Noname manuscript No.
(will be inserted by the editor)

Dynamic portfolio selection with mispricing and model ambiguity

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Received: date / Accepted: date

Abstract We investigate optimal portfolio selection problems with mispricing and model ambiguity under a financial market which contains a pair of mispriced stocks. We assume that the dynamics of the pair satisfies a "cointegrated system" advanced by Liu and Timmermann in a 2013 manuscript. The investor hopes to exploit the temporary mispricing by using a portfolio strategy under a utility function framework. Furthermore, she is ambiguity-averse and has a specific preference for model ambiguity robustness. The explicit solution for such a robust optimal strategy, and its value function, are derived. We analyze these robust strategies with mispricing in two cases: observed and unobserved mean-reverting (MR) stochastic risk premium. We show that the mispricing and model ambiguity have completely distinct impacts on the robust optimal portfolio selection, by comparing the utility losses. We also find that the ambiguity-averse investor (AAI) who ignores the mispricing or the model ambiguity, suffers a substantially larger utility loss if the risk premium is unobserved, compared to when it is observed.

Keywords Portfolio selection · Model ambiguity · Mispricing · Stochastic risk premium · Robust control · Utility maximization

JEL classification: C61, G11, G17, G22.

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1 Introduction

Theoretically, in a market without friction, two assets $P_1(t)$ and $P_2(t)$ with identical or nearly identical contingent claim values at a fixed future date should be traded at the same price or close to the same price during the trading period. Real markets are not frictionless. Hence there can exist remarkable price differences between certain pairs of assets, a phenomenon known as "mispricing". One can find vivid examples of mispricing in certain Chinese corporations traded on both Chinese stock exchanges as shares 'A' and Hong Kong stock exchanges as shares 'H': major examples include Bank of China, and Agricultural Bank of China. Another example is the pair of assets Shell and Royal-Dutch; derivatives on this pair of identical assets trade at significantly different prices despite being contingent claims on the same underlying asset, see Froot and Dabora (1999) for an empirical analysis.

By convention, one can consider that the higher-priced asset is (relatively) overpriced and the lower-priced one is (relatively) underpriced, even if only one price deviates from the normal price level of the asset. A common strategy for taking advantage of this kind of mispricing is to argue that at some (possibly predetermined) point in the future, the prices of the mispriced pair should coincide, and therefore it is wise to adopt "long-short" (LS) strategies. Such strategies take positions of equal size but opposite signs either in portfolio weight or in number of shares, see Shleifer and Vishny (1997), Mitchell and Pulvino (2001), Liu and longstaff (2004) and Jurek and Yang (2007) for LS-strategies work.

Although LS strategies are widely used in both industry and academia, they are designed to exploit long-term arbitrage opportunities, but ignore the exploration of temporary diversification benefits. To exploit mispricing optimally, Liu and Timmermann (2013) placed it under the portfolio maximization framework, and then derived optimal strategy for the investor. In their findings, the optimal strategy is not always of LS type.

On the other hand, a growing body of knowledge in academia is coming to grip with the notorious fact that portfolio optimization based on utility maximization is too sensitive to misspecification of drift components, considering how difficult estimating or calibrating these mean rates of return can be. Uncertainty on the mean rates of return of risky assets in stochastic models is a principal aspect of the so-called model ambiguity problem. It has significant impact on any quantitative method based on such models, and the exploitation of mispricing should be no stranger to this difficulty. The ambiguity-averse investor (AAI) should adjust her decision rules to guard against some adverse scenarios when facing model ambiguity. Rather than consider that the ambiguous parameters are in some bounded set with no information about their veracity, the AAI may instead think of alternative models as being close or distant to the estimated model, considering that the model given by her estimation or calibration technique is the only reference model, even if it contains some misspecification errors. Based on this, Anderson et al. (2000) and Hansen and Sargent (2001) pioneered a framework with model ambiguity or model misspecification for allowing the investor to consider a level of ambiguity aversion, i.e., a quantitative way of saying how confident she is in the reference model. Maenhout (2004, 2006) proposed a "homothetic robustness" assumption, and then derived an analytical optimal strategy for the portfolio optimization problem. Furthermore, he introduced a methodology to examine whether the homothetic robustness assumption is an empirically appropriate way of ambiguity aversion.

The phenomenon of so-called stochastic risk premium, often captured via mean-reverting (MR) models, which can be interpreted as non-stationarity of returns to some extent, and is often referred to as providing time-varying investment opportunities, is an important feature of stock price series, which is abundantly documented in the empirical literature on real market, see Chapter 20 in Cochrane (2001) and Rapach and Zhou (2013). Recently numerous papers studied the robust optimal strategy under both " homothetic robustness" and time-varying investment opportunities, see Maenhout (2006), Liu (2010), Branger et al. (2013), Munk and Rubtsov (2013) and Yi et al. (2013) to refer to only a few. Maenhout (2006) and Liu (2010) assumed that MR risk premium could be observed by the investor, which hardly holds in realistic markets. Branger et al. (2013) assumed that the stochastic risk premium consists of an observed part and an unobserved part estimated via a Bayesian learning method. In this paper, we focus on the combination of two time-varying stochastic factors, the pricing error and stochastic risk premium. We consider both the case when the risk premium process is observed, and the more realistic case where it is unobserved. Our methodology in the case of unobserved risk premium includes use of classical linear stochastic filtering, (Liptser and Shiryaev (2001) can be consulted for this method), which can be understood as a sequential Bayesian framework with explicit time-adaptive posteriors.

Notice that mispricing in a financial market plays a completely different role from model ambiguity. Mispricing can be observed by comparing the prices of a stock pair; but an investor does not know the real distance between the reference model and the true model. In addition, model ambiguity has a remarkable impact on the optimal strategy of the investor trying to exploit market mispricing. We will show that the optimal strategy is dependent on the liquidity parameters for stocks which are very difficult to estimate. Thus, a wise investor should consider ambiguity on the mispricing model when making decisions about investing in the mispriced assets. This is the task we investigate in this paper.

We will set up the model ambiguity along the line of Anderson et al. (2000) and Maenhout (2004). Our work uses Maenhout's "homothetic robustness" assumption to obtain solutions analytically. Specifically, we generate a financial market with pricing error modeled by a "cointegrated system", and propose a portfolio optimization problem under the CRRA utility framework to quantify precisely the performance of the strategy for the investor. Using the stochastic dynamic programming approach, we derive the explicit solution for a robust optimal strategy under market mispricing. Furthermore, our market's risk premium is of MR type. The robust optimal strategies under an observed risk premium and an unobserved risk premium are derived. To analyze the effects of mispricing and of model ambiguity on the portfolio selection, we calculate the utility losses which one would incur if one ignored model ambiguity, and if one ignored the investment opportunities afforded by mispricing. Adequate detection-error probabilities are used, and our analysis is based on empirical data, calibrating parameters from both Chinese and Hong Kong stock exchanges, under both cases of observed and unobserved MR risk premium. Comparing to the previous literature, we think this paper has the following two principal contributions:

(i) It is the first paper to consider model ambiguity for a market with mispricing. Liu and Timmermann (2013) abandoned the LS strategy and derived a non-LS strategy to exploit diversification benefits under the utility function framework. We follow this setup and investigate the effect of model ambiguity robustness on the optimal strategy. We find that model ambiguity does not always decrease the portfolio's absolute positions in the stocks; this is unlike other general settings with model ambiguity, even if the total market risk exposure is diminished by using a methodology which incorporates model-ambiguity robustness.

(ii) We investigate the impacts of introducing MR risk premium on mispricing and model ambiguity. Liu and Timmermann (2013) indicated that when one chooses identical liquidities for the stocks in the pair, the optimal strategy remains an LS strategy. However, that result does not extend to MR risk premium for the stock pair. Moreover, stochastic risk premium is an exogenous factor of the stock-price dynamics rather than an endogenous factor (like a pricing error). We find that the importance of mispricing and ambiguity are much more significant under the (realistic) unobserved case, than under the (idealized) observed case; this points to the importance of including both effects in practical applications.

The rest of this paper is organized as follows. The financial market model with mispriced stocks is described in Section 2. In Section 3, a robust problem for an AAI with CRRA utility is presented and solved under the market with mispricing and constant risk premium. Section 4 investigates the robust optimal strategies under observed and unobserved MR stochastic risk premium. Section 5 provides empirical examples and discusses the utility losses for ignoring mispricing and ignoring ambiguity. Section 6 concludes. The Appendix contains mathematical proofs.

2 Economy and assumptions

We consider a continuous-time financial market with the following assumptions: the investor can trade continuously in time, and trading in the market involves no extra costs or taxes. Let $(\boldsymbol{\Omega}, \mathcal{F}, P)$ be a complete probability space with filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$ generated by four standard one-dimension Brownian motions $\{Z_m(t)\}$, $\{Z(t)\}$, $\{Z_1(t)\}$ and $\{Z_2(t)\}$, which are all independent, and let a positive finite constant T be the maturity. Any decision made at time t is based on $\mathscr{F}_t$, i.e., the information available until time t. $T-t$ represents the horizon at time t (time to maturity).

Assume for the moment that the price of the risk-free asset P_0 is given by

$$dP_0(t) = rP_0(t)dt, \quad P_0(0) = p_0, \tag{1}$$

where constant $r > 0$ is the interest rate, and the price of the risky asset P_m representing the market index has the following diffusive dynamics,

$$\frac{dP_m(t)}{P_m(t)} = (r + \mu_m)dt + \sigma_m dZ_m(t), \quad P_m(0) = p_m, \tag{2}$$

where the risk-premium μ_m and the market volatility σ_m are both constants.

To introduce mispricing as in Liu and Timmermann (2013), we consider a stock pair, which should have similar contingent claims, despite a remarkable price difference in the market. The pair of price processes $(P_1(t), P_2(t))$ evolves according to the following system of stochastic differential equations

$$\frac{dP_1(t)}{P_1(t)} = (r + \beta\mu_m)dt + \beta\sigma_m dZ_m(t) + \sigma dZ(t) + bdZ_1(t) - \lambda_1 X(t)dt, \tag{3}$$
$$P_1(0) = p_1,$$

$$\frac{dP_2(t)}{P_2(t)} = (r + \beta\mu_m)dt + \beta\sigma_m dZ_m(t) + \sigma dZ(t) + bdZ_2(t) + \lambda_2 X(t)dt, \tag{4}$$
$$P_2(0) = p_2,$$

where β, b, λ_1 and λ_2 are constant. The part $\beta\sigma_m dZ_m(t)$ describes the systematic risk of the market, while $\sigma dZ(t) + bdZ_i(t), i = 1, 2$ correspond the idiosyncratic risks, where $\sigma dZ(t)$ is the common idiosyncratic part and $bdZ_i(t)$ is the individual risk for each stock of the pair. $\lambda_i X(t)dt$ represents the mispricing effect, where $X(t) = \ln P_1(t) - \ln P_2(t)$ is the pricing error between the two stocks. The constants $\lambda_i, i = 1, 2$ govern the liquidities of the two stocks, see details in Remark 1. According to Itô's formula, the dynamics of $X(t)$ satisfies the following equation

$$dX(t) = -(\lambda_1 + \lambda_2)X(t)dt + bdZ_1(t) - bdZ_2(t), \quad X(0) = X_0. \tag{5}$$

To capture the features of mispricing, we assume $\lambda_1 + \lambda_2 > 0$. Equation (5) is an MR process with zero long-term mean: the mean-reverting effect comes from the fact that a positive $X(t)$ would tend to pull X down to zero while a negative $X(t)$ has the tendency to push X back up to zero. The time until mean reversion is a random (uncertain) stopping time. These features reasonably capture the properties of financial markets.

Remark 1

(i) Since we assume that $\lambda_1 + \lambda_2 > 0$, hence λ_1 and λ_2 cannot be zero at the same time. It is important to avoid the case of $\lambda_1 = 0$ and $\lambda_2 = 0$, since this would represent a pair with the same expected return rate but higher risk than the market index.

(ii) The constants $\lambda_i, i = 1, 2$ govern the liquidities of two stocks, the situation $\lambda_1 = \lambda_2$ is an approximation for equally liquid stocks, see Section 2.1 in Liu and Timmermann (2013).

3 Robust problem with mispricing

An investor is allowed to invest in the risk-free asset and the market index, as well as in two stocks with mispricing. We assume the investor's decisions never influence the asset prices during the trading process (small investor, no market impact). Denote the portfolio weight on the market index by $\pi_m(t)$ and $(\pi_1(t), \pi_2(t))$ as the weigths of the pair allocated in two stocks. The dynamics of wealth process can be derived as

$$\begin{aligned}\frac{dW(t)}{W(t)} = &\{\mu_m [\pi_m(t) + \beta(\pi_1(t) + \pi_2(t))] - \pi_1(t)\lambda_1 X(t) + \pi_2(t)\lambda_2 X(t) + r\} \, dt \\ &+ \sigma_m [\pi_m(t) + \beta(\pi_1(t) + \pi_2(t))] \, dZ_m(t) + \sigma [\pi_1(t) + \pi_2(t)] \, dZ(t) \\ &+ \pi_1(t) b dZ_1(t) + \pi_2(t) b dZ_2(t), \quad W(0) = W_0.\end{aligned} \tag{6}$$

We assume that the investor's aim is to maximize her expected utility. In the previous literature, the investor is assumed to be an ambiguity-neutral investor (ANI), who pays no heed to model ambiguity, having full confidence in the reference model obtained by statistical estimation or calibration. The ANI maximizes the expected value of the utility function at the maturity T as

$$\max_{\pi \in \overline{\Pi}} \mathrm{E}_0^P \left[\frac{W(T)^{1-\gamma}}{1-\gamma} \right], \tag{7}$$

where $E_t^P[\cdot] = E^P[\cdot \mid \mathcal{F}_t]$ stands for the conditional expectation under a fixed probability measure P, $\gamma \in [0, 1) \bigcup (1, +\infty)$ represents the absolute risk-aversion coefficient and $\overline{\Pi}$ is the set of admissible strategies (see Definition 1). To incorporate model ambiguity, we believe that the investor does not have full confidence in the reference model and hopes to take other possible alternative models into account. Following the methodology proposed in Anderson et al. (2003), we assume that the probability measure P represents the investor's reference probability model, which is presumably estimated with significant errors or misspecification. Hence she is skeptical about this reference model. The alternative models which she hopes to consider are represented as probability measures Q equivalent to P (meaning that they share the same sets of measure 0 as P); she may, in principle, consider all Q in the set of probability measures $\mathcal{Q}$ defined by

$$\mathcal{Q} := \{Q | Q \sim P\}. \tag{8}$$

The celebrated Girsanov theorem provides the investor a solid mathematical framework to choose among all the Q's, by expliciting the relation between

P and any given Q. For each $Q \in \mathcal{Q}$ there exists progressively measurable process $\varphi(t) = (h_m(t), h(t), h_1(t), h_2(t))$ such that

$$\frac{dQ}{dP} = \nu(T),$$

where

$$\nu(t) = \exp\left\{\int_0^t \varphi(s)d\mathbf{Z}(s) - \frac{1}{2}\int_0^t \|\varphi(s)\|^2 ds\right\} \tag{9}$$

is a P-martingale with $d\mathbf{Z}(t) = (dZ_m(t), dZ(t), dZ_1(t), dZ_2(t))^{\mathbf{T}}$ and $\|\varphi(t)\|^2 = h_m^2(t) + h^2(t) + h_1^2(t) + h_2^2(t)$. The reference Karatzas and Shreve (1988) can be consulted for this theorem.

