to\infty} \frac{1}{C(T)} \int_{B(T)} \|U_t g \wedge g\|_\alpha^\alpha \mathrm{d}t = 0\,, \tag{5.3}$$

for all $g \in \overline{\vee\text{-span}}\{f_t : t \in \mathbb{T}^d\}$, where $U = \{U_t\}_{t\in\mathbb{T}^d}$ is the group action on $L_+^\alpha(S,\mu)$ defined as $U_t g := (d\mu\circ\phi_t/d\mu)^{1/\alpha} g\circ\phi_t,\ t\in\mathbb{T}^d$.

Proof. Observe that $\{Y_t^\alpha\}_{t\in\mathbb{T}^d}$ is a 1–Fréchet random field with spectral representation $\{f_t^\alpha\}_{t\in\mathbb{T}^d} \subset L_+^1(S,\mu)$. Thus, it is enough to focus on the case $\alpha = 1$ and in the sequel let $\|\cdot\|$ denote the L^1 norm (see also Remark 5.3). Without loss of generality, suppose $\mathbf{Y} = \{Y_t\}_{t\in\mathbb{T}^d}$ has the form $Y_t(\omega) = Y\circ\theta_t(\omega)$, where

$\{\theta_t\}_{t\in\mathbb{T}^d}$ is a measure–preserving $\mathbb{T}^d$–action on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, similarly as in (2.8), for all $\widetilde{Y} \in L^1(\Omega, \sigma_{\mathbf{Y}}, \mathbb{P})$, write

$$A_T\widetilde{Y}(\omega) \equiv \frac{1}{C(T)}\int_{B(T)} \widetilde{Y}\circ\theta_t(\omega)\lambda(\mathrm{d}t)\,,$$

and by Theorem 2.10,

$$\lim_{T\to\infty} \|A_T\widetilde{Y} - \mathbb{E}(\widetilde{Y}|\mathcal{I}_{\mathbf{Y}})\| = 0\,,$$

where $\mathcal{I}_{\mathbf{Y}}$ is the invariant σ–algebra consisting of sets in $\sigma_{\mathbf{Y}}$ invariant w.r.t. all shift operators. Then, $\{Y_t\}_{t\in\mathbb{T}^d}$ is ergodic, if and only if $\mathbb{E}(\widetilde{Y}|\mathcal{I}_{\mathbf{Y}}) = \mathbb{E}\widetilde{Y}$, $\mathbb{P}$-a.e. for all $\widetilde{Y} \in L^1(\Omega, \sigma_{\mathbf{Y}}, \mathbb{P})$ by Lemma 4.1, if and only if

$$\lim_{T\to\infty} \|A_T\widetilde{Y} - \mathbb{E}\widetilde{Y}\| = 0 \text{ for all } \widetilde{Y} \in L^1(\Omega, \sigma_{\mathbf{Y}}, \mathbb{P})\,. \tag{5.4}$$

In particular, it is equivalent to show that (5.4) holds for all $\widetilde{Y} \in \mathcal{H}$, where $\mathcal{H}$ is a linearly dense subset of $L^1(\Omega, \sigma_{\mathbf{Y}}, \mathbb{P})$, consisting of random variables of the following form:

$$\eta = \mathbf{1}_{\{Y_{t_1}\le a_1,\cdots,Y_{t_n}\le a_n\}}\,, n\in\mathbb{N}, t_i\in\mathbb{T}^d \text{ and } a_i\in\mathbb{R}. \tag{5.5}$$

Moreover, since η is bounded by 1 and so is $A_T\eta$, in (5.4) we can equivalently use the L^2–norm. Therefore, observing that

$$\begin{aligned}\|A_T\eta - \mathbb{E}\eta\|^2_{L^2(\Omega,\sigma_{\mathbf{Y}},\mathbb{P})} &= \mathbb{E}|A_T\eta - \mathbb{P}(Y_{t_1}\le a_1,\cdots,Y_{t_n}\le a_n)|^2\\ &= \mathbb{E}|A_T\eta - \exp\{-\|h_\eta\|\}|^2\end{aligned}$$

with $h_\eta(s) := \bigvee_{i=1}^n a_i^{-1}f_{t_i}(s) \in L^\alpha(S,\mu)$ corresponding to η given in (5.5), Condition (5.4) is equivalent to

$$\lim_{T\to\infty} \mathbb{E}|A_T\eta - \exp\{-\|h_\eta\|\}|^2 = 0\,, \quad \forall\eta\in\mathcal{H}\,. \tag{5.6}$$

By straightforward calculation, (5.6) becomes

$$\lim_{T\to\infty}\frac{\mathbb{E}|A_T\eta|^2}{\exp\{-2\|h_\eta\|\}} = 1 \text{ for all } \eta\in\mathcal{H}\,. \tag{5.7}$$

Observing that

$$\begin{aligned}\mathbb{E}|A_T\eta|^2 &= \frac{1}{C(T)^2}\int_{B(T)}\int_{B(T)} \mathbb{P}(Y_{t_i+t}\le a_i, Y_{t_i+\tau}\le a_i, 1\le i\le n)\mathrm{d}\tau\mathrm{d}t\\ &= \frac{1}{C(T)^2}\int_{B(T)}\int_{B(T)} \exp\{-\|U_{t-\tau}h_\eta\vee h_\eta\|\}\mathrm{d}\tau\mathrm{d}t\,,\end{aligned}$$

and

$$\frac{\exp\{-\|U_{t-\tau}h_\eta \vee h_\eta\|}{\exp\{-2\|h_\eta\|\}} = \exp\Big\{ -\|U_{t-\tau}h_\eta \vee h_\eta\| + \|h_\eta\| + \|U_{t-\tau}h_\eta\| \Big\},$$

we have

$$\frac{\mathbb{E}|A_T\eta|^2}{\exp\{-2\|h_\eta\|\}} = \frac{1}{C(T)^2}\int_{B(T)}\int_{B(T)} \exp\Big\{ -\|U_{t-\tau}h_\eta \wedge h_\eta\| \Big\}\mathrm{d}\tau\mathrm{d}t\,.$$

Plugging in the above equality and applying Lemma 4.3, we can show that (5.7) holds, if and only if

$$\lim_{T\to\infty}\frac{1}{C(T)^2}\int_{B(T)}\int_{B(T)} \|U_{t-\tau}h_\eta \wedge h_\eta\|\,\mathrm{d}\tau\mathrm{d}t = 0 \text{ for all } Y\in\mathcal{H}\,. \tag{5.8}$$

Since $R(t,\tau) := \|U_{t-\tau}h_\eta \wedge h_\eta\|$ is a nonnegative definite function on $\mathbb{T}^d\times\mathbb{T}^d$, by Bochner's theorem (see e.g. Corollary 2.3 in [15]), the l.h.s. of (5.8) becomes