Normally we would assume that $\varphi(t)$ satisfies the so-called Novikov condition to ensure the martingale property for $\nu(t)$. In this paper, for technical reason to ensure Theorem 1 (Verification Theorem), we simply assume that $\varphi(t) = (h_m(t), h(t), h_1(t), h_2(t))$ satisfies the linear growth condition[1] with respect to the pricing error X. Under this assumption, $\nu(t)$ is a P-martingale with filtration $\{\mathscr{F}_t\}_{t\in(0,T)}$. This result is derived in Lemma 2 in Honda and Kamimura (2011), also see Lemma 4.1.1 in Bensoussan (1992). It may seem that the linear growth condition is a restrictive assumption, i.e. restricting $\mathcal{Q}$ and thus constraining decisions artifically. However, we will see that the linear growth condition is actually not a restriction in terms of the AAI's decision, since we will find that the alternative model the AAI chooses is actually built on a $\varphi^*(t)$ which satisfies the linear growth condition. Also recall that, by Girsanov's theorem, under a fixed $Q \in \mathcal{Q}$ with its drift process $\varphi = (h_m, h, h_1, h_2)$, the four processes defined by

$$\begin{aligned} dZ_m^Q(t) =& dZ_m(t) - h_m(t)dt, \quad dZ^Q(t) = dZ(t) - h(t)dt, \\ dZ_1^Q(t) =& dZ_1(t) - h_1(t)dt, \quad dZ_2^Q(t) = dZ_2(t) - h_2(t)dt. \end{aligned}$$

are Brownian motions.

We assume that the AAI attains robustness by guarding against a worst-case scenario $Q^* \in \mathcal{Q}$ and wishes to decide on a robust strategy π^* when facing the worst-case scenario. The worst-case scenario and the robust strategy will be determined in the following way: fixing an admissible strategy π first, we propose a measure $Q^*(\pi) \in \mathcal{Q}$ which provides the smallest utility (worst-case model when strategy π is fixed). Based on Girsanov's theorem, $\varphi(t)$ can induce an alternative model $Q(\pi)$, and the worst-case model $Q^*(\pi)$ under π can be determined by minimizing over $\varphi^*(t)$. Note that this minimization defines $Q^*(\cdot)$ as a function from the set of admissible strategies into $\mathcal{Q}$. Then, we maximize the AAI's utility over all admissible strategies π, which gives us the maximized worst-case value function, attained at a specific optimal strategy π^*. Finally, we say that our worst-case model Q^* is the one corresponding to π^*, namely $Q^* = Q^*(\pi^*)$ by a slight abuse of notation which confuses the function $Q^*(\cdot)$ and its value at π^*.

[1] A function $h : [0, T] \times \mathbb{R}^k \to \mathbb{R}^l$ is said to satisfy the linear growth condition to x if $\|h(t, x)\| \le K(1 + \|x\|)$ for some $K > 0$.

By transferring Brownian motions, the dynamics of the wealth process (6) under an alternative model Q computes as

$$
\begin{aligned}
\frac{dW(t)}{W(t)} =&\Big[\mu_m\pi_f - \pi_1(t)\lambda_1 X(t) + \pi_2(t)\lambda_2 X(t) + r + h_m\sigma_m\pi_f + \sigma(\pi_1+\pi_2)h \\
&+ \pi_1 bh_1 + \pi_2 bh_2\Big]dt + \sigma_m\pi_f dZ_m^Q(t) + \sigma\left[\pi_1(t)+\pi_2(t)\right]dZ^Q(t) \\
&+ \pi_1(t)bdZ_1^Q(t) + \pi_2(t)bdZ_2^Q(t), \quad W(0) = W_0,
\end{aligned}
\tag{10}
$$

with $\pi_f = \pi_m(t) + \beta(\pi_1(t)+\pi_2(t))$. Meanwhile, the dynamics of the mispricing error under Q can be given by

$$
dX(t) = \left[-(\lambda_1+\lambda_2)X(t) + bh_1 - bh_2\right]dt + bdZ_1^Q(t) - bdZ_2^Q(t), \quad X(0) = X_0. \tag{11}
$$

Consequently, we can define the set of the admissible strategies related to our problem as follow.

Definition 1 A trading strategy $\pi(t) = \{(\pi_m(t), \pi_1(t), \pi_2(t)) : t \in [0,T]\}$ and an ambiguity control $\varphi(t)$ are said to be admissible, if
(i) $\forall t \in [0,T]$, $\varphi(t)$ satisfies the linear growth condition with respect to pricing error X,
(ii) $\forall (W_0, X_0) \in \mathbb{R}^+ \times \mathbb{R}$, the corresponding stochastic differential equation (10) has a pathwise unique solution $W^\pi(t)$,
(iii) the progressively measurable π satisfies $\mathrm{E}^{Q(\pi)}\left[\int_0^T |W^\pi(t)|^4 dt\right] < \infty$ and the linear growth condition with respect to X.
Denote by Π the set of all admissible strategies and $\mathcal{H}$ the set of all admissible ambiguity controls .

Note that the set of admissible strategies $\overline{\Pi}$ on page 6 would be a special case in Definition 1 with $\varphi(t) = \mathbf{0}$.

On the other hand, the investor recognizes that P is an approximation of the true model and thinks the alternative models should not deviate too much from the reference model. Therefore, one should use the distance between the alternative model and the reference model to penalize the utility. Combining all the above analysis, we adjust the original problem (7) to a robust control problem inspired by Anderson et al. (2000) and Hanson and Sargent (2001) for the AAI as follows,

$$
\sup_{\pi\in\Pi}\inf_{Q\in\mathcal{Q}} \mathrm{E}^Q\left\{\frac{W(T)^{1-\gamma}}{1-\gamma} + \int_0^T \frac{1}{\phi(s)}R(s)ds\right\}, \tag{12}
$$

where $\phi(t)$ stands for a preference parameter for ambiguity aversion, which measures the degree of suspicion in the reference model P at time t, and $R(t)$ measures the relative entropy between P and Q. Define $R(t) := \frac{1}{2}\|\varphi(t)\|^2$ following Hansen and Sargent (2001), then $E^Q\left[\int_0^T R(s)ds\right]$ measures the discrepancy between P and Q (also see, e.g., Dupuis and Ellis (1997)). With this specification, penalties are incurred for alternative models when they deviate from the reference model. Choosing this penalty in the optimization problem (12) allows one to be able to find an interior point minimizer, i.e. a robust optimal strategy, even if the parameter ranges are unbounded.

In the case $\phi \equiv 0$, the investor is entirely convinced that the true model is the reference model P, any deviation from P will be penalized by $\frac{1}{\phi}R$. Thus, $R \equiv 0$ must be satisfied to guarantee $\frac{1}{\phi}R \equiv 0$ and problem (12) reverts to problem (7), where no model ambiguity is allowed, as it should. At the other extreme, if $\phi \equiv \infty$, the investor has no information about the true model. Since the term $\frac{1}{\phi}R$ vanishes, the scenario will degenerate to an ill-posed robust problem due to the unbounded parameters.

We solve the robust problem (12) by dynamic programming. Define the value function J corresponding to problem (12) as

$$J(t, w, x) = \sup_{\pi \in \Pi} \inf_{Q \in \mathcal{Q}} E^Q_{t,w,x} \left\{ \frac{W(T)^{1-\gamma}}{1-\gamma} + \int_0^T \frac{1}{\phi(s)} R(s) ds \right\}, \tag{13}$$

where $E^Q_{t,w,x}[\cdot] = E^Q[\cdot \mid W(t) = w, X(t) = x]$. To obtain the analytical solution for problem (12), we use the "homothetic robustness" assumption proposed by Maenhout (2004, 2006), and set up the preference parameter ϕ as the function of the state variables (t, w, x) given by

$$\phi(t) = \frac{u}{(1-\gamma)J(t, w, x)}, \tag{14}$$

where $u \geq 0$ is the individual ambiguity-aversion level. The specific form of $\phi(t)$ in (14) gaurantees that the ambiguity penalty remains positive and has the so-called "homotheticity" property, which means that it remains independent of the wealth w. This has an advantage over using a constant ϕ: Maenhout (2004) indicated that the model ambiguity robustness would vanish as the wealth increases if ϕ is constant, which is unrealistic. The structure (14) imposes the desired homotheticity property as robustness will no longer wear off as wealth rises. Moreover, although scaling by some alternative function of wealth could work as well (such as $W^{1-\gamma}$), the specific form chosen here is especially convenient analytically, and as stated in Maenhout (2004), one would expect the same effects of model ambiguity robustness with alternative functions including factors like $W^{1-\gamma}$; more details can be found in Maenhout (2004), pages 959-961.

By penalizing the alternative models via the term $\frac{R(s)}{\phi(s)}$, the AAI is allowed to consider all alternative model drifts, and the "worst-case" Q^* is a specific model which deviates from the reference model P. By looking at the relative entropy measure R, we show how much we are willing to say that P is misspecified. A numerical example for the graph of R is provided in Figure 1. The interpretation here is not that we are computing the level of misspecification in our model, but rather that we are willing to admit that there is a certain amount of misspecification in P. Thus, the entropy R between P and Q^*, which will increase with the ambiguity-aversion level u (see Figure 1), is our allowable level of misspecification.

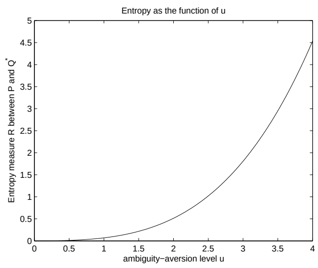


Fig. 1 Fixing $T-t=1$, R can be computed under the following benchmark-parameter values for China Citic Bank: $X=0.2, b=0.3115, \mu_m=0.1597, \lambda_1=-0.2595, \lambda_2=0.4580, \kappa=0.9494, \sigma=0.2687, r=0.02, \gamma=4$.

According to the principle of dynamic programming, the robust Hamilton-Jacobi-Bellmann (HJB) equation established by Anderson et al. (2000) to express the value function (13), can be derived as:

$$\begin{aligned}\sup_{\pi\in\Pi}\inf_{\varphi\in\mathcal{H}}\Bigg\{ & J_t + wJ_w(\mu_m\pi_f - \pi_1\lambda_1 x + \pi_2\lambda_2 x + r + h_m\sigma_m\pi_f + \sigma(\pi_1+\pi_2)h \\ & + \pi_1 bh_1 + \pi_2 bh_2) + \frac{1}{2}w^2 J_{ww}\left(\sigma_m^2{\pi_f}^2 + \sigma^2(\pi_1+\pi_2)^2\right. \\ & \left. + (\pi_1^2+\pi_2^2)b^2\right) + b^2 wJ_{wx}(\pi_1-\pi_2) + J_x\left[-(\lambda_1+\lambda_2)x\right. \\ & \left. + bh_1 - bh_2\right] + J_{xx}b^2 + \frac{(1-\gamma)J}{u}\left(\frac{1}{2}\|\varphi\|^2\right)\Bigg\} = 0, \end{aligned} \tag{15}$$

with the boundary condition $J(T,w,x)=\frac{W^{1-\gamma}}{1-\gamma}$ and the subscripts of J denoting partial derivatives. In Appendix A, the following proposition is established, which solves (15) analytically.

Proposition 1 *In the market with mispricing described by (1)-(4), the value function of the AAI with CRRA utility defined by (13) is given by*

$$J(t,w,x) = \frac{w^{1-\gamma}}{1-\gamma}\exp(G(t) + \frac{1}{2}N(t)x^2), \tag{16}$$

where $G(t)$ and $N(t)$ are time-dependent functions as

$$\begin{aligned} G(t) = & (1-\gamma)\left(r + \frac{\mu_m^2}{2\gamma\sigma_m^2}\right)(T-t) \\ & + \frac{b^2}{A}\ln\left(\frac{2k_2\exp\left((k_1+k_2)(T-t)/2\right)}{2k_2 + (k_1+k_2)\left(\exp\left(k_2(T-t)\right)-1\right)}\right), \end{aligned} \tag{17}$$

$$N(t) = \frac{\exp(k_2(T-t))-1}{2k_2 + (k_1+k_2)(\exp(k_2(T-t))-1)}k_3, \tag{18}$$

and an optimal strategy $\pi = (\pi_m, \pi_1, \pi_2)$ *for the AAI can be given by*

$$\pi_m^* = \frac{\mu_m}{\varGamma\sigma_m^2} + x\frac{\beta(\lambda_1 - \lambda_2)}{\varGamma(2\sigma^2 + b^2)}, \tag{19}$$

$$\pi_1^* = \frac{x}{(2\sigma^2 + b^2)b^2\varGamma}\left(-(b^2 + \sigma^2)\lambda_1 - \sigma^2\lambda_2 + \varGamma_1 b^2(2\sigma^2 + b^2)N(t)\right), \tag{20}$$

$$\pi_2^* = \frac{x}{(2\sigma^2 + b^2)b^2\varGamma}\left((b^2 + \sigma^2)\lambda_2 + \sigma^2\lambda_1 - \varGamma_1 b^2(2\sigma^2 + b^2)N(t)\right), \tag{21}$$

where

$$k_1 = -B, \quad k_2 = \sqrt{B^2 - 4AC}, \quad k_3 = 2C, \quad \varGamma = \gamma + u, \quad \varGamma_1 = \frac{1-\varGamma}{1-\gamma};$$
$$A = \frac{2b^2(1-\varGamma)}{\varGamma(1-\gamma)}, \quad B = -\frac{2(\lambda_1 + \lambda_2)}{\varGamma},$$
$$C = \frac{(1-\gamma)\left[(\lambda_1^2 + \lambda_2^2)(\sigma^2 + b^2) + 2\lambda_1\lambda_2\sigma^2\right]}{\varGamma b^2(2\sigma^2 + b^2)}.$$

The verification theorem (i.e. proving that the solution of the HJB equation is indeed the value function in (13)) is omitted here since we will provide one in Section 4 for a more complicated situation with an MR risk premium, which contains the verification result in this situation. Notice that

$$\pi_s := \pi_1^* + \pi_2^* = x\frac{\lambda_2 - \lambda_1}{(2\sigma^2 + b^2)\varGamma} \tag{22}$$

is the total investment weight in the stocks. The optimal strategy is not an LS strategy unless $x = 0$ or $\lambda_1 = \lambda_2$, which implies that the optimal strategy could be an LS strategy only if the difference between two stocks is eliminated or two stocks have the same market liquidity.

As a general rule in the setting of model ambiguity (see Maenhout (2004, 2006) and Liu (2010)), model ambiguity should lead to a decrease in risk exposures. In our situation, this does holds, provided we interpret the risk exposures as containing two parts: the market index π_m^* and the total investment in the stocks π_s. [2]

Now we analyze the weights in the two stocks with mispricing. Without loss of generality, we assume that $X(t) > 0$ implying $P_1(t) > P_2(t)$. A surprising result discovered by Liu and Timmermann (2013) is that the short-term ANI should take a short position in both stocks when $\lambda_2 < 0$ and a long position in both stocks when $\lambda_1 < 0$, which indicates that the short-term ANI acts myopically to exploit the divergence: see Section 4.3 in Liu and Timmermann (2013). When we focus on the combination of the model ambiguity and mispricing, one can see that the optimal portfolio allocation in the stocks for short-term AAI is also myopic, i.e., the ambiguity aversion would not change the long or short position for the AAI. Furthermore, inserting the specific form

[2] Changes for each stock will be analyzed in Section 4 for a more complex and realistic situation with an MR risk premium.

for $N(t)$ from (18), we find that the "observationally equivalent" property (i.e., Proposition 2 in Maenhout (2006)) is still in effect in the optimal portfolio in stocks for the AAI. Specifically we have the following proposition.

Proposition 2 *The optimal strategy for an ANI under CRRA utility function with risk-aversion coefficient $\Gamma := u+\gamma$ is the same as the optimal strategy for an AAI with ambiguity-aversion level u under CRRA utility with risk-aversion coefficient γ.*

4 Robust optimal portfolio with mispricing and mean-reverting risk premium

In this section, a more realistic market is considered with mispricing: two stocks with an MR risk premium. We show below that even if the two stocks, have the same liquidity, the optimal strategy cannot be of LS type under stochastic MR risk premium. To simplify the model, and to focus on the two mispriced stocks, investing in the market index is not allowed in this section. Specifically, the dynamics of two stocks with mispricing and MR risk premium are

$$\frac{dP_1(t)}{P_1(t)} = (\tilde{r} + a(t))dt + \sigma dZ(t) + bdZ_1(t) - \lambda_1 X(t)dt, \quad P_1(0) = p_1, \tag{23}$$

$$\frac{dP_2(t)}{P_2(t)} = (\tilde{r} + a(t))dt + \sigma dZ(t) + bdZ_2(t) + \lambda_2 X(t)dt, \quad P_2(0) = p_2, \tag{24}$$

where the premium for the common risk part $\sigma dZ(t)$ is the MR process $a(t)$ whose dynamics are given by

$$da(t) = \kappa(\theta - a(t))dt + \sigma_a dZ_a(t), \quad a(0) = a_0, \tag{25}$$

with positive constants: κ, θ, σ_a and Brownian motion $Z_a(t)$ independent of $\{\mathscr{F}_t\}$[3]. Notice the stochastic systemic risk premium $\sigma_m dZ_m(t)$ is incorporated into the common risk part $\sigma dZ(t)$ and $\tilde{r} = r + \beta\mu_m$. Beyond the mathematics, the introduction of $a(t)$ cannot be seen as a simple extension from a one-dimensional $X(t)$ to two dimensions, even though $a(t)$ satisfies MR dynamics similarly to the pricing error $X(t)$. Indeed $X(t)$ can be directly computed from the prices of mispriced stocks while $a(t)$ is an exogenous economic factor which is independent of the stock dynamics (23)-(24). Although many papers assume that the stochastic risk premium is completely observed, this is a theoretical construct; it is far from realistic to make this assumption in the practice of finance. Thus in following subsections, we will investigate the optimal strategies for two situations: observed or unobserved risk premium. We choose to preserve the case of observed MR in our study, for the purpose of comparing it with the unobserved case, and for didactic purposes.

[3] This independence assumption is only for expositional simplicity. Allowing for correlations is a straightforward extension.

4.1 Optimal strategy with observed risk premium

In this subsection, we assume $a(t)$ can be completely observed by the AAI. We apply a methodology simlar to that of Section 3 to derive the explicit optimal strategy for the AAI. Under an alternative model Q, a new Brownian motion Z_a^Q can be defined as

$$dZ_a^Q(t) = dZ_a(t) - h_a(t)dt. \tag{26}$$

Therefore, the wealth process is given by

$$\begin{aligned}\frac{dW(t)}{W(t)} =& \Big[X(t)(\lambda_2\pi_2 - \lambda_1\pi_1) + \tilde{r} + (a(t) + \sigma h)(\pi_1 + \pi_2) + \pi_1 bh_1 + \pi_2 bh_2\Big]dt \\ &+ \sigma\left[\pi_1(t) + \pi_2(t)\right]dZ^Q(t) + \pi_1(t)bdZ_1^Q(t) + \pi_2(t)bdZ_2^Q(t), \quad W(0) = W_0.\end{aligned} \tag{27}$$

The dynamics of the pricing error $X(t)$ can be shown to be

$$dX(t) = \left[-(\lambda_1 + \lambda_2)X(t) + bh_1 - bh_2\right]dt + bdZ_1^Q(t) - bdZ_2^Q(t), \quad X(0) = X_0, \tag{28}$$

while (25) can be modified to

$$da(t) = \left[\kappa(\theta - a(t)) + h_a\sigma_a\right]dt + \sigma_a dZ_a^Q(t), \quad a(0) = a_0. \tag{29}$$

We keep using the notation Π and $\mathcal{H}$ to represent the sets of admissible s-trategies and controls. Notice the definition of admissible strategy is parallel to Definition 1, where the control $\varphi(t)$ is redefined as $\varphi = (h(t), h_1(t), h_2(t), h_a(t))$ and the linear growth condition is extended to the pair (X, a). The AAI's objective is still given by (12), where now the market model contains an observed MR risk premium. Then the HJB equation can be derived as:

$$\begin{aligned}\sup_{\pi\in\Pi}\inf_{\varphi\in\mathcal{H}}\Bigg\{& J_t + wJ_w\left[(\lambda_2\pi_2 - \lambda_1\pi_1)x + \tilde{r} + (a + \sigma h)(\pi_1 + \pi_2) + \pi_1 bh_1 + \pi_2 bh_2\right] \\ &+ \frac{1}{2}w^2 J_{ww}\left(\sigma^2(\pi_1 + \pi_2)^2 + (\pi_1^2 + \pi_2^2)b^2\right) + b^2 wJ_{wx}(\pi_1 - \pi_2) \\ &+ J_{xx}b^2 + \frac{1}{2}J_{aa}\sigma_a^2 + J_x\left(-(\lambda_1 + \lambda_2)x + bh_1 - bh_2\right) \\ &+ J_a\left(\kappa(\theta - a) + h_a\sigma_a\right) + \frac{(1-\gamma)J}{u}\left(\frac{1}{2}\|\varphi\|^2\right)\Bigg\} = 0.\end{aligned} \tag{30}$$

We provide the following verification theorem for problem (12) under the MR risk premium environment.

Theorem 1 *Suppose there exists a function $G(y) \in C^{1,2}(S)$ with $S = [0, T] \times \mathbb{R} \times \mathbb{R}$, and a control $(\pi, \varphi) \in \Pi \times \mathcal{H}$ such that*
(i) the equality in (30) turns to "$\geq$" with π^* for all $\phi \in \mathcal{H}$ for all $y \in S$;

(ii) the equality in (30) turns to "$\leq$" with φ^* for all $\pi \in \Pi$ for all $y \in S$;
(iii) equation (30) holds with π^* and φ^* for all $y \in S$.
Then G is the value function for problem (12) and (π^*, φ^*) is an optimal control.

Proof We may try to adapt the standard verification theorem provided by Theorem 3.2 in Mataramvura and Øksendal (2008); this would mean ignoring a uniform integrability condition for the function $G(y)$ and a convergence condition for the intertemporal discrepancy between two measures: $\int_0^T \frac{1}{\phi(s)} R(s)ds$. However, the proof method in Honda and Kamimura (2010) can be used to avoid the verification of these two conditions. We omit the details. Notice that the linear growth condition ensures that the stochastic process formed as ν in (9) is a martingale, which is the key reason for being able to circumvent the two conditions in Mataramvura and Øksendal (2008).

By searching for a typical ansatz, we derive a closed-form solution to HJB (30), which will be provided in the Appendix A. We obtain the following result.

Proposition 3 *In the market with mispricing and an observed MR risk premium described by (29), the value function of the AAI has the structure*

$$J(t,w,x,a) = \frac{w^{1-\gamma}}{1-\gamma} \exp\left(\frac{1}{2}A_1(t)a^2 + A_2(t)a + A_0(t) + \frac{1}{2}B_1(t)x^2 + B_3(t)ax\right). \tag{31}$$

where A_0, A_1, A_2, B_1 and B_3 are functions of t which satisfy a system of ODEs (70)-(74) in Appendix B. An optimal strategy $\pi = (\pi_1, \pi_2)$ for the AAI is given by

$$\begin{aligned}
\pi_1^*(t) &= \left(\frac{\lambda_2 - \lambda_1}{2(2\sigma^2 + b^2)\Gamma} - \frac{\lambda_1 + \lambda_2}{2\Gamma b^2} + \frac{\Gamma_1 B_1(t)}{\Gamma}\right) x(t) \\
&\quad + \left(\frac{1}{(2\sigma^2 + b^2)\Gamma} + \frac{\Gamma_1 B_3(t)}{\Gamma}\right) a(t), && (32)\\
\pi_2^*(t) &= \left(\frac{\lambda_2 - \lambda_1}{2(2\sigma^2 + b^2)\Gamma} + \frac{\lambda_1 + \lambda_2}{2\Gamma b^2} - \frac{\Gamma_1 B_1(t)}{\Gamma}\right) x(t) \\
&\quad + \left(\frac{1}{(2\sigma^2 + b^2)\Gamma} - \frac{\Gamma_1 B_3(t)}{\Gamma}\right) a(t), && (33)
\end{aligned}$$

where

$$\Gamma = \gamma + u, \quad \Gamma_1 = \frac{1-\Gamma}{1-\gamma}.$$

The optimal strategy on stocks can be divided into two components for each stock. For example, for π_1, the first component is

$$\pi_1^x(t) = \left(\frac{\lambda_2 - \lambda_1}{2(2\sigma^2 + b^2)\Gamma} - \frac{\lambda_1 + \lambda_2}{2\Gamma b^2} + \frac{\Gamma_1 B_1(t)}{\Gamma}\right) x(t).$$

It reflects the exploration of mispricing corresponding the robust optimal strategy derived in (20) in the case of a risk premium with no stochasticity. A related analysis can be found in Section 3. The second component for π_1 is

$$\pi_1^a(t) = \left(\frac{1}{(2\sigma^2 + b^2)\Gamma} + \frac{\Gamma_1 B_3(t)}{\Gamma} \right) a(t).$$

This hedges the stochastic risk premium, linearly in a. This term could be called the "demand for hedging the risk premium", or "MR demand". It disappears and the robust optimal strategy (32) degenerates to (20) if a is identically zero. As stated in Chacko and Viceira (2005), this demand itself can be separated into two parts: the myopic demand $a(t)/[(2\sigma^2 + b^2)\Gamma]$, and intertemporal demand $\Gamma_1 B_3(t)a(t)/\Gamma$ for hedging the stochasticity of the risk premium. For further detail on myopic and intertemporal demands, see Liu and Pan (2003) and Chacko and Viceira (2005).

Although the two stocks share the same MR risk premium, the total investment weight in the stocks with mispricing and stochastic risk premium is not zero even if two stocks have the same liquidity. Specifically, combining (32) and (33), the total investment in stocks can be computed as

$$\pi_s = \pi_1^*(t) + \pi_2^*(t) = \frac{\lambda_2 - \lambda_1}{(2\sigma^2 + b^2)\Gamma} x(t) + \frac{2a(t)}{(2\sigma^2 + b^2)\Gamma}, \tag{34}$$

which implies, even with $\lambda_1 = \lambda_2 = \lambda$, that the robust optimal strategy is not an LS strategy. This conclusion is consistent with the analysis on the structure of demand in the last paragraph above: the AAI should demand that a greater proportion of the total investment be devoted to hedge the time-varying investment opportunity if $a(t)$ is positive. Interestingly, the only part of the total investment in stocks which is independent of investment horizon $T - t$ is the myopic demand. Additionally, the risk exposure of the investment in stocks decreases w.r.t. the ambiguity-aversion level u (as it should), since it is inversely proportional to $\Gamma = \gamma + u$.

4.2 Optimal strategy with unobserved risk premium

In a realistic market, the risk premium is unobserved and the investor can only learn about it from observing realized stock prices. Bayesian learning is a convenient framework for extracting information about a latent continuous-space process via observations. In our linear situation, the density of the unobserved risk premium process conditional on the observed prices is Gaussian, and is given dynamically in time by classical stochastic (Kalman-Bucy) filtering. Specifically, the mean $\hat{a}$ and variance of this Gaussian conditional law are deterministic functions of time; the mean solves a linear ordinary differential equation depending on the observations, and the variance solves a non-linear (Ricatti) equation which is observation-independent, see Kalman and Bucy (1961). To simplify our presentation, as proposed for instance in Branger et al. (2013), we will assume that the variance has already reached its steady state η; this variance indeed converges to this value, and thus it is sufficient to assume that the stock price data has been available for sufficiently long past period.

Then applying Theorem 12.7 in Liptser and Shiryaev (2001), the filtered model of our MR risk premium dynamics (25) is stationary and the entire market has the following dynamics which depend on three Brownian motions, and in which the mean value $\hat{a}$ can be considered as a state variable process:

$$\frac{dP_1(t)}{P_1(t)} = (\tilde{r} + \hat{a}(t))dt + \sigma d\hat{Z}(t) + bd\hat{Z}_1(t) - \lambda_1 X(t)dt, \tag{35}$$

$$\frac{dP_2(t)}{P_2(t)} = (\tilde{r} + \hat{a}(t))dt + \sigma d\hat{Z}(t) + bd\hat{Z}_2(t) + \lambda_2 X(t)dt, \tag{36}$$

$$d\hat{a}(t) = \kappa(\theta - \hat{a}(t))dt + \eta\left(Kd\hat{Z} + K_1 d\hat{Z}_1 + K_2 d\hat{Z}_2\right), \tag{37}$$

where the variance of the filtered process (which coincides with the variance of the estimation error) is

$$\eta = \left(\kappa + \sqrt{\kappa^2 + \frac{2\sigma_a^2}{b^2 + \sigma^2}}\right)(b^2 + \sigma^2), \tag{38}$$

and

$$K = \frac{\sigma}{b^2 + \sigma^2}\left(1 + \frac{1-\rho}{\rho_1}\right), \quad K_1 = \frac{b}{b^2 + \sigma^2}, \quad K_2 = \frac{(1-\rho)b}{\rho_1(b^2 + \sigma^2)}, \tag{39}$$

with $(\rho, \rho_1) = \left(\frac{\sigma^2}{b^2+\sigma^2}, \sqrt{\frac{b^2}{b^2+\sigma^2}}\right)$. Here, the so-called innovations processes $\hat{Z}(t)$, $\hat{Z}_1(t)$, $\hat{Z}_2(t)$ are three independent standard Brownian motions under P. They can be related to the original noises and the unobserved process a by the formulas

$$d\hat{Z}(t) = \frac{\sigma}{\sigma^2 + b^2}(a(t) - \hat{a}(t))dt + dZ(t), \tag{40}$$

$$d\hat{Z}_1(t) = \frac{b}{\sigma^2 + b^2}(a(t) - \hat{a}(t))dt + dZ_1(t), \tag{41}$$

$$d\hat{Z}_2(t) = \frac{b}{\sigma^2 + b^2}(a(t) - \hat{a}(t))dt + dZ_2(t). \tag{42}$$

The filtered model (35)-(37) above is, in the sense of least squares, the best estimate available to the investor given the observed information and the benchmark model. The AAI may still be skeptical about the estimated risk premium, and hopes to use robustness on the filtered model itself. Thus, the AAI will find the robust optimal strategy under objective (12) over the admissible strategy parallel to Definition 1, where $\varphi(t)$ is redefined as $\varphi = (h(t), h_1(t), h_2(t))$ and the linear growth condition is extended to $(X, \hat{a})$. Using a Girsanov shift and the dynamic programming approach to derive a robust optimal strategy under unobserved risk premium, which will be provided in the Appendix C, we achieve the following proposition