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{C(T)^2}\int_{B(T)}\int_{B(T)}\int_{\mathbb{T}^d} &\mathrm{e}^{i(t-\tau)^\top x}\nu(\mathrm{d}x)\mathrm{d}\tau\mathrm{d}t \\ &= \lim_{T\to\infty}\frac{1}{C(T)^2}\int_{\mathbb{T}^d} |\mathrm{e}^{it^\top x}\mathrm{d}t|^2\nu(\mathrm{d}x) = \nu(\{0\})\,,\end{aligned} \tag{5.9}$$

for some finite symmetric measure ν on $\mathbb{T}^d$. At the same time, by the inversion formula of the Fourier transform,

$$\nu(\{0\}) = \lim_{T\to\infty}\frac{1}{C(T)}\int_{B(T)} \|U_t h_\eta \wedge h_\eta\|\,\mathrm{d}t\,. \tag{5.10}$$

Combining (5.8), (5.9) and (5.10), we have proved the desired result. □

The ergodicity of stationary α–Fréchet random fields $\{Y_t\}_{t\in\mathbb{T}^d}$ is closely related to the recurrence properties of the underlying $\mathbb{T}^d$–action. As in the sum–stable case, we have

Theorem 5.5. *$\{Y_t\}_{t\in\mathbb{T}^d}$ is ergodic, if and only if the $\mathbb{T}^d$–action $\{\phi_t\}_{t\in\mathbb{T}^d}$ has no nontrivial positive component.*

Proof. Theorem 5.4 and the multiparameter stochastic ergodic theorem (Theorem 2.9) allow us to extend the proof of Theorem 8 in [10] to the multiparameter setting. □

The following theorem gives a simple necessary and sufficient condition for the mixing of measurable stationary α–Fréchet random field.

Theorem 5.6. $\{Y_t\}_{t\in\mathbb{T}^d}$ *is mixing, if and only if*

$$\lim_{n\to\infty} \|f_{t_n} \wedge f_0\| = 0 \ \text{ for all } \{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}\,.$$

The result follows by using similar arguments as in the proofs of Theorem 3.3 and Theorem 3.4 in [37].

Remark 5.7. The recent work of Kabluchko and Schlather [11] provides simple characterizations of ergodicity and weak mixing (of all orders) for general classes of stationary max–infinitely divisible processes. Their results apply to the max–stable setting. For the case of processes, they provide an alternative characterization of positive recurrence to that in Theorem 5.1.

We conclude this section with an interesting result showing the equivalence of weak mixing and ergodicity for the general class of positively dependent stationary random fields, which includes as particular cases max–infinitely divisible and max–stable random fields (and processes). Recall that the field $\mathbf{X} = \{X_t\}_{t\in\mathbb{T}^d}$ is said to be *positively dependent* or *associated* if all its finite–dimensional distributions are associated. Namely, for all $t_i \in T,\ 1 \le i \le n$, we have

$$\mathrm{Cov}(g_1(\widetilde{\mathbf{X}}), g_2(\widetilde{\mathbf{X}})) \ge 0, \quad \text{with } \widetilde{\mathbf{X}} = \{X_{t_i}\}_{i=1}^n,$$

for all coordinate–wise monotone non–decreasing functions g_1 and g_2 such that the above covariance is well–defined. All max–infinitely divisible processes (fields) are associated (see e.g. [21]). This implies, in particular that, for all $\widetilde{\mathbf{X}} = \{X_{t_j}\}_{j=1}^n$,

$$\mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{x}, \widetilde{\mathbf{X}} \le \mathbf{y}\} - \mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{x}\}\mathbb{P}\{\ \widetilde{\mathbf{X}} \le \mathbf{y}\} = \mathrm{Cov}(g_1(\widetilde{\mathbf{X}}), g_2(\widetilde{\mathbf{X}})) \ge 0, \quad (5.11)$$

where $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$, $g_1(\mathbf{u}) = \mathbf{1}_{\{\mathbf{u}\le\mathbf{x}\}}$, $g_2(\mathbf{u}) = \mathbf{1}_{\{\mathbf{u}\le\mathbf{y}\}}$, and where the inequalities are coordinatewise.

Theorem 5.8. *Let* $\mathbf{X} = \{X_t\}_{t\in\mathbb{T}^d}$ *be a measurable stationary random field, which is positively dependent (i.e. associated). Then,* $\mathbf{X}$ *is ergodic, if and only if it is weakly mixing.*

Proof. We will only consider the case $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$ being simpler. The convergence (4.5) implies (4.2) and thus weak mixing implies ergodicity.

Suppose now that $\mathbf{X}$ is ergodic. Let μ be the distribution of the process $\mathbf{X}$ defined on $\mathcal{B}_{\mathbb{R}^{\mathbb{R}^d}}$ as follows:

$$\mu(A) := \mathbb{P}\{\mathbf{X} \in A\}, \quad \forall A \in \mathcal{B}_{\mathbb{R}^{\mathbb{R}^d}}.$$

Consider the shift–operators $\theta_\tau(x) \equiv x(\tau) := \{x_{t+\tau}\}_{t\in\mathbb{R}^d},\ \tau \in \mathbb{R}^d, x \in \mathbb{R}^{\mathbb{R}^d}$. The stationarity of the process $\mathbf{X}$ implies that $\{\theta_\tau\}_{\tau\in\mathbb{R}^d}$ is a $\mu-$measure preserving $\mathbb{R}^d-$action on $\mathbb{R}^{\mathbb{R}^d}$.

To prove weak mixing, it is enough to show that

$$\frac{1}{T^d}\int_{(0,T)^d} |\mu(A\cap B(\tau)) - \mu(A)\mu(B)|\mathrm{d}\tau \longrightarrow 0, \quad \text{as } T\to\infty, \tag{5.12}$$

for all fixed $A,\ B \in \mathcal{B}_{\mathbb{R}^{\mathbb{R}^d}}$, where $B(\tau) \equiv \theta_\tau(B)$. Let $\mathcal{C} = \{\{x \in \mathbb{R}^{\mathbb{R}^d} : a_i < x_{t_i} \le b_i,\ 1 \le i \le n\} :\ n \in \mathbb{N}, a_i, b_i \in \mathbb{R}\}$ be the semiring of cylinder sets with finite–dimensional rectangular faces. For all $\epsilon > 0$, there exist cylinder sets from the ring generated by $\mathcal{C}$, namely, $A_\epsilon = \cup_{k=1}^m A_{\epsilon,k}$ and $B_\epsilon = \cup_{k=1}^n B_{\epsilon,k}$, with $A_{\epsilon,k},\ B_{\epsilon,k} \in \mathcal{C}$, such that

$$\mu(A\Delta A_\epsilon) < \epsilon \quad \text{and} \quad \mu(B\Delta B_\epsilon) = \mu(B(\tau)\Delta B_\epsilon(\tau)) < \epsilon.$$

Observe that

$$\begin{aligned}
&|\mu(A\cap B(\tau)) - \mu(A)\mu(B)| \le |\mu(A_\epsilon\cap B_\epsilon(\tau)) - \mu(A_\epsilon)\mu(B_\epsilon)| \\
&\qquad +|\mu(A\cap B(\tau)) - \mu(A_\epsilon\cap B_\epsilon(\tau))| + |\mu(A) - \mu(A_\epsilon)| + |\mu(B) - \mu(B_\epsilon)| \\
&\quad \le |\mu(A_\epsilon\cap B_\epsilon(\tau)) - \mu(A_\epsilon)\mu(B_\epsilon)| + 3\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it suffices to show that the convergence in (5.12) holds with A and B replaced by A_ϵ and B_ϵ. Without loss of generality, suppose that $A_{\epsilon,k},\ 1 \le k \le m$ and also $B_{\epsilon,k},\ 1 \le k \le n$ are disjoint. We then have that

$$\mathbb{P}(A_\epsilon\cap B_\epsilon(\tau)) - \mathbb{P}(A_\epsilon)\mathbb{P}(B_\epsilon) = \sum_{k_1=1}^m\sum_{k_2=1}^n \mathbb{P}(A_{\epsilon,k_1}\cap B_{\epsilon,k_2}(\tau)) - \mathbb{P}(A_{\epsilon,k_1})\mathbb{P}(B_{\epsilon,k_2}).$$

This, since m and n are fixed, implies that it suffices to show that (5.12) holds for all A and B in the semiring $\mathcal{C}$.

Let $A = \{x \in \mathbb{R}^{\mathbb{R}^d} : a_{1,i} < x_{t_i} \le a_{2,i},\ 1 \le i \le r\}$ and $B = \{x \in \mathbb{R}^{\mathbb{R}^d} : b_{1,i} < x_{t_i} \le b_{2,i},\ 1 \le i \le r\}$, where $\mathbf{a}_k = \{a_{k,i}\}_{i=1}^r$ and $\mathbf{b}_k = \{b_{k,i}\}_{i=1}^r,\ k = 1,2$ are *fixed*. Let also $\widetilde{\mathbf{X}} = \{X_{t_i}\}_{i=1}^r$, and observe that

$$\mathbf{1}_{\{\mathbf{a}_1<\widetilde{\mathbf{X}}\le\mathbf{a}_2\}} = \sum_{i\in\{1,2\}^r} \delta_i \mathbf{1}_{\{\widetilde{\mathbf{X}}\le\mathbf{a}_{(i)}\}},$$

where $\delta_i \in \{-1,+1\}$, $i = \{i(j)\}_{j=1}^r,\ i(j) \in \{1,2\}$ and $\mathbf{a}_{(i)} = \{a_{i(j),j}\}_{j=1}^r$.

By using a similar expression for $\mathbf{1}_{\{\mathbf{b}_1<\widetilde{\mathbf{X}}\le\mathbf{b}_2\}}$, (involving the *same* δ_i's) we obtain

$$\begin{aligned}
\mathbb{P}(A\cap B(\tau)) &\equiv \mathbb{P}\{\mathbf{a}_1 < \widetilde{\mathbf{X}} \le \mathbf{a}_2,\ \mathbf{b}_1 < \widetilde{\mathbf{X}}(\tau) \le \mathbf{b}_2\} \\
&= \sum_{i\in\{1,2\}^r}\sum_{j\in\{1,2\}^r} \delta_i\delta_j \mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{a}_{(i)},\ \widetilde{\mathbf{X}}(\tau) \le \mathbf{b}_{(j)}\},
\end{aligned} \tag{5.13}$$

where $\widetilde{\mathbf{X}}(\tau) = \{X_{\tau+t_i}\}_{i=1}^r$ and also

$$\begin{aligned} \mathbb{P}(A)\mathbb{P}(B) &\equiv \mathbb{P}\{\mathbf{a}_1 < \widetilde{\mathbf{X}} \le \mathbf{a}_2\}\mathbb{P}\{\mathbf{b}_1 < \widetilde{\mathbf{X}} \le \mathbf{b}_2\} \\ &= \sum_{i\in\{1,2\}^r}\sum_{j\in\{1,2\}^r} \delta_i\delta_j \mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{a}_{(i)}\}\mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{b}_{(j)}\}, \end{aligned} \tag{5.14}$$

where the probabilities stand for expectations of indicators. Now, by subtracting (5.14) from (5.13), and applying the triangle inequality, we get

$$\begin{aligned} &|\mathbb{P}(A\cap B(\tau)) - \mathbb{P}(A)\mathbb{P}(B)| \qquad (5.15) \\ \le &\sum_{i,j\in\{1,2\}^r} |\mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{a}_{(i)},\ \widetilde{\mathbf{X}}(\tau) \le \mathbf{b}_{(j)}\} - \mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{a}_{(i)}\}\mathbb{P}\{\widetilde{\mathbf{X}}(\tau) \le \mathbf{b}_{(j)}\}|. \end{aligned}$$

The ergodicity of the field $\mathbf{X}$, implies that for all $i, j \in \{1,2\}^r$,

$$\frac{1}{T^d}\int_{(0,T)^d} \Big(\mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{a}_{(i)},\ \widetilde{\mathbf{X}}(\tau) \le \mathbf{b}_{(j)}\} - \mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{a}_{(i)}\}\mathbb{P}\{\widetilde{\mathbf{X}}(\tau) \le \mathbf{b}_{(j)}\}\Big)\mathrm{d}\tau \longrightarrow 0,$$

as $T \to \infty$. Since $\mathbf{X}$ is *associated*, however, the last integrand is non–negative, for all $\tau \in \mathbb{R}^d$ (recall (5.11), above) and thus

$$\frac{1}{T^d}\int_{(0,T)^d} \Big|\mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{a}_{(i)},\ \widetilde{\mathbf{X}}(\tau) \le \mathbf{b}_{(j)}\} - \mathbb{P}\{\widetilde{\mathbf{X}} \le \mathbf{a}_{(i)}\}\mathbb{P}\{\widetilde{\mathbf{X}}(\tau) \le \mathbf{b}_{(j)}\}\Big|\mathrm{d}\tau \longrightarrow 0,$$

as $T \to \infty$. By considering the integral average in (5.15), we get

$$\frac{1}{T^d}\int_{(0,T)^d} |\mathbb{P}(A\cap B(\tau)) - \mathbb{P}(A)\mathbb{P}(B)|\mathrm{d}\tau \longrightarrow 0,$$

as $T \to \infty$. We have thus shown that for *associated* fields the convergence in (4.2) implies (4.5) for all A and B in the semiring $\mathcal{C}$. The above approximation arguments show that this is so for all A and B in $\mathcal{B}_{\mathbb{R}^{\mathbb{R}^d}}$. □

6 Examples

This section contains two examples of stable random fields and their ergodic properties via the positive–null decomposition of the underlying action. These examples show the usefulness of our results to check whether or not a stationary SαS (or max–stable) random field is ergodic (or equivalently, weakly mixing).

The first example is based on a self–similar SαS processes with stationary increments introduced by [3] as a stochastic integral with respect to an

SαS random measure, with the integrand being the local time process of a fractional Brownian motion. We extend these processes by replacing the fractional Brownian motion by a Brownian sheet. We can call it a *Brownian sheet local time fractional SαS random field* following the terminology of [3].