Proposition 4 *In the market with mispricing and an unobserved MR risk premium learned from the Kalman-Bucy filter, the value function of the AAI has the structure*

$$\begin{aligned} J(t,w,x,\hat{a}) = \frac{w^{1-\gamma}}{1-\gamma}\exp\bigg(&\frac{1}{2}\hat{A}_1(t)\hat{a}^2+\hat{A}_2(t)\hat{a}+\hat{A}_0(t) \\ &+\frac{1}{2}\hat{B}_1(t)x^2+\hat{B}_2(t)x+\hat{B}_3(t)\hat{a}x\bigg), \end{aligned} \tag{43}$$

where $\hat{A}_0$, $\hat{A}_1$, $\hat{A}_2$, $\hat{B}_1$, $\hat{B}_2$ and $\hat{B}_3$ are functions of t which satisfy the ODE system (82)-(87) in Appendix C. An optimal strategy $\pi=(\pi_1,\pi_2)$ for the AAI can be given by

$$\begin{aligned} \pi_1^*(t) =& \frac{\hat{a}(t)}{(2\sigma^2+b^2)\Gamma}+\frac{(\lambda_2-\lambda_1)x(t)}{2(2\sigma^2+b^2)\Gamma}-\frac{(\lambda_1+\lambda_2)x(t)}{2\Gamma b^2} \\ &+\frac{[\hat{B}_1(t)x(t)+\hat{B}_2(t)+\hat{B}_3\hat{a}(t)]\Gamma_1}{\Gamma} \\ &+\frac{\eta[\hat{A}_1(t)\hat{a}(t)+\hat{A}_2(t)+\hat{B}_3(t)x(t)]\Gamma_1}{(2\sigma^2+b^2)\Gamma}, \end{aligned} \tag{44}$$

$$\begin{aligned} \pi_2^*(t) =& \frac{\hat{a}(t)}{(2\sigma^2+b^2)\Gamma}+\frac{(\lambda_2-\lambda_1)x(t)}{2(2\sigma^2+b^2)\Gamma}+\frac{(\lambda_1+\lambda_2)x(t)}{2\Gamma b^2} \\ &-\frac{[\hat{B}_1(t)x(t)+\hat{B}_2(t)+\hat{B}_3\hat{a}(t)]\Gamma_1}{\Gamma} \\ &+\frac{\eta[\hat{A}_1(t)\hat{a}(t)+\hat{A}_2(t)+\hat{B}_3(t)x(t)]\Gamma_1}{(2\sigma^2+b^2)\Gamma}, \end{aligned} \tag{45}$$

where $\Gamma=\gamma+u$ and $\Gamma_1=(1-\Gamma)/(1-\gamma)$ and η is given by (38).

Comparing to the robust optimal strategy under observed risk premium which is provided by (32)-(33), we separate π_1 into three components. The first component

$$\pi_1^{mpc} = \frac{\hat{a}}{(2\sigma^2+b^2)\Gamma}+\frac{(\lambda_2-\lambda_1)x(t)}{2(2\sigma^2+b^2)\Gamma}-\frac{(\lambda_1+\lambda_2)x(t)}{2\Gamma b^2} \tag{46}$$

stands for the myopic demand independent of horizon $T-t$. The second component

$$\pi_1^{inter} = \frac{[\hat{B}_1(t)x(t)+\hat{B}_2(t)+\hat{B}_3\hat{a}(t)]\Gamma_1}{\Gamma} \tag{47}$$

corresponds the intertemporal demand which hedges the stochastic changes of investment environment and pricing error. These first and second components are parallel to the decomposition of the robust optimal strategy given in (32) for the case of observed risk premium. The novelty in the case of unobserved a is the third component

$$\pi_1^{uobv} = \frac{\eta[\hat{A}_1(t)\hat{a}(t)+\hat{A}_2(t)+\hat{B}_3(t)x(t)]\Gamma_1}{(2\sigma^2+b^2)\Gamma} \tag{48}$$

which hedges changes in the unobserved risk premium $a(t)$. It will vanish if the AAI can obtain absolutely accurate observation of a, i.e. if $\eta = 0$. The total investment in stocks is the sum of π_1^* and π_2^*, which computes as

$$\pi_s = \frac{2\hat{a}(t)}{(2\sigma^2 + b^2)\Gamma} + \frac{(\lambda_2 - \lambda_1)x(t)}{(2\sigma^2 + b^2)\Gamma} + \frac{2\eta[\hat{A}_1(t)\hat{a}(t) + \hat{A}_2(t) + \hat{B}_3(t)x(t)]\Gamma_1}{(2\sigma^2 + b^2)\Gamma}. \tag{49}$$

We will verify in practice that $|\pi_s|$ is a decreasing function of the ambiguity-aversion level u, which implies that the model-ambiguity robustness diminishes the AAI's exposure to market risk.

5 Numerical examples

In this section, we investigate the quantitative effects of mispricing and model ambiguity on the optimal strategy and value function under unobserved risk premium, by examining some numerical examples. Moreover, to show the difference between mispricing and model ambiguity, we compare the utility loss caused by each feature individually.

To connect the empirical data and our theoretical results, we calibrate our parameters to a pair of Chinese bank stocks traded simultaneously as A-shares on the Shanghai stock exchange and as H-shares in Hong Kong. Both shares have the same dividend and voting rights. Consequently any differences in prices when converting these shares into the same currency can be labeled as mispricing, and thus are candidates for our cointegrated model setup.

Table 1 Cointegration estimates for pairs of Chinese Bank A and H shares

Stock	$\tilde{r}$	λ_1	λ_2	σ	b	κ	θ	σ_a
CCB	0.1817	-0.2595	0.4850	0.2687	0.3115	0.9494	0.0132	0.0372
ABC	0.1597	0.8726	-0.6682	0.1601	0.2106	0.4448	0.0112	0.0308
ICBC	0.1817	-0.2595	0.4850	0.2687	0.3115	0.9494	0.0132	0.0372
BC	0.2303	-0.2979	0.4110	0.2051	0.2543	0.9723	0.0404	0.0357

We estimate the parameters for 4 Chinese bank stocks traded in both China A shares and Hong Kong H shares. CCB, ABC, ICBC and BC stands for China Citic Bank (sample period: Apr. 2007-Feb. 2013), Agricultural Bank of China (Jul. 2010-Feb. 2013), Industrial and Commercial Bank of China (Oct. 2006-Feb. 2013) and Bank of China (Jul. 2006-Feb. 2013), respectively. The detailed calculations leading to our statistical methodology (maximum likelihood estimation (MLE)) are in Appendix E. In the remainder of this section, without loss of generality, we focus on the parameters for CCB with a negative λ_1 and a positive λ_2. According to (35)-(36), with a positive pricing error X, the overpriced price tends to increase but the underpriced price will increase even more since we assumed positivity of $\lambda_1 + \lambda_2$, thus ensuring stability (non-explosion) of the MR dynamics in (5), including the existence of a limiting stationary distribution for $X(t)$.

The AAI may have limited confidence on these estimated values, especially for the drifts parameters. One validation measure that would be helpful for her to gauge how accurate the estimations might be, is the set of the estimators' standard errors in a controlled experiment. To compute these, we use a simulation. Assuming the true parameters are $\tilde{r} = 0.15$, $\lambda_1 = \lambda_2 = 0.5$, $\sigma = b = 0.3$, $\kappa = 0.7$, $\theta = 0$, $\sigma_a = 0.5$ and $a_0 = 0$, we simulate 10000 sample paths of $P_1(t)$ and $P_2(t)$ using the Euler-Maruyama scheme, see Maruyama (1955), where the time step $\Delta t = 5$ minutes, then we estimate the parameters of the mispricing model by MLE using data observed at various frequencies. The standard errors of the estimated parameters are shown here.

Table 2 Standard Error of Parameter Estimation

Frequency	Period	$\tilde{r}$	λ_1	λ_2	σ	b	κ	θ	σ_a	a_0
15 mins	1 month	0.6269	0.3128	0.3066	0.0183	0.0117	0.1605	0.2296	0.0225	0.2706
30 mins	1 month	0.6776	0.3443	0.3375	0.0259	0.0165	0.1617	0.2321	0.0224	0.2969
1 hour	1 month	0.7005	0.3958	0.3872	0.0372	0.0232	0.1612	0.2347	0.0224	0.3136
2 hours	1 month	0.7114	0.4642	0.4552	0.0536	0.0327	0.1645	0.2457	0.0223	0.3233
daily	1 month	0.7181	0.5389	0.5283	0.0791	0.0456	0.1710	0.2729	0.0229	0.3339
daily	3 months	0.5547	0.4982	0.5024	0.0432	0.0266	0.1649	0.2975	0.0213	0.2865
daily	6 months	0.4442	0.4826	0.4743	0.0306	0.0191	0.1608	0.3080	0.0201	0.2879
daily	1 year	0.3412	0.4294	0.4115	0.0213	0.0135	0.1572	0.2965	0.0197	0.2833
daily	2 years	0.2808	0.3582	0.3496	0.0149	0.0095	0.1589	0.2570	0.0228	0.2620

Using 1 month of daily data as the benchmark, we can see slight improvements in the standard errors of λ_1 and λ_2 as the observation frequency increases from once a day to once every 15 minutes, and also as the length of the period increases from 1 year to 2 years. Even though λ_1 and λ_2 appear in the drift, they affect both mean and variance of the likelihood function. Surprisingly, the estimation of $\tilde{r}$ also benefits slightly from the high frequency data due to the improvement in λ_1 and λ_2. However, even under the rather high frequency of one data every 15 minutes, the errors for $\tilde{r}$, λ_1, λ_2, κ and θ are quite substantial. This shows that the AAI is correct to be suspicious of the precision of the estimated drift parameters: to be safe, she will want to use a method which is robust against this model ambiguity.

5.1 Detection-error probabilities

To obtain reasonable numerical applications in practice, we should decide upon an adequate value for the AAI's ambiguity-aversion level in our model. Following the principle and methodology proposed by Anderson et al. (2003) and Maenhout (2006), the ambiguity-aversion level u in (14) should be chosen such that the alternative model describing the worst-case scenario seems sufficiently similar to the reference model for the AAI with that u. To make this idea quantitative, one selects u in such a way that the AAI be capable of mistaking one model for the other with sufficiently high probability. Anderson et al. (2003) recommend that this "detection-error probability" be larger than 10%.

We follow the methodology proposed by Maenhout (2006) for comparing P and Q^*. Denote $\xi_{1,t}$ and $\xi_{2,t}$ the logs of the Radon-Nikodym derivatives $\frac{dQ^*}{dP}$ and $\frac{dP}{dQ^*}$ respectively. They compute as

$$\xi_{1,t} = \log\left[\frac{dQ^*}{dP}\right] = \int_0^t \varphi^*(s)\cdot d\mathbf{Z} - \frac{1}{2}\int_0^t \|\varphi^*(s)\|^2 ds, \tag{50}$$

$$\xi_{2,t} = \log\left[\frac{dP}{dQ^*}\right] = -\int_0^t \varphi^*(s)\cdot d\mathbf{Z} + \frac{1}{2}\int_0^t \|\varphi^*(s)\|^2 ds \tag{51}$$

with $d\mathbf{Z}(t) = (dZ_a(t), dZ(t), dZ_1(t), dZ_2(t))^{\mathbf{T}}$ and $\varphi^*(s) = (h_a^*, h^*, h_1^*, h_2^*)$ given by (69) in Appendix B for observed risk premium and $d\mathbf{Z}(t) = (d\hat{Z}_3(t), d\hat{Z}_4(t))^{\mathbf{T}}$ and $\varphi^*(s) = (h_1^*, h_2^*)$ given by (80)-(81) in Appendix C for unobserved risk premium. According to Maenhout (2006), the probability of mistaking P for Q^* or vice-versa is defined as

$$\xi_N(u) = \frac{1}{2}Pr(\xi_{1,N} > 0|P, \mathcal{F}_0) + \frac{1}{2}Pr(\xi_{2,N} > 0|Q^*, \mathcal{F}_0). \tag{52}$$

As mentioned above, Anderson et al. (2003) advocate choosing a value of u such that $\xi_N(u)$ is no less than 10%. Based on the parameters in Table 1 for CCB, we can derive the worst-case scenario Q^* given by (69) for the observed case and (80)-(81) for the unobserved case. As stated in Maenhout (2006), Fourier inversion can then be introduced to compute the detection-error probabilities for various values of u and γ, which are summarized in the table below.

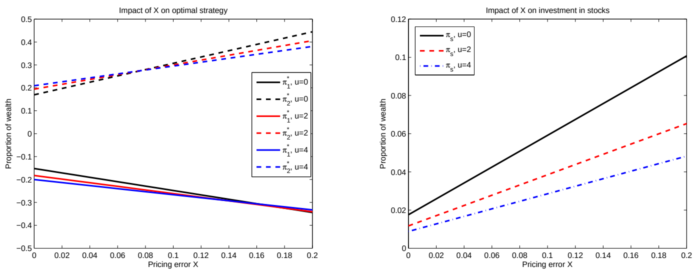
Table 3 Detection-error probabilities

	observed				unobserved			
u	1	2	3	10	1	2	3	10
$\gamma = 2$	0.375	0.324	0.286	0.124	0.452	0.398	0.334	0.297
$\gamma = 4$	0.412	0.382	0.323	0.282	0.475	0.423	0.387	0.321
$\gamma = 20$	0.487	0.423	0.365	0.284	0.554	0.523	0.489	0.427

As we can see from Table 3, where we used $N = 6$ years to coincide with the maximal length of CCB data available to us, any choice of the pair u, γ with our set of benchmark parameters leads to cases where investigating our model's ambiguity robustness is legitimate and even desirable.

5.2 Robust optimal strategy

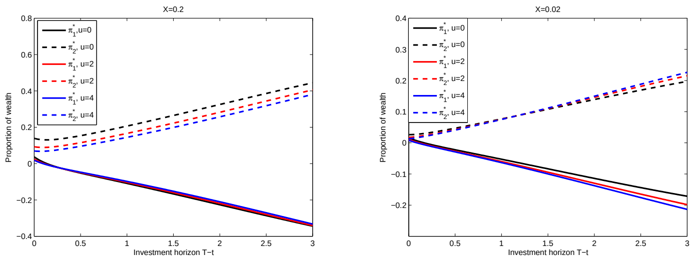
In this subsection, we focus on the impacts of the pricing error X and model ambiguity on the robust optimal strategy (44)-(45). Notice that an AAI with ambiguity-averse level $u = 0$ turns into an ANI, see the explanation for the extreme situation on Page 9. We omit the analysis of the impact of different pairs (λ_1, λ_2) on the optimal strategy, which Liu and Timmermann (2013) emphasized. Maenhout (2004, 2006) showed that model ambiguity can be regarded as another aspect of risk, which decreases exposures to risky assets. Figure 2 shows various effects of model ambiguity on the optimal strategy and the total investment in stocks as a function of the pricing error X. We fix horizon $T - t = 3$ and let the pricing error X vary from zero to 20%. If we look at the total investment in stocks from Figure 2(b), a familiar result for model ambiguity holds: the risk exposure of the AAI in the whole market decreases as the ambiguity-aversion level u increases. On the other hand, by delving into the detail of the portfolio, Figure 2(a) discloses an interesting result which cannot be predicted from a standard model-ambiguity setting: with larger pricing error X, the absolute positions of π_1^* and π_2^* decrease when the ambiguity-aversion level u increases, but when the pricing error X is smaller, higher ambiguity aversion increases the absolute position in each stock.