Example 6.1. Suppose $(\Omega', \mathcal{F}', P')$ is a probability space supporting a Brownian sheet $\{B_u\}_{u\in\mathbb{R}^d_+}$. By [4], $\{B_u\}$ has a jointly continuous local time field $\{l(x,u) : x \in \mathbb{R},\ u \in \mathbb{R}^d_+\}$ defined on the same probability space. We will define an SαS random field based on this local time field, which inherits the stationary increments property from $\{B_u\}_{u\in\mathbb{R}^d_+}$. Let M_α be an SαS random measure on $\Omega'\times\mathbb{R}$ with control measure $P'\times Leb$ living on another probability space $(\Omega, \mathcal{F}, P)$. Following verbatim the calculations of [3] we have

$$Z_u = \int_{\Omega'\times\mathbb{R}} l(x,u)(\omega') M_\alpha(\mathrm{d}\omega', \mathrm{d}x), \quad u \in \mathbb{R}^d_+$$

is a well–defined $S\alpha S$ random field which has stationary increments over d–dimensional rectangles.

We now concentrate on the increments of $\{Z_u\}$ taken over d–dimensional rectangles. For any $t \in \mathbb{Z}^d_+$, define

$$X_t = \Delta Z_t := \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_d=0}^{1}(-1)^{i_1+i_2+\cdots+i_d+d} Z_{t+(i_1,i_2,\ldots,i_d)}\,. \tag{6.1}$$

Clearly $\{X_t\}_{t\in\mathbb{Z}^d_+}$ is a stationary $S\alpha S$ random field, which can be extended (in law) to a stationary $S\alpha S$ random field $\mathbf{X} := \{X_t\}_{t\in\mathbb{Z}^d}$ by Kolmogorov's extension theorem. We claim that $\mathbf{X}$ is generated by a null $\mathbb{Z}^d$–action. To prove this, define, for all $n \geq 1$, $\tau^{(n)} := (n^{4/d}, n^{4/d}, \ldots, n^{4/d})$, and for all $n \geq 1$ and $t \in \mathbb{Z}^d_+$,

$$T_{n,t} := \left\{s : t_i + n^{4/d} \leq s_i \leq 1 + t_i + n^{4/d} \text{ for all } i = 1, 2, \ldots, d\right\}.$$

For each $t \in \mathbb{Z}^d_+$, take a positive real number a_t in such a way that $\sum_{t\in\mathbb{Z}^d_+} a_t = 1$. Defining $\Delta l(x,t)$ in parallel to (6.1) and following the proof of (4.7) in [3], we can establish that

$$\begin{aligned}
&\int_{\Omega'}\int_{\mathbb{R}} e^{-x^2/2} \sum_{t\in\mathbb{Z}^d_+}\sum_{n=1}^{\infty} a_t \Delta l\left(x, t+\tau^{(n)}\right) \mathrm{d}x\mathrm{d}P' \\
&= \sum_{t\in\mathbb{Z}^d_+} a_t \sum_{n=1}^{\infty}\int_{T_{n,t}} \frac{\mathrm{d}s}{\sqrt{1+\prod_{i=1}^d s_i}} \\
&\leq \sum_{n=1}^{\infty}\frac{1}{\sqrt{1+n^4}} < \infty\,.
\end{aligned}$$

This shows, in particular, that $\sum_{t\in\mathbb{Z}_+^d}\sum_{n=1}^{\infty} a_t \Delta l\left(x, t+\tau^{(n)}\right)(\omega') < \infty$ for $P' \times$ *Leb*–almost all $(\omega', x) \in \Omega \times \mathbb{R}$. Besides, it can be easily shown that $\sum_{t\in\mathbb{Z}_+^d} a_t \Delta l\,(x,t)\,(\omega') > 0$ for $P' \times$ *Leb*–almost all $(\omega', x) \in \Omega \times \mathbb{R}$ (see, for example, [41]). Hence by Theorem 3.1, it follows that $\mathbf{X}$ is generated by a null action and hence is weakly mixing.

The next example is based on a class of mixing stationary SαS process considered in [24]. We look at a stationary SαS random field generated by d independent recurrent Markov chains at least one of which is null–recurrent. This is a class of stationary SαS random fields which are weakly mixing as a field but not necessarily ergodic in every direction.

Example 6.2. We start with d irreducible aperiodic recurrent Markov chains on $\mathbb{Z}$ with laws $P_i^{(1)}(\cdot)$, $P_i^{(2)}(\cdot), \ldots, P_i^{(d)}(\cdot)$, $i \in \mathbb{Z}$ and transition probabilities $(p_{jk}^{(1)}), (p_{jk}^{(2)}), \ldots, (p_{jk}^{(d)})$ respectively. For all $l = 1, 2, \ldots, d$, let $\pi^{(l)} = (\pi_i^{(l)})_{i\in\mathbb{Z}}$ be a σ–finite invariant measure corresponding to the family $(P_i^{(l)})$. Let $\tilde{P}_i^{(l)}$ be the lateral extension of $P_i^{(l)}$ to $\mathbb{Z}^{\mathbb{Z}}$; that is under $\tilde{P}_i^{(l)}$, $x(0) = i$, $(x(0), x(1), \ldots)$ is a Markov chain with transition probabilities $(p_{jk}^{(l)})$ and $(x(0), x(-1), \ldots)$ is a Markov chain with transition probabilities $(\pi_k^{(l)}\, p_{kj}^{(l)}/\pi_j^{(l)})$. Assume at least one (say, the first one) of the Markov chains is null–recurrent and define a σ–finite measure μ on $S = \left(\mathbb{Z}^{\mathbb{Z}}\right)^d$ by

$$\mu(A_1 \times A_2 \times \cdots \times A_d) = \prod_{l=1}^{d}\left(\sum_{i=-\infty}^{\infty} \pi_i^{(l)}\, \tilde{P}_i^{(l)}(A_l)\right),$$

and observe that μ is invariant under the $\mathbb{Z}^d$–action $\{\phi_{(i_1,i_2,\ldots,i_d)}\}$ on S defined as the coordinatewise left shift, that is,

$$\phi_{(i_1,\ldots,i_d)}\left(a^{(1)}, \ldots, a^{(d)}\right)(u_1, \ldots, u_d) = \left(a^{(1)}(u_1+i_1)), \ldots, a^{(d)}(u_d+i_d)\right) \quad (6.2)$$

for all $\left(a^{(1)}, \ldots, a^{(d)}\right) \in S$ and $u_1, \ldots, u_d \in \mathbb{Z}$.

Let $\mathbf{X} = \{X_{(i_1,i_2,\ldots,i_d)}\}_{(i_1,\ldots,i_d)\in\mathbb{Z}^d}$ be a stationary SαS random field defined by the integral representation (3.1) with M_α being a SαS random measure on S with control measure μ and

$$f_{(i_1,i_2,\ldots,i_d)} = f \circ \phi_{(i_1,i_2,\ldots,i_d)}\,, \quad i_1, i_2, \ldots, i_d \in \mathbb{Z}$$

with

$$f(x^{(1)}, x^{(2)}, \ldots, x^{(d)}) = \mathbf{1}_{\{x^{(1)}(0)=x^{(2)}(0)=\cdots=x^{(d)}(0)=0\}}\,,\ x^{(1)}, x^{(2)}, \ldots, x^{(d)} \in \mathbb{Z}^{\mathbb{Z}}\,.$$

Clearly, the restriction of (6.2) to the first coordinate is a null flow because the first Markov chain is null–recurrent (see Example 4.1 in [35]) and hence (6.2) is a null $\mathbb{Z}^d$–action. This shows, in particular, that $\mathbf{X}$ is weakly mixing. However, if $d > 1$ and some of the Markov chains are positive–recurrent then the restriction of μ in the corresponding coordinate directions are finite and hence by Theorem 4.5, $\mathbf{X}$ is not ergodic along those directions. In this case, the random field cannot be mixing because it is not mixing in every coordinate direction. This gives examples of stationary d–dimensional ($d > 1$) SαS random fields which are weakly mixing but not mixing. See Example 4.2 in [6] for such an example in the $d = 1$ case.