(a) (b)

Fig. 2 The impacts of mispricing on optimal strategies

To examine the aforementioned distinction between how ambiguity aversion affects risk exposure depending on the pricing error X, we hold X at 0.2, and at 0.02, and the estimated risk premium $\hat{a}$ at the mean $\theta = 0.0132$. Figure 3(a) graphs the risk exposure for various ambiguity levels as a function of the horizon $T-t$, for $X = 0.2$: it diminishes as the ambiguity-aversion increasing, for all horizons. However, Figure 3(b) for $X = 0.02$ shows that the opposite holds for the long-horizon investor. We investigate an explanation for this by analyzing the components of the robust optimal strategy. From the discussion on Page 17, the robust optimal strategy in stocks can be divided to three components: π^{mpc}, π^{inter} and π^{uobv}. Figure 4 examines the changes of each components caused by model ambiguity. Under the case $X = 0.2$, Figures 4(a) and 4(b) show that the risk exposures for all three components of π_1^* or π_2^* decrease with the model ambiguity, which explains the result in Figure 3(a). For $X = 0.02$, considering π_1 as an example, π_1^{mpc} and π_1^{uobv} remain unchanged compared to the case of $X = 0.2$, but the absolute value of π_1^{inter} increases w.r.t. the ambiguity-aversion level; since it dominates the other two components, this explains the inverse impact of ambiguity on the optimal strategy with long horizon. However, the intertemporal components of the two stocks will cancel each other out when one looks at the total investment in stocks, which explains why one still gets the regular result of decreasing risk exposure for the whole market when ambiguity aversion increases.



(a) (b)

Fig. 3 The impacts of model ambiguity on optimal strategies

On the other hand, for positive pricing error X and negative λ_1, (meaning that stock 1 is overpriced and tends to increase, but stock 2 will increase faster to ensure asymptotic stability), although the myopic investment demands a positive position in stock 1 due to the negative λ_1, the intertemporal demand in stock 1 is negative because of the positive pricing error, and will dominate the other demands for long-horizon investment. This coincides with the intuition that the position in the overpriced stock should be negative and the position in the underpriced stock should be positive in the long-horizon investment.

5.3 Utility losses

When the investor applies a non-optimal strategy, a loss of utility will be incurred. Given a specific strategy α, we can define the suboptimal value function under α as

$$J^{\alpha}(t,w,x) = \inf_{Q\in\mathcal{Q}} E^{Q}_{t,W,V}\left\{\frac{W^{\alpha}(T)^{1-\gamma}}{1-\gamma} + \int_0^T \frac{1}{\phi^{\alpha}(s)}R(s)ds\right\}, \tag{53}$$

where $W^{\alpha}(T)$, $\phi^{\alpha}(s)$ are the terminal wealth and a specific preference parameter under strategy α. Furthermore, following Branger et al. (2013), we define the wealth-equivalent utility loss under this specific strategy α as

$$L^{\alpha}(t) := 1 - \left(\frac{J^{\alpha}}{J}\right)^{\frac{1}{1-\gamma}}, \tag{54}$$

where J is the value functions given by (31) or (43) for observed or unobserved risk premium.

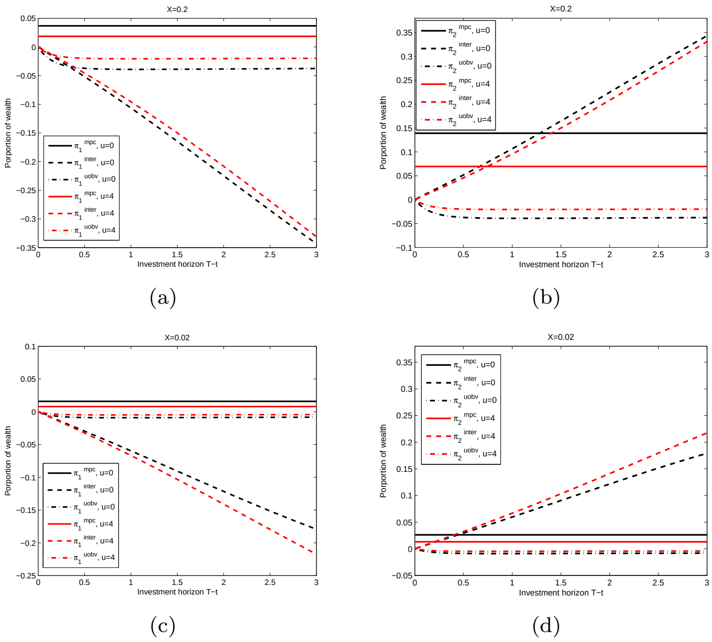


Fig. 4 The impacts of model ambiguity on components of optimal strategies

In this subsection, we quantify the impacts of mispricing and model ambiguity on utility loss. First in the case of the observed risk premium, if the AAI ignores mispricing in the financial market and behaves like an investor in a market without mispricing error $x(t)$, according to (32)-(33), she will follow the specific strategy $\pi^{IM} = (\pi_1^{IM}, \pi_2^{IM})$ given by

$$\pi^{IM}(t) = \left(\left(\frac{1}{(2\sigma^2 + b^2)\Gamma} + \frac{\Gamma_1 B_3(t)}{\Gamma} \right) a(t), \left(\frac{1}{(2\sigma^2 + b^2)\Gamma} - \frac{\Gamma_1 B_3(t)}{\Gamma} \right) a(t) \right). \tag{55}$$

Writing J^{IM} instead of $J^{\pi^{IM}}$ to lighten notation (and similarly elsewhere), consistent with (14), we assume that

$$\phi^{IM}(t) = \frac{u}{(1-\gamma) J^{IM}(t, w, x)}. \tag{56}$$

With the above setup, J^{IM} has the following structure

$$\begin{aligned} J^{IM}(t, w, x, a) = \frac{w^{1-\gamma}}{1-\gamma} \exp \bigg(& \frac{1}{2} A_1^{IM}(t) a^2 + A_2^{IM}(t) a + A_0^{IM}(t) \\ & + \frac{1}{2} B_1^{IM}(t) x^2 + B_3^{IM}(t) a x \bigg), \end{aligned} \tag{57}$$

where $A_1^{IM}(t)$, $A_2^{IM}(t)$, $A_0^{IM}(t)$, $B_1^{IM}(t)$ and $B_3^{IM}(t)$ satisfy an ODE system (88)-(92) in Appendix D. With this closed-form expression for J^{IM}, the utility loss $L^{IM}(t)$ can be obtained via direct calculation.

The quantitative impacts of ignoring model ambiguity are entirely distinct from those of ignoring mispricing. Model ambiguity is a sentiment factor for the AAI instead of an objective factor in the market such as mispricing. Even if the AAI understands mispricing well, if she insists unwisely to invest like an ANI by ignoring model ambiguity, significant utility losses will be incurred: this is equivalent to inserting $u = 0$ into (32)-(33), yielding the optimal strategy (π_1^{IA}, π_2^{IA}) for an ANI as

$$\begin{aligned}\pi_1^{IA}(t) = &\left(\frac{\lambda_2 - \lambda_1}{2(2\sigma^2 + b^2)\gamma} - \frac{\lambda_1 + \lambda_2}{2\gamma b^2} + \frac{B_1(t)}{\gamma}\right) x(t)\\ &+ \left(\frac{1}{(2\sigma^2 + b^2)\gamma} + \frac{B_3(t)}{\gamma}\right) a(t), \end{aligned} \tag{58}$$

$$\begin{aligned}\pi_2^{IA}(t) = &\left(\frac{\lambda_2 - \lambda_1}{2(2\sigma^2 + b^2)\gamma} + \frac{\lambda_1 + \lambda_2}{2\gamma b^2} - \frac{B_1(t)}{\gamma}\right) x(t)\\ &+ \left(\frac{1}{(2\sigma^2 + b^2)\gamma} - \frac{B_3(t)}{\gamma}\right) a(t), \end{aligned} \tag{59}$$

The corresponding utility loss $L^{IA}(t)$ is obtained explicitly from its corresponding value function $J^{IA}(t, w, x, a)$ which computes as

$$\begin{aligned}J^{IA}(t, w, x, a) = \frac{w^{1-\gamma}}{1-\gamma} \exp\Bigg(&\frac{1}{2} A_1^{IA}(t) a^2 + A_2^{IA}(t) a + A_0^{IA}(t)\\ &+ \frac{1}{2} B_1^{IA}(t) x^2 + B_3^{IA}(t) a x\Bigg)\end{aligned} \tag{60}$$

where the functions A_1^{IA}, A_2^{IA}, A_0^{IA}, B_1^{IA} and B_3^{IA} satisfy the system of ODEs (93)-(97) given in Appendix D.

Next we consider the situation of unobserved risk premium, where all relevant quantities are expressed using the state variable $\hat{a}(t)$ rather than $a(t)$. Again, after some effort, all expressions can be given explicitly. If the AAI invests in the pair of stocks as if the market had no mispricing, based on (44)-(45), her strategy is given by

$$\pi_1^{UIM}(t) = \frac{\hat{a}}{(2\sigma^2 + b^2)\Gamma} + \frac{[\hat{B}_2(t) + \hat{B}_3 \hat{a}(t)]\Gamma_1}{\Gamma} + \frac{\eta[\hat{A}_1(t)\hat{a}(t) + \hat{A}_2(t)]\Gamma_1}{(2\sigma^2 + b^2)\Gamma}, \tag{61}$$

$$\pi_2^{UIM}(t) = \frac{\hat{a}}{(2\sigma^2 + b^2)\Gamma} - \frac{[\hat{B}_2(t) + \hat{B}_3 \hat{a}(t)]\Gamma_1}{\Gamma} + \frac{\eta[\hat{A}_1(t)\hat{a}(t) + \hat{A}_2(t)]\Gamma_1}{(2\sigma^2 + b^2)\Gamma}. \tag{62}$$

and the corresponding value function J^{UIM} is

$$J^{UIM}(t,w,x,\hat{a}) = \frac{w^{1-\gamma}}{1-\gamma}\exp\bigg(\frac{1}{2}\hat{A}_1^{UIM}(t)\hat{a}^2 + \hat{A}_2^{UIM}(t)\hat{a} + \hat{A}_0^{UIM}(t) + \frac{1}{2}\hat{B}_1^{UIM}(t)x^2 + \hat{B}_2^{UIM}(t)x + \hat{B}_3^{UIM}(t)\hat{a}x\bigg), \tag{63}$$

with functions $\hat{A}_1^{UIM}$, $\hat{A}_2^{UIM}$, $\hat{A}_0^{UIM}$, $\hat{B}_1^{UIM}$, $\hat{B}_2^{UIM}$ and $\hat{B}_3^{UIM}$ satisfying the system of ODEs (98)-(103) given in Appendix D. If she ignores model ambiguity and acts like an ANI, the strategy computes as

$$\begin{aligned}\pi_1^{UIA}(t) =& \frac{\hat{a}}{(2\sigma^2+b^2)\gamma} + \frac{(\lambda_2-\lambda_1)x(t)}{2(2\sigma^2+b^2)\gamma} - \frac{(\lambda_1+\lambda_2)x(t)}{2\gamma b^2} \\ &+ \frac{[\hat{B}_1(t)x(t)+\hat{B}_2(t)+\hat{B}_3\hat{a}(t)]}{\gamma} \\ &+ \frac{\eta[\hat{A}_1(t)\hat{a}(t)+\hat{A}_2(t)+\hat{B}_3(t)x(t)]}{(2\sigma^2+b^2)\gamma},\end{aligned} \tag{64}$$

$$\begin{aligned}\pi_2^{UIA}(t) =& \frac{\hat{a}}{(2\sigma^2+b^2)\gamma} + \frac{(\lambda_2-\lambda_1)x(t)}{2(2\sigma^2+b^2)\gamma} + \frac{(\lambda_1+\lambda_2)x(t)}{2\gamma b^2} \\ &- \frac{[\hat{B}_1(t)x(t)+\hat{B}_2(t)+\hat{B}_3\hat{a}(t)]}{\gamma} \\ &+ \frac{\eta[\hat{A}_1(t)\hat{a}(t)+\hat{A}_2(t)+\hat{B}_3(t)x(t)]}{(2\sigma^2+b^2)\gamma},\end{aligned} \tag{65}$$

with the corresponding value function J^{UIM} given by

$$\begin{aligned}J^{UIA}(t,w,x,\hat{a}) =& \frac{w^{1-\gamma}}{1-\gamma}\exp\bigg(\frac{1}{2}\hat{A}_1^{UIA}(t)\hat{a}^2 + \hat{A}_2^{UIA}(t)\hat{a} + \hat{A}_0^{UIA}(t) \\ &+ \frac{1}{2}\hat{B}_1^{UIA}(t)x^2 + \hat{B}_2^{UIA}(t)x + \hat{B}_3^{UIA}(t)\hat{a}x\bigg),\end{aligned} \tag{66}$$

where $\hat{A}_1^{UIA}$, $\hat{A}_2^{UIA}$, $\hat{A}_0^{UIA}$, $\hat{B}_1^{UIA}$, $\hat{B}_2^{UIA}$ and $\hat{B}_3^{UIA}$ satisfy a system of ODEs (104)-(109) given in Appendix D. The utility loss functions L^{IA}, L^{UIM} and L^{UIA} are then given via definition (54).

We illustrate these utility losses quantitatively in Figure 5, under various ambiguity-aversion levels, as functions of time to maturity, by holding the market states as $X(t) = 0.2, \hat{a}(t) = a(t) = 0.0132$. Figure 5(a) and Figure 5(b) show that when one ignores mispricing, the utility loss decreases as the investor becomes more ambiguity-averse; this is consistent with the intuition that ambiguity forces the risk exposure downward, so the AAI will reduce the level of exploration of the pricing error. Figure 5(c) and Figure 5(d) show that when one ignores ambiguity, the utility loss increases w.r.t. the ambiguity-aversion level; thus for more conservative investors, acting as an ANI will suffer

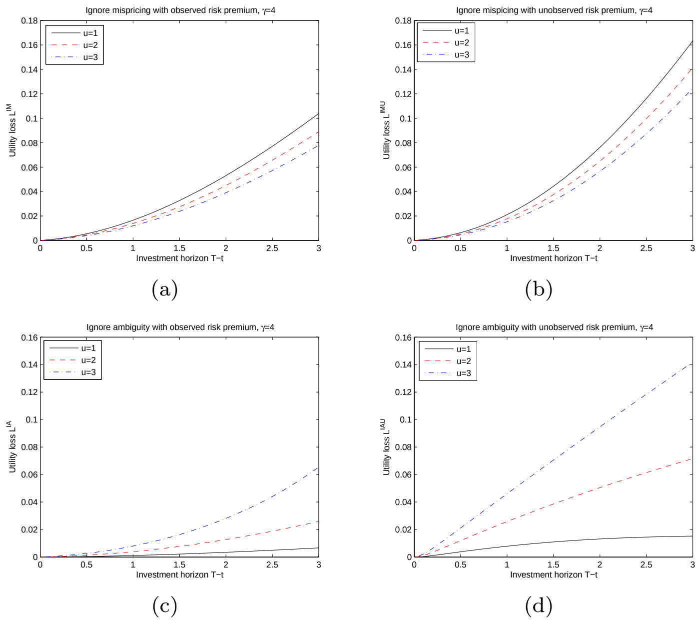


Fig. 5 Utility Losses

more utility loss. In fact, these results are consistent with Figure 2(b) which discloses that risk exposure decreases w.r.t. the ambiguity-aversion level and increases w.r.t. the pricing error.

A comparison between the observed and unobserved cases shows that the utility loss, whether by ignoring mispring or ambiguity, is larger for unobserved risk premium than when the risk-premium is observed; the difference is quite large for longer investment horizons (see $T - t = 3$). This result implies an important recommendation for practitioners: in the realistic situation where risk premia cannot be observed directly, investors should be quite wary of ignoring either mispricing or ambiguity, to avoid significant impacts on their bottom line.

6 Conclusions

In this paper, we investigated optimal strategies for an AAI in a market with mispricing. The prices of a stock pair follow a “cointegrated system” proposed by Liu and Timmermann (2013). At the same time, the AAI may lack full confidence in the model describing the economy, and chooses to consider a “homethetic robust” utility maximization problem in order to model her aversion to ambiguity. Furthermore, we have considered the realistic situation where the AAI can invest in a time-varying financial environment, where specifically the risk premium model is of stochastic mean-reverting (MR) type, and is not necessarily observed.

Under the above assumptions, we derived the explicit robust optimal strategies and corresponding value functions, starting with the case without stochastic risk premium. For the case with MR stochastic risk premium, we allowed two situations: observed risk premium and unobserved risk premium. By analyzing the impacts of mispricing and model ambiguity on optimal strategies and utility losses, the main findings are as follows: (i) Although model ambiguity does not always reduce the absolute value of the position in each stock, it does decrease the overall market exposure, consistent with Maenhout (2004, 2006). (ii) Unlike the result in Liu and Timmermann (2013), under stochastic risk premium, the optimal strategy for the AAI is not a Long-Short strategy even if stock liquidities are identical. (iii) The comparisons of utility losses show that mispricing and model ambiguity are not substitutes for one another. Utility loss by ignoring mispricing decreases w.r.t. ambiguity-aversion level while utility loss by ignoring ambiguity increases w.r.t. ambiguity-aversion level. (iv) Ignoring either source of error is highly unrecommended for longer investment horizons under the realistic situation of unobserved MR risk premium.

Acknowledgement: This research is supported by National Science Foundation of China under Grant (No. 71201173), Humanity and Social Science Foundation of Ministry of Education of China (No. 12YJCZH267), Distinguished Young Talents in Higher Education of Guangdong of China (2012WYM-001) and Fundamental Research Funds for the Central Universities (13wkpy28).

The authors thank the referee for comments and suggestions which improved the paper.

Appendix A

The proof of Proposition 1 and 2: To solve problem (13), we conjecture the corresponding value function has the structure (16). Inserting this structure into the HJB equation (15), by the first-order conditions, the functions h_m^*, h^*, h_1^* and h_2^* which realize the minimum in (15) are given by

$$\begin{aligned}
&h_m^* = -u\sigma_m\pi_f, \quad h^* = -(\pi_1+\pi_2)u\sigma, \\
&h_1^* = -b\pi_1 u - Nxbu/(1-\gamma), \quad h_2^* = Nxbu/(1-\gamma) - b\pi_2 u.
\end{aligned}$$

Substituting the above expressions for h_m^*, h^*, h_1^* and h_2^* into equation (15), by first-order conditions, we can achieve the expression for the optimal strategy π^* shown in (19)-(21). Plugging π^* into (15) implies

$$\begin{aligned}
&\left(\frac{1}{2(1-\gamma)}N_t + \frac{1-\gamma-u}{(\gamma+u)(1-\gamma)^2}b^2N^2 - \frac{\lambda_1+\lambda_2}{(1-\gamma)(\gamma+u)}N\right. \\
&\left.+\frac{(b^2+\sigma^2)(\lambda_1^2+\lambda_2^2)+2\sigma^2\lambda_1\lambda_2}{2(\gamma+u)b^2(2\sigma^2+b^2)}\right)x^2 + \left(\frac{1}{1-\gamma}G_t + \frac{b^2N}{1-\gamma} + \frac{\mu_m^2}{2(\gamma+u)\sigma_m^2} + r\right) = 0
\end{aligned}$$

The above equation is ensured if the following equations are satisfied:

$$
\begin{aligned}
&\frac{1}{2(1-\gamma)}N_t + \frac{1-\gamma-u}{(\gamma+u)(1-\gamma)^2}b^2N^2 - \frac{\lambda_1+\lambda_2}{(1-\gamma)(\gamma+u)}N \\
&+ \frac{(b^2+\sigma^2)(\lambda_1^2+\lambda_2^2)+2\sigma^2\lambda_1\lambda_2}{2(\gamma+u)b^2(2\sigma^2+b^2)} = 0, \qquad (67) \\
&\frac{1}{1-\gamma}G_t + \frac{b^2N}{1-\gamma} + \frac{\mu_m^2}{2(\gamma+u)\sigma_m^2} + r = 0. \qquad (68)
\end{aligned}
$$

One can easily verify that expressions (17) and (18) are the solutions to the above two equations respectively. Moreover, inserting the specific expressions for k_1, k_2 and k_3 to (18), γ would be eliminated in expressions (20) and (21), which implies Proposition 3.3.

Appendix B

The proof of Proposition 3: To solve the HJB equation (30), we conjecture the solution has the structure (31). Inserting (31) to (30), by the first-order condition, we acquire

$$
\begin{aligned}
h^* &= -(\pi_1+\pi_2)u\sigma, \\
h_a^* &= -\frac{\sigma_a u}{1-\gamma}(A_1a + A_2 + B_3x), \\
h_1^* &= -b\pi_1 u - (B_1x + B_3a)bu/(1-\gamma), \\
h_2^* &= (B_1x + B_3a)bu/(1-\gamma) - b\pi_2 u.
\end{aligned} \tag{69}
$$

Parallel to the proof of Proposition 3.2, we substitute the above expressions in the HJB equation (30). By the first-order condition, calculation yields an optimal strategy $\pi^* = (\pi_1^*, \pi_2^*)$ shown in (32)-(33). Plugging the optimal strategy into (30), the right side of equation (30) becomes an affine function of a^2, a, x^2 and ax. The equation has to be satisfied for all values of a and x, which leads to the following system of ODEs:

$$
\begin{aligned}
A_{1t} + & \frac{4(1-\gamma)}{\Gamma(2\sigma^2+b^2)} - \Gamma(1-\gamma)\left[\frac{4\sigma^2}{\Gamma^2(2\sigma^2+b^2)^2} + b^2\left(\frac{1}{\Gamma(2\sigma^2+b^2)} - \frac{\Gamma_1 B_3}{\Gamma}\right)^2\right. \\
& \left. + b^2\left(\frac{1}{\Gamma(2\sigma^2+b^2)} + \frac{\Gamma_1 B_3}{\Gamma}\right)^2\right] + (1-\gamma)\frac{4b^2\Gamma_1^2 B_3^2}{\Gamma} \qquad (70) \\
& + 2b^2 B_3^2 \Gamma_1 + \sigma_a^2 A_1^2 \Gamma_1 - 2\kappa A_1 = 0,
\end{aligned}
$$

$$
A_{2t} + \sigma_a^2 \Gamma_1 A_1 A_2 + \kappa(A_1\theta - A_2) = 0, \qquad (71)
$$

$$
A_{0t} + (1-\gamma)\tilde{r} + b^2 B_1 + \frac{1}{2}\sigma_a^2 \Gamma_1 A_2^2 + \frac{1}{2}\sigma_a^2 A_1 + \kappa A_2 \theta = 0, \qquad (72)
$$

$$
\begin{aligned}
B_{1t} + & (1-\gamma)\left[\frac{(\lambda_2-\lambda_1)^2}{\Gamma(2\sigma^2+b^2)} + \frac{(\lambda_1+\lambda_2)^2}{\Gamma b^2} - \frac{2\Gamma_1 B_1(\lambda_1+\lambda_2)}{\Gamma} - \frac{\sigma^2(\lambda_2-\lambda_1)^2}{\Gamma(2\sigma^2+b^2)^2}\right. \\
& - \Gamma b^2\left(\left(\frac{\lambda_2-\lambda_1}{2\Gamma(2\sigma^2+b^2)} - \frac{\lambda_1+\lambda_2}{2\Gamma b^2} + \frac{\Gamma_1 B_1}{\Gamma}\right)^2\right. \qquad (73) \\
& \left. + \left(\frac{\lambda_2-\lambda_1}{2\Gamma(2\sigma^2+b^2)} + \frac{\lambda_1+\lambda_2}{2\Gamma b^2} - \frac{\Gamma_1 B_1}{\Gamma}\right)^2\right) \\
& \left. + 2\Gamma_1 B_1 \frac{2\Gamma_1 B_1 - (\lambda_1+\lambda_2)}{\Gamma b^2}\right] + 2b^2\Gamma_1 B_1^2 + \sigma_a^2 \Gamma_1 B_3^2 - 2(\lambda_1+\lambda_2)B_1 = 0,
\end{aligned}
$$

$$
\begin{aligned}
B_{3t} + & (1-\gamma)\left\{\frac{\lambda_2-\lambda_1}{\Gamma(2\sigma^2+b^2)} - \frac{\Gamma_1 B_3(\lambda_1+\lambda_2)}{\Gamma} + \frac{\lambda_2-\lambda_1}{\Gamma(2\sigma^2+b^2)} - \frac{2\sigma^2(\lambda_2-\lambda_1)}{\Gamma(2\sigma^2+b^2)^2}\right. \\
& - b^2\left[\frac{2(\lambda_2-\lambda_1)}{\Gamma(2\sigma^2+b^2)^2} + 4\Gamma_1 B_3\left(\frac{\lambda_1+\lambda_2}{2\Gamma b^2} - \frac{\Gamma_1 B_1}{\Gamma}\right)\right] \\
& \left. + b^2\Gamma_1\left(\frac{4\Gamma_1 B_1 B_3}{\Gamma} - \frac{(\lambda_1+\lambda_2)B_3}{\Gamma b^2}\right)\right\} \qquad (74) \\
& + 2\Gamma_1 b^2 B_1 B_3 + \sigma_a^2 \Gamma_1 A_1 B_3 - (\lambda_1+\lambda_2)B_3 - \kappa B_3 = 0,
\end{aligned}
$$

with terminal conditions $A_1(T) = A_2(T) = A_0(T) = B_1(T) = B_3(T) = 0$.

Appendix C

The proof of Proposition 4: To simplify the calculation, we define a new Brownian motion $\hat{Z}_3$ under P as

$$
d\hat{Z}_3 = \frac{\sigma}{\sqrt{\sigma^2+b^2}} d\hat{Z}(t) + \frac{b}{\sqrt{\sigma^2+b^2}} d\hat{Z}_1(t). \qquad (75)
$$

Using standard Gaussian linear regression, the filtered model can be rewritten as

$$\frac{dP_1(t)}{P_1(t)} = (\tilde{r} + \hat{a}(t) - \lambda_1 X(t))dt + \hat{\sigma} d\hat{Z}_3(t), \tag{76}$$

$$\frac{dP_2(t)}{P_2(t)} = (\tilde{r} + \hat{a}(t) + \lambda_2 X(t))dt + \rho\hat{\sigma} d\hat{Z}_3(t) + \sqrt{1-\rho^2}\hat{\sigma} d\hat{Z}_4(t), \tag{77}$$

$$d\hat{a}(t) = \kappa(\theta - \hat{a}(t))dt + \eta\left(\frac{1}{\hat{\sigma}} d\hat{Z}_3(t) + \frac{1-\rho}{\rho_1\hat{\sigma}} d\hat{Z}_4(t)\right), \tag{78}$$

where $\hat{Z}_4$ is a Brownian motion under P independent of $\hat{Z}_3$ and $\hat{\sigma} = \sqrt{\sigma^2 + b^2}$, η is given by (38) and $(\rho, \rho_1) = \left(\frac{\sigma^2}{b^2+\sigma^2}, \sqrt{\frac{b^2}{b^2+\sigma^2}}\right)$. Based on Girsanov's theorem, we define a new Brownian motions under Q as

$$d\hat{Z}_3^Q(t) = d\hat{Z}_3(t) - h_1(t)dt, \quad d\hat{Z}_4^Q(t) = d\hat{Z}_4(t) - h_2(t)dt.$$

Applying the dynamic programming principle, the robust HJB equation for the filtered model can be derived as:

$$\begin{aligned}\sup_{\pi\in\Pi}\inf_{\varphi\in\mathcal{H}} \Bigg\{ & J_t + wJ_w\left[\tilde{r} + (\lambda_2\pi_2 - \lambda_1\pi_1)x + \hat{a}(\pi_1+\pi_2) + h_1\hat{\sigma}(\pi_1 + \rho\pi_2) + h_2\pi_2\rho_1\hat{\sigma}\right] \\ & + J_x\left(-(\lambda_1+\lambda_2)x(t) + h_1\hat{\sigma}(1-\rho) - h_2\rho_1\hat{\sigma}\right) + J_{\hat{a}}\left(\kappa(\theta-\hat{a}) + h_1\frac{\eta}{\hat{\sigma}} + h_2\frac{\eta(1-\rho)}{\rho_1\hat{\sigma}}\right) \\ & + \frac{1}{2}w^2 J_{ww}\hat{\sigma}^2(\pi_1^2 + \pi_2^2 + 2\rho\pi_1\pi_2) + wJ_{wx}\hat{\sigma}(1-\rho)(\pi_1-\pi_2) + \eta(\pi_1+\pi_2)J_{w\hat{a}} \\ & + J_{xx}\hat{\sigma}^2(1-\rho) + \frac{\eta^2(1-\rho)}{\rho_1^2\hat{\sigma}^2}J_{\hat{a}\hat{a}} + \frac{(1-\gamma)J}{2u}\left(h_1^2+h_2^2\right)\Bigg\} = 0.\end{aligned} \tag{79}$$