Remark 6.3. Note that, in the above examples, the kernels are nonnegative functions and the cocycles are trivial and hence we can define α–Fréchet analogues of these fields by replacing the integrals with respect to the SαS random measures by extremal integrals with respect to α–Fréchet random sup–measures with the same control measures as above. Since the underlying action is the same, using Theorem 5.5, we can establish that the corresponding α-Fréchet fields are weakly mixing. In particular, when $d > 1$, we can obtain an example of an α–Fréchet field which is weakly mixing but not mixing.

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A Proofs of Auxiliary Results

A.1 Proof of Lemma 2.2

Set

$$u(I(\mathcal{G})) := \sup_{\nu\in\Lambda(\mathcal{G})} \mu(S_\nu)\,. \tag{A.1}$$

Without loss of generality, we assume $\mu(S) < \infty$ (recall that μ is σ–finite), whence $u(I(\mathcal{G})) < \infty$. Then, there exists a sequence of measures $\{\nu_n\}_{n\in\mathbb{N}} \subset \Lambda(\mathcal{G})$, such that $u_n := \mu(S_{\nu_n}) \to u(I(\mathcal{G}))$ as $n \to \infty$. Set

$$P_{\mathcal{G}} := \bigcup_{n=1}^{\infty} S_{\nu_n}\,.$$

Clearly, $P_{\mathcal{G}}$ is measurable. We show that there exists $\nu_{\mathcal{G}} \in \Lambda(\mathcal{G})$ such that $S_{\nu_{\mathcal{G}}} = P_{\mathcal{G}}$ and $\mu(P_{\mathcal{G}}) = u(I(\mathcal{G}))$. Indeed, we can define on $(S, \mathcal{B})$ the measure

$$\nu_{\mathcal{G}}(A) := \sum_{n=1}^{\infty} \frac{1}{2^n u_n} \nu_n(A) \text{ for all } A \in \mathcal{B}. \tag{A.2}$$

Clearly, $\nu_{\mathcal{G}} \in \Lambda(\mathcal{G})$, $S_{\nu_{\mathcal{G}}} = P_{\mathcal{G}} \mod \mu$, and $\mu(P_{\mathcal{G}}) \leq u(I(\mathcal{G}))$ by (A.1). It is also clear that for all $n \in \mathbb{N}$, $\nu_n \ll \nu_{\mathcal{G}}$, and hence $P_{\mathcal{G}} \supset S_{\nu_n} \mod \mu$. This implies $\mu(P_{\mathcal{G}}) \geq u_n$ for all $n \in \mathbb{N}$. We have thus shown that $\mu(P_{\mathcal{G}}) = u(I(\mathcal{G}))$.

To complete the proof, we show $P_{\mathcal{G}}$ is unique modulo μ–null sets. Suppose there exist $P_{\mathcal{G}}^{(1)}$ and $P_{\mathcal{G}}^{(2)}$ such that $\mu(P_{\mathcal{G}}^{(1)}) = \mu(P_{\mathcal{G}}^{(2)}) = u(I(\mathcal{G}))$ and $\mu(P_{\mathcal{G}}^{(1)} \triangle P_{\mathcal{G}}^{(2)}) > 0$. Suppose $\nu^{(1)}, \nu^{(2)} \in \Lambda(\mathcal{G})$ are defined as in (A.2), so that $S_{\nu^{(i)}} = P_{\mathcal{G}}^{(i)}$ for $i = 1, 2$. Clearly, $\nu^{(1)} + \nu^{(2)} \in \Lambda(\mathcal{G})$. Then, we have $P_{\mathcal{G}}^{(1)} \cup P_{\mathcal{G}}^{(2)} \subset I(\mathcal{G})$ and $\mu(P_{\mathcal{G}}^{(1)} \cup P_{\mathcal{G}}^{(2)}) > u(I(\mathcal{G}))$, which contradicts (A.1). The proof is thus complete.

A.2 Proof of Theorem 2.3

First we introduce some notations. For all transformation ϕ on $(S, \mathcal{B}, \mu)$, write

$$\Lambda(\phi) := \{\nu \ll \mu : \nu \text{ finite positive measure on } S, \nu \circ \phi^{-1} = \nu\}.$$

We need the following lemma.

Lemma A.1. *Suppose ϕ is an arbitrary invertible, bi-measurable and non–singular transformation on $(S, \mathcal{B}, \mu)$. Then*

$$\mu(\phi^{-1}(S_\nu) \triangle S_\nu) = 0 \text{ for all } \nu \in \Lambda(\phi).$$

Proof. First, we show for all $\nu \in \Lambda(\phi)$, $\mu(\phi^{-1}(S_\nu) \triangle S_\nu) = 0$. If not, then set $E_0 := \phi^{-1}(S_\nu) \setminus S_\nu$, $F_0 = \phi(E_0)$ and suppose $\mu(E_0) > 0$. Since ϕ is non–singular, $\mu(F_0) > 0$. Note that $F_0 \subset S_\nu$ and $\mu \sim \nu$ on S_ν, whence $\nu(F_0) > 0$. Note also that $\nu(S_\nu^c) = 0$ and $\nu \circ \phi^{-1} = \nu$ imply $\nu(F_0) = \nu \circ \phi^{-1}(F_0) = \nu(E_0) \leq \nu(S_\nu^c) = 0$. This contradicts $\nu(F_0) > 0$. We have thus shown that $\mu(\phi^{-1}(S_\nu) \setminus S_\nu) = 0$.

Next, we show that $\mu(S_\nu \setminus \phi^{-1}(S_\nu)) = 0$. Indeed, set $E_1 := S_\nu \setminus \phi^{-1}(S_\nu)$, we have $\nu(S_\nu) = \nu(E_1) + \nu(\phi^{-1}(S_\nu) \cap S_\nu)$. At the same time, $\nu(S_\nu) = \nu \circ \phi^{-1}(S_\nu) = \nu(\phi^{-1}(S_\nu) \cap S_\nu) + \nu(E_0)$, where $E_0 := \phi^{-1}(S_\nu) \setminus S_\nu$. Since $\nu(E_0) = 0$ as shown in the first part of the proof, the two equations above imply $\nu(E_1) = 0$, since ν is finite. Finally, by the fact that $\nu \sim \mu$ on S_ν, we have $\mu(S_\nu \setminus \phi^{-1}(S_\nu)) \equiv \mu(E_1) = 0$. □

Now we prove Theorem 2.3. (i) Fix $\phi \in \mathcal{G}$. Note that by Lemma 2.2, there exists $\nu_{\mathcal{G}} \in \Lambda(\phi) \subset I(\mathcal{G})$ such that $S_{\nu_{\mathcal{G}}} = P_{\mathcal{G}}$. Then, by Lemma A.1, $\mu(\phi^{-1}(P_{\mathcal{G}}) \triangle P_{\mathcal{G}}) = 0$. By the fact that all $\phi \in \mathcal{G}$ are invertible, we have that $\phi^{-1}(N_{\mathcal{G}})^c = \phi^{-1}(N_{\mathcal{G}}^c)$ and by the identity $A \triangle B = A^c \triangle B^c$, we have $\mu(\phi^{-1}(N_{\mathcal{G}}) \triangle N_{\mathcal{G}}) = 0$. The previous argument is valid for all $\phi \in \mathcal{G}$.