Substituting the conjecture of the value function (43) into (79), the first-order condition yields

$$\begin{aligned} h_1^* = & -\frac{\eta u}{(1-\gamma)\hat{\sigma}}(\hat{A}_1\hat{a} + \hat{A}_2 + \hat{B}_3 x) - \hat{\sigma}(\pi_1 + \rho\pi_2)u \\ & - \frac{\hat{\sigma}(1-\rho)u}{1-\gamma}(\hat{B}_1 x + \hat{B}_2 + \hat{B}_3\hat{a}), \end{aligned} \tag{80}$$

$$\begin{aligned} h_2^* = & -\frac{\eta u(1-\rho)}{(1-\gamma)\hat{\sigma}\rho_1}(\hat{A}_1\hat{a} + \hat{A}_2 + \hat{B}_3 x) - \hat{\sigma}\rho_1\pi_2 u \\ & + \frac{\rho_1\hat{\sigma}u}{1-\gamma}(\hat{B}_1 x + \hat{B}_2 + \hat{B}_3\hat{a}), \end{aligned} \tag{81}$$

and an optimal strategy shown in (44)-(45). Plugging these into (30), it turns out that our conjecture (43) indeed solves the HJB equation if the functions satisfy the following ODE system:

$$
\begin{aligned}
&\hat{A}_{1t}+\frac{2(1+\eta\Gamma_1\hat{A}_1)^2}{\hat{\sigma}^2(1+\rho)\Gamma}(1-\gamma)-2\kappa\hat{A}_1+2\hat{\sigma}^2(1-\rho)(1-\gamma)\frac{\Gamma_1^2}{\Gamma}\hat{B}_3^2+2\hat{\sigma}^2\Gamma_1(1-\rho)\hat{B}_3^2\\
&\quad+\frac{2(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\hat{A}_1^2=0, \qquad (82)\\
&\hat{A}_{2t}+\kappa\theta\hat{A}_1-\kappa\hat{A}_2+2(1-\rho)(1-\gamma)\hat{\sigma}^2\frac{\Gamma_1^2}{\Gamma}\hat{B}_2\hat{B}_3+2\eta\Gamma_1(1-\gamma)\frac{\hat{A}_2+\eta\Gamma_1\hat{A}_1\hat{A}_2}{\hat{\sigma}^2(1+\rho)\Gamma}\\
&\quad+2\hat{\sigma}^2(1-\rho)\Gamma_1\hat{B}_2\hat{B}_3+\frac{2(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\hat{A}_1\hat{A}_2=0, \qquad (83)\\
&\hat{A}_{0t}+\tilde{r}(1-\gamma)+\kappa\theta\hat{A}_2+\hat{\sigma}^2(1-\rho)\frac{\Gamma_1^2}{\Gamma}\hat{B}_2^2+(1-\gamma)\frac{(\eta\Gamma_1\hat{A}_2)^2}{\hat{\sigma}^2(1+\rho)\Gamma}+\hat{B}_2^2\hat{\sigma}^2\Gamma_1(1-\rho)\\
&\quad+\hat{\sigma}^2(1-\rho)\hat{B}_1+\frac{(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\hat{A}_2^2+\frac{\eta^2(1-\rho)}{\rho_1^2\hat{\sigma}^2}\hat{A}_1=0, \qquad (84)\\
&\hat{B}_{1t}+\frac{(\lambda_2-\lambda_1)^2}{2\hat{\sigma}^2(1+\rho)\Gamma}(1-\gamma)+\frac{(\lambda_1+\lambda_2)^2}{2\hat{\sigma}^2(1-\rho)\Gamma}(1-\gamma)+\frac{2\hat{\sigma}^2(1-\rho)\Gamma_1^2\hat{B}_1^2}{\Gamma}(1-\gamma)\\
&\quad-\frac{2\Gamma_1\hat{B}_1(\lambda_1+\lambda_2)}{\Gamma}(1-\gamma)+\frac{2\eta^2\Gamma_1^2\hat{B}_3^2}{\hat{\sigma}^2(1+\rho)\Gamma}(1-\gamma)+2\hat{\sigma}^2\Gamma_1(1-\rho)\hat{B}_1^2\\
&\quad+2\Gamma_1\frac{(1-\rho)\eta^2}{\rho_1^2\hat{\sigma}^2}\hat{B}_3^2=0, \qquad (85)\\
&\hat{B}_{2t}-(\lambda_1+\lambda_2)+\kappa\theta\hat{B}_3+2\hat{\sigma}^2(1-\gamma)(1-\rho)\frac{\Gamma_1^2}{\Gamma}\hat{B}_1\hat{B}_2-(\lambda_1+\lambda_2)(1-\gamma)\frac{\Gamma_1}{\Gamma}\hat{B}_2\\
&\quad+(1-\gamma)\frac{2\eta^2\Gamma_1^2\hat{A}_2\hat{B}_3}{\hat{\sigma}^2(1+\rho)\Gamma}+2\hat{\sigma}^2(1-\rho)\Gamma_1\hat{B}_1\hat{B}_2+\frac{2(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\hat{A}_2\hat{B}_3\\
&\quad+(1-\gamma)\frac{\eta\Gamma_1(\lambda_2-\lambda_1)}{\hat{\sigma}^2(1+\rho)\Gamma}\hat{A}_2=0, \qquad (86)\\
&\hat{B}_{3t}-\kappa\hat{B}_3+2(1-\gamma)(1-\rho)\hat{\sigma}^2\frac{\Gamma_1^2}{\Gamma}\hat{B}_1\hat{B}_3-(1-\gamma)(\lambda_1+\lambda_2)\frac{\Gamma_1}{\Gamma}\hat{B}_3\\
&\quad+(1-\gamma)(1+\eta\Gamma_1\hat{A}_1)\frac{\lambda_2-\lambda_1}{\hat{\sigma}^2(1+\rho)\Gamma}+(1-\gamma)\frac{2\eta^2\Gamma_1^2\hat{A}_1\hat{B}_3+2\eta\Gamma_1\hat{B}_3}{\hat{\sigma}^2(1+\rho)\Gamma}\\
&\quad+2\hat{\sigma}^2(1-\rho)\Gamma_1\hat{B}_1\hat{B}_3+\frac{2(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\hat{A}_1\hat{B}_3=0, \qquad (87)
\end{aligned}
$$

with terminal conditions $\hat{A}_1(T)=\hat{A}_2(T)=\hat{A}_0(T)=\hat{B}_1(T)=\hat{B}_2(T)=\hat{B}_3(T)=0$.

Appendix D

This section details the calculation of the four value functions under the strategies π^{IM}, π^{IA}, π^{UIM}, π^{UIA} in Section 5.3. We conjecture that the structures of value functions J^{IM} and J^{IA} are given by (57) and (60) for the observed case and (63) and (66) for unobserved case. Set π^{IM} as an example, we hope to solve for the value function J^{IM} under the given strategy for ignoring mispricing π^{IM}. Inserting (55), (69) and (57) into HJB equation (30), we repeat the variables separation method similarly to Appendix B. The following system of ODEs can be derived for (57):

$$
\begin{aligned}
&\left(A_1^{IM}\right)_t + \frac{4(1-\gamma)}{\Gamma(2\sigma^2+b^2)} - \Gamma(1-\gamma)\left[\frac{4\sigma^2}{\Gamma^2(2\sigma^2+b^2)^2} + b^2\left(\frac{1}{\Gamma(2\sigma^2+b^2)} - \frac{\Gamma_1 B_3}{\Gamma}\right)^2\right.\\
&\qquad \left. + b^2\left(\frac{1}{\Gamma(2\sigma^2+b^2)} + \frac{\Gamma_1 B_3}{\Gamma}\right)^2\right] + (1-\gamma)\frac{2b^2\Gamma_1^2 B_3\left(B_3^{IM}\right)}{\Gamma} \qquad (88)\\
&\qquad + 2b^2\left(B_3^{IM}\right)^2\Gamma_1 + \sigma_a^2\left(A_1^{IM}\right)^2\Gamma_1 - 2\kappa\left(A_1^{IM}\right) = 0,\\
&\left(A_2^{IM}\right)_t + \sigma_a^2\Gamma_1\left(A_1^{IM}\right)\left(A_2^{IM}\right) + \kappa\left[\left(A_1^{IM}\right)\theta - \left(A_2^{IM}\right)\right] = 0, \qquad (89)\\
&\left(A_0^{IM}\right)_t + (1-\gamma)\tilde{r} + b^2\left(B_1^{IM}\right) + \frac{1}{2}\sigma_a^2\Gamma_1\left(A_2^{IM}\right)^2 + \frac{1}{2}\sigma_a^2\left(A_1^{IM}\right) + \kappa\theta\left(A_2^{IM}\right) = 0, \qquad (90)\\
&\left(B_1^{IM}\right)_t + 2b^2\Gamma_1\left(B_1^{IM}\right)^2 + \sigma_a^2\Gamma_1\left(B_3^{IM}\right)^2 - 2(\lambda_1+\lambda_2)\left(B_1^{IM}\right) = 0, \qquad (91)\\
&\left(B_3^{IM}\right)_t + (1-\gamma)\left\{\frac{\lambda_2-\lambda_1}{\Gamma(2\sigma^2+b^2)} - \frac{\Gamma_1 B_3(\lambda_1+\lambda_2)}{\Gamma} + \frac{2b^2\Gamma_1\left(B_1^{IM}\right)B_3}{\Gamma}\right\}\\
&\qquad + 2\Gamma_1 b^2\left(B_1^{IM}\right)\left(B_3^{IM}\right) + \sigma_a^2\Gamma_1\left(A_1^{IM}\right)\left(B_3^{IM}\right)\\
&\qquad - (\lambda_1+\lambda_2)\left(B_3^{IM}\right) - \kappa\left(B_3^{IM}\right) = 0, \qquad (92)
\end{aligned}
$$

with terminal conditions $A_1^{IM}(T) = A_2^{IM}(T) = A_0^{IM}(T) = B_1^{IM}(T) = B_3^{IM}(T) = 0$ and A_1, A_2, A_3, B_1 and B_3 satisfy (70)-(74). Similarly, given the strategy π^{IA}, we obtain the following system of ODEs for the functions used in J^{IA} in (60).

$$
\begin{aligned}
&\left(A_1^{IA}\right)_t + \frac{4(1-\gamma)}{\gamma(2\sigma^2+b^2)} - \Gamma(1-\gamma)\left[\frac{4\sigma^2}{\gamma^2(2\sigma^2+b^2)^2} + b^2\left(\frac{1}{\gamma(2\sigma^2+b^2)} - \frac{\gamma_1 B_3}{\gamma}\right)^2\right.\\
&\qquad \left. + b^2\left(\frac{1}{\gamma(2\sigma^2+b^2)} + \frac{B_3}{\gamma}\right)^2\right] + (1-\gamma)\frac{4b^2\Gamma_1^2 B_3\left(B_3^{IA}\right)}{\Gamma}\\
&\qquad + 2b^2\left(B_3^{IA}\right)^2\Gamma_1 + \sigma_a^2\left(A_1^{IA}\right)^2\Gamma_1 - 2\kappa\left(A_1^{IA}\right) = 0, \qquad (93)\\
&\left(A_2^{IA}\right)_t + \sigma_a^2\Gamma_1\left(A_1^{IA}\right)\left(A_2^{IA}\right) + \kappa\left[\left(A_1^{IA}\right)\theta - \left(A_2^{IA}\right)\right] = 0, \qquad (94)\\
&\left(A_0^{IA}\right)_t + (1-\gamma)\tilde{r} + b^2\left(B_1^{IA}\right) + \frac{1}{2}\sigma_a^2\Gamma_1\left(A_2^{IA}\right)^2 + \frac{1}{2}\sigma_a^2\left(A_1^{IA}\right) + \kappa\theta\left(A_2^{IA}\right) = 0, \qquad (95)
\end{aligned}
$$

$$
\begin{aligned}
&\left(B_1^{IA}\right)_t + (1-\gamma)\Bigg\{ \frac{(\lambda_2-\lambda_1)^2}{\gamma(2\sigma^2+b^2)} + \frac{(\lambda_1+\lambda_2)^2}{\gamma b^2} - \frac{2B_1(\lambda_1+\lambda_2)}{\gamma} - \frac{\sigma^2(\lambda_2-\lambda_1)^2}{\gamma(2\sigma^2+b^2)^2} \\
&\quad - \Gamma b^2 \left[\left(\frac{\lambda_2-\lambda_1}{2\gamma(2\sigma^2+b^2)} - \frac{\lambda_1+\lambda_2}{2\gamma b^2} + \frac{\Gamma_1 B_1}{\gamma} \right)^2 \right. \\
&\quad \left. + \left(\frac{\lambda_2-\lambda_1}{2\gamma(2\sigma^2+b^2)} + \frac{\lambda_1+\lambda_2}{2\gamma b^2} - \frac{\Gamma_1 B_1}{\gamma} \right)^2 \right] \\
&\quad + 2\Gamma_1 \left(B_1^{IA}\right) \frac{2\Gamma_1 B_1 - (\lambda_1+\lambda_2)}{\gamma b^2} \Bigg\} + 2b^2\Gamma_1 \left(B_1^{IA}\right)^2 \\
&\quad + \sigma_a^2 \Gamma_1 \left(B_3^{IA}\right)^2 - 2(\lambda_1+\lambda_2)\left(B_1^{IA}\right) = 0,
\end{aligned}
\tag{96}
$$

$$
\begin{aligned}
&\left(B_3^{IA}\right)_t + (1-\gamma)\Bigg\{ \frac{\lambda_2-\lambda_1}{\gamma(2\sigma^2+b^2)} - \frac{B_3(\lambda_1+\lambda_2)}{\gamma} + \frac{\lambda_2-\lambda_1}{\gamma(2\sigma^2+b^2)} - \frac{2\sigma^2(\lambda_2-\lambda_1)}{\gamma(2\sigma^2+b^2)^2} \\
&\quad - b^2\Gamma \left[\frac{2(\lambda_2-\lambda_1)}{\gamma(2\sigma^2+b^2)^2} + \frac{4B_3}{\gamma} \left(\frac{\lambda_1+\lambda_2}{2\gamma b^2} - \frac{\Gamma_1 B_1}{\gamma} \right) \right] \\
&\quad + b^2\Gamma_1 \left(\frac{2\Gamma_1 B_1 \left(B_3^{IA}\right) + 2\Gamma_1 B_3 \left(B_1^{IA}\right)}{\gamma} - \frac{(\lambda_1+\lambda_2)B_3}{\gamma b^2} \right) \Bigg\} \\
&\quad + 2\Gamma_1 b^2 \left(B_1^{IA}\right)\left(B_3^{IA}\right) + \sigma_a^2\Gamma_1 \left(A_1^{IA}\right)\left(B_3^{IA}\right) - (\lambda_1+\lambda_2)\left(B_3^{IA}\right) - \kappa\left(B_3^{IA}\right) = 0,
\end{aligned}
\tag{97}
$$

with terminal conditions $A_1^{IA}(T) = A_2^{IA}(T) = A_0^{IA}(T) = B_1^{IA}(T) = B_3^{IA}(T) = 0$ and A_1, A_2, A_3, B_1 and B_3 satisfy (70)-(74). In the unobserved case, given the strategy π^{UIM}, the system of ODEs needed to express the function J^{UIM} in (63) is

$$
\begin{aligned}
&\left(\hat{A}_1^{UIM}\right)_t + \frac{4(1+\eta\Gamma_1\hat{A}_1)}{\hat{\sigma}^2(1+\rho)\Gamma}(1-\gamma) + 4\eta\Gamma_1\left(\hat{A}_1^{UIM}\right)\frac{(1+\eta\Gamma_1\hat{A}_1)}{\hat{\sigma}^2(1+\rho)\Gamma}(1-\gamma) \\
&\quad + 2\hat{\sigma}^2(1-\rho)(1-\gamma)\frac{\Gamma_1^2}{\Gamma}\hat{B}_3\left(\hat{B}_3^{UIM}\right) + 2\hat{\sigma}^2\Gamma_1(1-\rho)\left(\hat{B}_3^{UIM}\right)^2 \\
&\quad + \frac{2(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\left(\hat{A}_1^{UIM}\right)^2 - \hat{\sigma}^2\Gamma\left[\left(\frac{(1+\eta\Gamma_1\hat{A}_1)}{\hat{\sigma}^2(1+\rho)\Gamma}\right)^2(1+\rho)\right. \\
&\quad \left. + \left(\frac{\hat{B}_3\Gamma_1}{\Gamma}\right)^2(1-\rho) - 2\kappa\left(\hat{A}_1^{UIM}\right)\right] = 0,
\end{aligned}
\tag{98}
$$