(ii) Consider $L^1(P_{\mathcal{G}}, \mathcal{B} \cap P_{\mathcal{G}}, \mu|_{P_{\mathcal{G}}})$, where $\mathcal{B} \cap P_{\mathcal{G}} := \{A \cap P_{\mathcal{G}} : A \in \mathcal{B}\}$ and $\mu|_{P_{\mathcal{G}}}$ is the restriction of μ tn $\mathcal{B} \cap P_{\mathcal{G}}$. Define

$$\widetilde{\phi} f(s) \equiv \left[\widetilde{\phi}(f)\right](s) := \frac{\mathrm{d}(\mu \circ \phi^{-1})}{\mathrm{d}\mu}(s) f \circ \phi^{-1}(s) \mathbf{1}_{P_{\mathcal{G}} \cap \phi(P_{\mathcal{G}})}(s) \quad \text{for all } f \in L^1(P_{\mathcal{G}}, \mu|_{P_{\mathcal{G}}}) \,. \tag{A.3}$$

In this way, the mapping $\widetilde{\phi}$ is a restricted version of $\widehat{\phi}$ on $L^1(P_{\mathcal{G}}, \mu|_{P_{\mathcal{G}}})$ in the sense that

$$\widetilde{\phi} f = \widehat{\phi} f \,,\ \mu|_{P_{\mathcal{G}}}\text{-a.e. for all } f \in L^1(P_{\mathcal{G}}, \mu|_{P_{\mathcal{G}}}) \subset L^1(S, \mu) \,. \tag{A.4}$$

Recall that by Lemma 2.2 there exists $\nu \in \Lambda(\mathcal{G})$ such that $\widehat{\phi}(\mathrm{d}\nu/\mathrm{d}\mu) = \mathrm{d}\nu/\mathrm{d}\mu$ for all $\phi \in \mathcal{G}$ and $\mathrm{supp}(\nu) = P_{\mathcal{G}}$. Whence, for $\widetilde{\nu} := \nu|_{P_{\mathcal{G}}}$, we have $\widetilde{\phi}(\mathrm{d}\widetilde{\nu}/\mathrm{d}\mu|_{P_{\mathcal{G}}}) = \mathrm{d}\widetilde{\nu}/\mathrm{d}\mu|_{P_{\mathcal{G}}})$ for all $\phi \in \mathcal{G}$ and $\widetilde{\nu} \sim \mu|_{P_{\mathcal{G}}}$. Note that all locally compact abelian groups are amenable (see, e.g., Example 1.1.5(c) in [32]). Thus, Theorems 1 (part (1) and (8)) in [40] applied to $\widetilde{G}$ and f, implies that

$$\sum_{n=1}^{\infty} \widetilde{\phi}_{u_n} f(s) = \infty \,,\ \mu|_{P_{\mathcal{G}}}\text{-a.e. for all } \{\widetilde{\phi}_{u_n}\}_{n \in \mathbb{N}} \subset \widetilde{\mathcal{G}} \,,$$

which, by (A.4), is equivalent to (2.3).

(iii) Similarly as in (ii), restrict $\mathcal{G}$ to $L^1(N_{\mathcal{G}}, \mathcal{B} \cap N_{\mathcal{G}}, \mu|_{N_{\mathcal{G}}})$ and apply Theorem 2 (part (1) and (8)) in [40].

A.3 Proof of Theorem 2.4

We only sketch the proof of this result.

(i) We apply Theorem 1 (part (1) and (6)) in [40]. Recall that the adjoint operator of $\widehat{\phi}$

$$\widehat{\phi}^* : (L^1)^* \to (L^1)^* \,, \text{ where } (L^1)^* = L^\infty$$

is such that for all $f \in L^1(S, \mu)$ and $h \in L^\infty(S, \mu)$,

$$\int_S f(s)[\widehat{\phi}^*(h)](s)\mu(\mathrm{d}s) = \int_S [\widehat{\phi}(f)](s) h(s) \mu(\mathrm{d}s) \,.$$

The last integral equals

$$\int_S \frac{\mathrm{d}(\mu\circ\phi^{-1})}{\mathrm{d}\mu}(s) f\circ\phi^{-1}(s) h\circ\phi\circ\phi^{-1}(s)\mu(\mathrm{d}s) = \int_S f(s) h\circ\phi(s)\mu(\mathrm{d}s)\,,$$

whence $[\widehat{\phi}^*(h)](s) = h\circ\phi(s)$, μ-a.e.. Thus, if W is a weakly wandering set w.r.t. $\mathcal{G}$, we have

$$\sum_{n=1}^{\infty} \widehat{\phi}^*_{t_n}\mathbf{1}_W(s) < 2 \text{ for some } \{\phi_{t_n}\}_{n\in\mathbb{N}} \subset \mathcal{G}\,.$$

Now, part (6) of Theorem 1 in [40] is equivalent to the nonexistence of a weakly wandering set of positive measure.

(ii) The proof is similar to the proof of Proposition 1.4.7 in [1].

A.4 Proof of Lemma 4.1

Suppose (4.2) holds and $D \in \sigma_{\mathbf{x}}$ such that $\phi_t^{-1}(D) = D$ for all $t \in \mathbb{T}^d$. Taking $A = B = D$ in (4.2) we get $\mathbb{P}(D) = \mathbb{P}(D)^2$, from which $\mathbb{P}(D) = 0$ or 1.

To prove the converse, observe that by Theorem 2.10, it follows that

$$\frac{1}{C(T)}\int_{B(T)} \mathbf{1}_{\phi_t^{-1}(A)}\lambda(\mathrm{d}t) \overset{\text{a.s.}}{\to} \mathbb{P}(A)$$

as $n \to \infty$. Multiplying above by $\mathbf{1}_B$ and using dominated convergence theorem, (4.2) follows.

A.5 Proof of Theorem 4.7.

To prove Theorem 4.7, we first need the following lemma.

Lemma A.2. *Assume $\{X_t\}_{t\in\mathbb{T}^d}$ is a stationary $S\alpha S$ random field with spectral representation $\{f_t\}_{t\in\mathbb{T}^d} \subset L^\alpha(S,\mathcal{B},\mu)$, $\alpha \in (0,2)$. Then, $\{X_t\}_{t\in\mathbb{T}^d}$ is weakly mixing, if and only if, there exists a density one set $D \subset \mathbb{T}^d$, such that*

$$\lim_{n\to\infty} \mu\Big\{s : \Big|\sum_{j=1}^p \beta_j f_{\tau_j}(s)\Big| \in K\,, \Big|\sum_{k=1}^q \gamma_k f_{t_k+t_n^*}(s)\Big| > \epsilon\Big\} = 0$$
$$\text{for all } p,q\in\mathbb{N}, \beta_j,\gamma_k\in\mathbb{R}, \tau_j, t_k \in \mathbb{T}^d\,,$$
$$\text{compact } K \subset \mathbb{R}\setminus\{0\}, \epsilon > 0 \text{ and } \{t_n^*\}_{n\in\mathbb{N}} \in \mathcal{T}_D\,. \quad \text{(A.5)}$$

Proof. It transpires from the proofs in [16] that a stationary process $\{X_t\}_{t\in\mathbb{T}^d}$ is weakly mixing if and only if there exists a density one set $D \subset \mathbb{T}^d$ such that

$$\lim_{n\to\infty} \mathbb{E}\Big[\exp(i\sum_{j=1}^{p}\beta_j X_{\tau_j})\exp(i\sum_{k=1}^{q}\gamma_k X_{t_k+t_n^*})\Big] = \mathbb{E}\exp(i\sum_{j=1}^{p}\beta_j X_{\tau_j})\,\mathbb{E}\exp(i\sum_{k=1}^{q}\gamma_k X_{t_k})$$
$$\text{for all } p,q\in\mathbb{N}, \beta_j,\gamma_k\in\mathbb{R}\,, \tau_j, t_k\in\mathbb{T} \text{ and } \{t_n^*\}_{n\in\mathbb{N}}\in\mathcal{T}_D\,. \quad \text{(A.6)}$$

To prove Lemma A.2 from (A.6), it suffices to follow closely and carefully (see Remark A.3 below) the argument of Gross in [5] (Section 2 therein). □

Remark A.3. Gross's argument, however, is based on the following weaker condition for weak mixing:

$$\lim_{n\to\infty} \mathbb{E}\Big[\exp(i\theta_1 X_0)\exp(i\theta_2 X_{t_n})\Big] = \mathbb{E}\exp(i\theta_1 X_0)\,\mathbb{E}\exp(i\theta_2 X_0)$$
$$\text{for all } \theta_1,\theta_2\in\mathbb{R}, \{t_n\}_{n\in\mathbb{N}}\in\mathcal{T}_D\,, \quad \text{(A.7)}$$

which, according to Gross, follows from the proof of main result in [16]. The equivalence of (A.6) and (A.7) seems nontrivial and yet not mentioned in [5].

Now, in order to complete the proof of Theorem 4.7, it suffices to prove the following lemma.

Lemma A.4. *Assume $\alpha\in(0,2)$ and $\{X_t\}_{t\in\mathbb{T}^d}$ is a stationary $S\alpha S$ process with spectral representation $\{f_t\}_{t\in\mathbb{T}^d}\subset L^\alpha(S,\mathcal{B},\mu)$. Then* (A.5) *is true if and only if* (4.8) *is true.*

Proof. Clearly (A.5) implies (4.8). Now suppose that (4.8) is true. we will show (A.5). For any $p,q\in\mathbb{N}$ and $\tau_j, t_k\in\mathbb{T}^d$, write

$$g_p(s) := \sum_{j=1}^{p}\beta_j f_{\tau_j}(s) \quad \text{and} \quad h_q(s) := \sum_{k=1}^{q}\gamma_k f_{t_k}(s)\,, \quad \text{(A.8)}$$

We will prove (A.5) by induction on (p,q). By (4.8), we have that (A.5) holds for $(p,q)=(1,1)$.

(i) Suppose for fixed (p,q) (A.5) holds, we will show that (A.5) holds for $(p+1,q)$. If not, then there exists $\{t_n^*\}_{n\in\mathbb{N}}\in\mathcal{T}_D$ such that for some compact $K\subset\mathbb{R}\setminus\{0\}$ and $\delta>0$, we have $\mu(E_n)\ge\delta$ with

$$E_n := \Big\{s : |g_p(s)+\beta_{p+1}f_{\tau_{p+1}}(s)|\in K\,, |U_{t_n^*}h_q(s)|>\epsilon\Big\}\,.$$

Here for all $t \in \mathbb{T}^d$,

$$U_t\Big(\sum_{k=1}^{q} \gamma_k f_{t_k}\Big)(s) := \sum_{k=1}^{q} \gamma_k f_{t_k+t}(s)\,.$$

Without loss of generality, we can assume $K \subset (0,\infty)$. Then, since K is compact, there exists $0 < d_K < M$ such that $K \subset [d_K, M]$. Since $f_{\tau_1}, \cdots, f_{\tau_{p+1}} \in L^\alpha(S,\mu)$, we can also choose M to be large enough so that $\mu(E_M^0) \le \delta/2$, where

$$E_M^0 := \Big\{s : |g_p(s)| > M \text{ or } |\beta_{p+1} f_{\tau_{p+1}}(s)| > M\Big\}\,.$$

Then, we claim that for each n, either of the two sets

$$E_n^p := \Big\{s : |g_p(s)| \in \Big[\frac{d_K}{2}, M\Big], |U_{t_n^*} h_q(s)| > \epsilon\Big\}$$

and

$$E_n^{p+1} := \Big\{s : |\beta_{p+1} f_{\tau_{p+1}}(s)| \in \Big[\frac{d_K}{2}, M\Big]\,, |U_{t_n^*} h_g(s)| > \epsilon\Big\}$$

has measure larger than $\delta/4$. Otherwise, observe that

$$E_n \subset E_n^p \cup E_n^{p+1} \cup E_M^0\,,$$

which implies that $\mu(E_n) < \delta$, a contradiction.

It then follows that either $\{E_n^p\}_{n\in\mathbb{N}}$ or $\{E_n^{p+1}\}_{n\in\mathbb{N}}$ will have a subsequence with measures larger than $\delta/4$. Namely, there exists $\{t_{n_k}^*\}_{k\in\mathbb{N}} \in \mathcal{T}_D$ such that

$$\mu(E_{n_k}^p) \ge \frac{\delta}{4} \text{ for all } k \in \mathbb{N} \quad \text{ or } \quad \mu(E_{n_k}^{p+1}) \ge \frac{\delta}{4} \text{ for all } k \in \mathbb{N}\,.$$

But the first case contradicts the assumption that (A.5) holds for (p,q) and the second case contradicts (4.8). We have thus shown that (A.5) holds for $(p+1,q)$.

(ii) Next, suppose (A.5) holds for (p,q) and we show that it holds for $(p,q+1)$. If not, then there exists a compact $K \subset \mathbb{R}\setminus\{0\}$ such that

$$\mu\Big\{s : |g_p(s)| \in K\,, |U_{t_n^*}(h_q + \gamma_{q+1} f_{t_{q+1}})(s)| > \epsilon\Big\} \nrightarrow 0 \text{ as } n \to \infty\,.$$

Then, by a similar argument as in part (i), one can show that for all $\epsilon > 0$, there exists $\{t_n^*\}_{n\in\mathbb{N}} \in \mathcal{T}_D$ and $\delta > 0$ such that we have either

$$\mu\Big\{s : |g_p(s)| \in K, |U_{t_n^*} h_q(s)| > \frac{\epsilon}{2}\Big\} \ge \delta > 0$$

or

$$\mu\Big\{s : |g_p(s)| \in K, |\gamma_{q+1} f_{t_{q+1}+t_n^*}(s)| > \frac{\epsilon}{2}\Big\} \ge \delta > 0\,.$$

Both cases lead to contradictions. We have thus shown that (A.5) holds for $(p,q+1)$. The proof is thus complete. □

References

[1] J. Aaronson. *An Introduction to Infinite Ergodic Theory.* American Mathematical Society, 1997.

[2] S. Cambanis, C.D. Hardin and A. Weron. Ergodic Properties of Stationary Stable Processes, *Stochastic Processes and their Applications*, 24:1–18, 1987.

[3] S. Cohen and G. Samorodnitsky. Random rewards, fractional Brownian local times and stable self-similar processes. *Ann. Appl. Probab.*, 16(3):1432–1461, 2006.

[4] W. Ehm. Sample function properties of multiparameter stable processes. *Z. Wahrsch. Verw. Gebiete*, 56(2):195–228, 1981.

[5] A. Gross. Some mixing conditions for stationary symmetric stable stochastic processes. *Stochastic Process. Appl.*, 51(2):277–295, 1994.

[6] A. Gross and J. B. Robertson. Ergodic properties of random measures on stationary sequences of sets. *Stochastic Process. Appl.*, 46(2):249–265, 1993.

[7] L. de Haan. A spectral representation for max-stable processes. *Ann. Probab.*, 12(4):1194–1204, 1984.

[8] L. de Haan and J. Pickands III, Stationary min–stable stochastic processes, *Probability Theory and Related Fields*, 72(4):477–492, 1986.

[9] L. K. Jones and U. Krengel. On transformations without finite invariant measure. *Advances in Math.*, 12:275–295, 1974.

[10] Z. Kabluchko. Spectral representations of sum– and max–stable processes, to appear in *Extremes (DOI 10.1007/s10687-009-0083-9)*, 2008.
`http://www.springerlink.com/content/ah5622k118686267/fulltext.pdf`

[11] Z. Kabluchko and M. Schlather Ergodic properties of max–infinitely divisible processes *Preprint*, 2009. `http://arxiv.org/abs/0905.4196`

[12] S. Kolodynski and J. Rosinski. Group self-similar stable processes in $\mathbb{R}^d$, *Journal of Theoretical Probability*, 16:855-876, 2003.

[13] U. Krengel. Classification of states for operators. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 2*, pages 415–429. Univ. California Press, Berkeley, Calif., 1967.

[14] U. Krengel. *Ergodic Theorems*. de Gruyter, Berlin, 1985.

[15] G. Lumer. Bochner's theorem, states, and the Fourier transforms of measures. *Studia Math.*, 46:135–140, 1973.

[16] G. Maruyama. Infinitely divisible processes, *Theory of Probability and its Applications*,15(1):1–22, 1970.

[17] J. Neveu. Existence of bounded invariant measures in ergodic theory. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 2*, pages 461–472. Univ. California Press, Berkeley, Calif., 1967.

[18] K. Petersen. *Ergodic theory*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1983.

[19] V. Pipiras and M. S. Taqqu. Stable stationary processes related to cyclic flows. *Ann. Probab.*, 32(3A):2222–2260, 2004.

[20] K. Podgórski. A note on ergodic symmetric stable processes. *Stochastic Process. Appl.*, 43(2):355–362, 1992.

[21] S.I. Resnick *Extreme Values, Regular Variation, and Point Processes* Springer, 1987.

[22] J. Rosiński. On the structure of stationary stable processes. *Ann. Probab.*, 23(3):1163–1187, 1995.

[23] J. Rosiński. Decomposition of stationary α–stable random fields. *Ann. Probab.*, 28:1797–1813, 2000.

[24] J. Rosiński and G. Samorodnitsky. Classes of mixing stable processes. *Bernoulli*, 2(4):365–377, 1996.

[25] J. Rosiński and T Żak. Simple conditions for mixing of infinitely divisible processes, *Stochastic Processes and Their Applications*, 61:277-288, 1996.

[26] J. Rosiński and T Żak. The equivalence of ergodicity and weak mixing for infinitely divisible processes, *Journal of Theoretical Probability*, 10:73-86, 1997.

[27] E. Roy. Ergodic properties of Poissonian ID processes, *Annals of Probability*, 35(2): 551-576, 2007.

[28] E. Roy. Poisson suspensions and infinite ergodic theory, *Ergodic Theory and Dynamical Systems*, 29(2): 667-683, 2009.

[29] P. Roy. Ergodic theory, abelian groups, and point processes induced by stable random fields, to appear in *Annals of Probability*, 2007. http://arxiv.org/abs/0712.0688

[30] P. Roy. Nonsingular Group Actions and Stationary SαS Random Fields, to appear in *Proceedings of the American Mathematical Society*, 2009. http://arxiv.org/abs/0910.2186

[31] P. Roy and G. Samorodnitsky. Stationary symmetric α-stable discrete parameter random fields. *Journal of Theoretical Probability* 21:212–233, 2008.

[32] V. Runde. *Lectures on amenability*, volume 1774 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[33] G. Samorodnitsky. Extreme value theory, ergodic theory, and the boundary between short memory and long memory for stationary stable processes. *Annals of Probability* 32:1438–1468, 2004.

[34] G. Samorodnitsky. Maxima of continuous time stationary stable processes. *Advances in Applied Probability* 36:805–823, 2004.

[35] G. Samorodnitsky. Null flows, positive flows and the structure of stationary symmetric stable processes. *Ann. Probab.*, 33:1782–1803, 2005.

[36] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Random Processes*. Chapman & Hall, 1994.

[37] S. A. Stoev. On the ergodicity and mixing of max-stable processes. *Stochastic Process. Appl.*, 118(9):1679–1705, 2008.

[38] S. A. Stoev and M. S. Taqqu. Extremal stochastic integrals: a parallel between max-stable and alpha-stable processes. *Extremes*, 8(3):237–266, 2006.

[39] D. Surgailis, J. Rosiński, V. Mandrekar and S. Cambanis. Stable mixed moving averages *Probab. Theory Related Fields*, 97:543–558, 1993.

[40] W. Takahashi. Invariant functions for amenable semigroups of positive contractions on L^1. *Kōdai Math. Sem. Rep.*, 23:131–143, 1971.

[41] L. T. Tran. On a problem posed by Orey and Pruitt related to the range of the N-parameter Wiener process in R^d. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 37(1):27–33, 1976/77.

[42] Y. Wang and S. A. Stoev. Association of max–stable processes and sum–stable processes. *Technical Report 488, Department of Statistics, University of Michigan*, 2009. http://arxiv.org/abs/0910.2069

[43] Y. Wang and S. A. Stoev. On the structure and representations of max–stable processes, *Technical Report 487, Department of Statistics, University of Michigan*, 2009. http://arxiv.org/abs/0903.3594

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