$$
\begin{aligned}
&\left(\hat{A}_2^{UIM}\right)_t + (1-\gamma)\frac{2\eta\hat{A}_2\Gamma_1}{\hat{\sigma}^2(1+\rho)\Gamma} + \kappa\theta\left(\hat{A}_1^{UIM}\right) - \kappa\left(\hat{A}_2^{UIM}\right) \\
&\qquad + 2(1-\rho)(1-\gamma)\hat{\sigma}^2\frac{\Gamma_1^2}{\Gamma}\left[\left(\hat{B}_2^{UIM}\right)\hat{B}_3 + \left(\hat{B}_3^{UIM}\right)\hat{B}_2\right] \\
&\qquad + 2\eta(1-\gamma)\frac{\Gamma_1}{\Gamma}\left[\frac{1+\eta\hat{A}_1\Gamma_1}{\hat{\sigma}^2(1+\rho)}\left(\hat{A}_2^{UIM}\right) + \frac{2\eta\Gamma_1\hat{A}_2}{\hat{\sigma}^2(1+\rho)}\left(\hat{A}_1^{UIM}\right)\right] \\
&\qquad + 2\hat{\sigma}^2(1-\rho)\Gamma_1\left(\hat{B}_2^{UIM}\right)\left(\hat{B}_3^{UIM}\right) + \frac{2(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\left(\hat{A}_1^{UIM}\right)\left(\hat{A}_2^{UIM}\right) \\
&\qquad - \frac{2(1+\eta\Gamma_1\hat{A}_1)\eta\hat{A}_2\Gamma_1}{\hat{\sigma}^2(1+\rho)\Gamma} - 2(1-\rho)\hat{\sigma}^2\frac{\hat{B}_2\hat{B}_3\Gamma_1^2}{\Gamma} = 0,
\end{aligned} \tag{99}
$$

$$
\begin{aligned}
&\left(\hat{A}_0^{UIM}\right)_t + \tilde{r}(1-\gamma) + \kappa\theta\left(\hat{A}_2^{UIM}\right) + \hat{\sigma}^2(1-\rho)\frac{\Gamma_1^2}{\Gamma}\hat{B}_2\hat{B}_2^{UIM} \\
&\qquad + \hat{\sigma}^2(1-\rho)\left(\hat{B}_1^{UIM}\right) + \frac{(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\left(\hat{A}_2^{UIM}\right)^2 + \frac{\eta^2(1-\rho)}{\rho_1^2\hat{\sigma}^2}\left(\hat{A}_1^{UIM}\right) \\
&\qquad + \hat{\sigma}^2\Gamma_1(1-\rho)\left(\hat{B}_2^{UIM}\right)^2 + (1-\gamma)\frac{2\eta^2\Gamma_1^2\hat{A}_2}{\hat{\sigma}^2(1+\rho)\Gamma}\hat{A}_2^{UIM} \\
&\qquad - \hat{\sigma}^2\Gamma\left[\left(\frac{\hat{A}_2\eta\Gamma_1}{\hat{\sigma}^2(1+\rho)\Gamma}\right)^2 + \left(\frac{\hat{B}_2\Gamma_1}{\Gamma}\right)^2(1-\rho)\right] = 0,
\end{aligned} \tag{100}
$$

$$
\left(\hat{B}_1^{UIM}\right)_t + 2\hat{\sigma}^2\Gamma_1(1-\rho)\left(\hat{B}_1^{UIM}\right)^2 + 2\Gamma_1\frac{(1-\rho)\eta^2}{\rho_1^2\hat{\sigma}^2}\left(\hat{B}_3^{UIM}\right)^2 = 0, \tag{101}
$$

$$
\begin{aligned}
&\left(\hat{B}_2^{UIM}\right)_t - (\lambda_1+\lambda_2) + \kappa\theta\left(\hat{B}_3^{UIM}\right) + 2\hat{\sigma}^2(1-\gamma)(1-\rho)\frac{\Gamma_1^2}{\Gamma}\left(\hat{B}_1^{UIM}\right)\hat{B}_2 \\
&- (\lambda_1+\lambda_2)(1-\gamma)\frac{\Gamma_1}{\Gamma}\hat{B}_2 + (1-\gamma)\frac{2\eta^2\Gamma_1^2\left(\hat{A}_2^{UIM}\right)\left(\hat{B}_3^{UIM}\right)}{\hat{\sigma}^2(1+\rho)\Gamma} \\
&+ 2\hat{\sigma}^2(1-\rho)\Gamma_1\left(\hat{B}_1^{UIM}\right)\left(\hat{B}_2^{UIM}\right) + \frac{2(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\left(\hat{A}_2^{UIM}\right)\left(\hat{B}_3^{UIM}\right) \\
&+ (1-\gamma)\frac{\eta\Gamma_1(\lambda_2-\lambda_1)}{\hat{\sigma}^2(1+\rho)\Gamma}\hat{A}_2 = 0,
\end{aligned} \tag{102}
$$

$$
\begin{aligned}
&\left(\hat{B}_3^{UIM}\right)_t - \kappa\left(\hat{B}_3^{UIM}\right) + 2(1-\gamma)(1-\rho)\hat{\sigma}^2\frac{\Gamma_1^2}{\Gamma}\left(\hat{B}_1^{UIM}\right)\hat{B}_3 - (1-\gamma)(\lambda_1+\lambda_2)\frac{\Gamma_1}{\Gamma}\hat{B}_3 \\
&\qquad + (1-\gamma)(1+\eta\Gamma_1\hat{A}_1)\frac{\lambda_2-\lambda_1}{\hat{\sigma}^2(1+\rho)\Gamma} + (1-\gamma)\frac{2\eta^2\Gamma_1^2\hat{A}_1 + 2\eta\Gamma_1}{\hat{\sigma}^2(1+\rho)\Gamma}\left(\hat{B}_3^{UIM}\right) \\
&\qquad + 2\hat{\sigma}^2(1-\rho)\Gamma_1\left(\hat{B}_1^{UIM}\right)\left(\hat{B}_3^{UIM}\right) + \frac{2(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\left(\hat{A}_1^{UIM}\right)\left(\hat{B}_3^{UIM}\right) = 0,
\end{aligned} \tag{103}
$$

with terminal conditions $\hat{A}_1^{UIM}(T) = \hat{A}_2^{UIM}(T) = \hat{A}_0^{UIM}(T) = \hat{B}_1^{UIM}(T) = \hat{B}_2^{UIM}(T) = \hat{B}_3^{UIM}(T) = 0$ and $\hat{A}_1$, $\hat{A}_2$, $\hat{A}_3$, $\hat{B}_1$, $\hat{B}_2$ and $\hat{B}_3$ satisfy (82)-(87). Finally, again in the unobserved case, given the strategy π^{UIA}, the functions needed to express J^{UIA} in (66) solve the following system of ODEs:

$$\begin{aligned}
&\left(\hat{A}_1^{UIA}\right)_t + \frac{4(1+\eta\hat{A}_1)}{\hat{\sigma}^2(1+\rho)\gamma}(1-\gamma) + 4\eta\Gamma_1\left(\hat{A}_1^{UIA}\right)\frac{(1+\eta\hat{A}_1)}{\hat{\sigma}^2(1+\rho)\gamma}(1-\gamma) - 2\kappa\left(\hat{A}_1^{UIA}\right)\\
&\quad + 2\hat{\sigma}^2(1-\rho)(1-\gamma)\frac{\Gamma_1}{\gamma}\hat{B}_3\left(\hat{B}_3^{UIA}\right) + 2\hat{\sigma}^2\Gamma_1(1-\rho)\left(\hat{B}_3^{UIA}\right)^2\\
&\quad + \frac{2(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\left(\hat{A}_1^{UIA}\right)^2 - \hat{\sigma}^2\Gamma\left[\left(\frac{(1+\eta\hat{A}_1)}{\hat{\sigma}^2(1+\rho)\gamma}\right)^2(1+\rho)\right.\\
&\quad \left. + \left(\frac{\hat{B}_3}{\gamma}\right)^2(1-\rho)\right] = 0,
\end{aligned} \tag{104}$$

$$\begin{aligned}
&\left(\hat{A}_2^{UIA}\right)_t + (1-\gamma)\frac{2\eta\hat{A}_2}{\hat{\sigma}^2(1+\rho)\gamma} + \kappa\theta\left(\hat{A}_1^{UIA}\right) - \kappa\left(\hat{A}_2^{UIA}\right)\\
&\quad + 2(1-\rho)(1-\gamma)\hat{\sigma}^2\frac{\Gamma_1}{\gamma}\left[\left(\hat{B}_2^{UIA}\right)\hat{B}_3 + \left(\hat{B}_3^{UIA}\right)\hat{B}_2\right]\\
&\quad + 2\eta(1-\gamma)\frac{\Gamma_1}{\gamma}\left[\frac{1+\eta\hat{A}_1}{\hat{\sigma}^2(1+\rho)}\left(\hat{A}_2^{UIA}\right) + \frac{2\eta\ \hat{A}_2}{\hat{\sigma}^2(1+\rho)}\left(\hat{A}_1^{UIA}\right)\right]\\
&\quad + 2\hat{\sigma}^2(1-\rho)\Gamma_1\left(\hat{B}_2^{UIA}\right)\left(\hat{B}_3^{UIA}\right) + \frac{2(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\left(\hat{A}_1^{UIA}\right)\left(\hat{A}_2^{UIA}\right)\\
&\quad - \frac{2(1+\eta\hat{A}_1)\eta\hat{A}_2\Gamma}{\hat{\sigma}^2(1+\rho)\gamma^2} - 2(1-\rho)\hat{\sigma}^2\frac{\hat{B}_2\hat{B}_3\Gamma}{\gamma^2} = 0,
\end{aligned} \tag{105}$$

$$\begin{aligned}
&\left(\hat{A}_0^{UIA}\right)_t + \tilde{r}(1-\gamma) + \kappa\theta\left(\hat{A}_2^{UIA}\right) + \hat{\sigma}^2(1-\rho)\frac{\Gamma_1}{\gamma}\hat{B}_2\hat{B}_2^{UIA} + (1-\gamma)\frac{2\eta^2\Gamma_1\hat{A}_2}{\hat{\sigma}^2(1+\rho)\gamma}\hat{A}_2^{UIA}\\
&\quad + \hat{\sigma}^2(1-\rho)\left(\hat{B}_1^{UIA}\right) + \frac{(1-\rho)\eta^2\Gamma_1}{\rho_1^2\hat{\sigma}^2}\left(\hat{A}_2^{UIA}\right)^2 + \frac{\eta^2(1-\rho)}{\rho_1^2\hat{\sigma}^2}\left(\hat{A}_1^{UIA}\right)\\
&\quad + \hat{\sigma}^2\Gamma_1(1-\rho)\left(\hat{B}_2^{UIA}\right)^2 - \hat{\sigma}^2\Gamma\left[\left(\frac{\hat{A}_2\eta}{\hat{\sigma}^2(1+\rho)\gamma}\right)^2 + \left(\frac{\hat{B}_2}{\gamma}\right)^2(1-\rho)\right] = 0,
\end{aligned} \tag{106}$$

$$\left(\hat{B}_1^{UIA}\right)_t + 2(1-\gamma)\left[\frac{\eta(\lambda_2-\lambda_1)\hat{B}_3}{\hat{\sigma}^2(1+\rho)\gamma} - \frac{\lambda_1+\lambda_2}{\gamma}\hat{B}_1 + \frac{(\lambda_2-\lambda_1)^2}{2\hat{\sigma}^2(1+\rho)\gamma} + \frac{(\lambda_1+\lambda_2)^2}{2\hat{\sigma}^2\gamma(1-\rho)}\right.$$
$$+ \hat{\sigma}^2(1-\rho)\Gamma_1\left(\frac{2\hat{B}_1}{\gamma} - \frac{\lambda_1+\lambda_2}{\hat{\sigma}^2\gamma(1-\rho)}\right)\left(\hat{B}_1^{UIA}\right) + \frac{2\eta^2\Gamma_1\hat{B}_3\left(\hat{B}_3^{UIA}\right)}{\hat{\sigma}^2(1+\rho)\gamma}$$
$$+ \left(\hat{B}_3^{UIA}\right)^2\Gamma_1\frac{\eta^2(1-\rho)}{\rho_1^2(1-\gamma)\hat{\sigma}^2} - \hat{\sigma}^2\Gamma(1+\rho)\left(\frac{\frac{\lambda_2-\lambda_1}{2}+\eta\hat{B}_3}{\hat{\sigma}^2(1+\rho)\gamma}\right)^2$$
$$\left. -\hat{\sigma}^2\Gamma(1-\rho)\left(\frac{\hat{B}_1}{\gamma} + \hat{\sigma}^2\Gamma_1\left(\hat{B}_1^{UIA}\right)^2\frac{1-\rho}{1-\gamma} - \frac{\lambda_1+\lambda_2}{2\hat{\sigma}^2(1-\rho)\gamma}\right)^2\right] = 0, \tag{107}$$

$$\left(\hat{B}_2^{UIA}\right)_t + (1-\gamma)\left[\frac{\eta(\lambda_2-\lambda_1)\hat{A}_2}{\hat{\sigma}^2(1+\rho)\gamma} - \frac{\lambda_1+\lambda_2}{\gamma}\hat{B}_2 - \frac{\lambda_1+\lambda_2}{1-\gamma} + \frac{\kappa\theta\left(\hat{B}_3^{UIA}\right)}{1-\gamma}\right.$$
$$\hat{\sigma}^2(1-\rho)\Gamma_1\left(\frac{2\hat{B}_1\left(\hat{B}_2^{UIA}\right)}{\gamma} + \frac{2\hat{B}_1\left(\hat{B}_2^{UIA}\right)}{\gamma} - \left(\hat{B}_2^{UIA}\right)\frac{\lambda_1+\lambda_2}{\hat{\sigma}^2\gamma(1-\rho)}\right)$$
$$+ \eta\Gamma_1\frac{2\eta\left(\hat{B}_3^{UIA}\right)\hat{A}_1 + 2\eta\left(\hat{B}_3^{UIA}\right)\hat{A}_2}{\hat{\sigma}^2(1+\rho)\gamma} + 2\left(\hat{B}_1^{UIA}\right)\left(\hat{B}_2^{UIA}\right)\hat{\sigma}^2\Gamma_1\frac{1-\rho}{1-\gamma}$$
$$+ 2\left(\hat{A}_2^{UIA}\right)\left(\hat{B}_3^{UIA}\right)\frac{(1-\rho)\eta^2\Gamma_1}{\rho_1^2(1-\gamma)\hat{\sigma}^2} - 2\hat{\sigma}^2\Gamma(1-\rho)\left(\frac{\hat{B}_1\hat{B}_2}{\gamma^2} - \frac{\hat{B}_2(\lambda_1+\lambda_2)}{2\hat{\sigma}^2\gamma(1-\rho)}\right)$$
$$\left. + \frac{2\eta\hat{A}_2\Gamma}{\hat{\sigma}^2\gamma}\left(\frac{\eta\hat{B}_3 + \frac{\lambda_2-\lambda_1}{2}}{\hat{\sigma}^2(1+\rho)\gamma}\right) + \eta\Gamma_1\frac{\lambda_2-\lambda_1}{\hat{\sigma}^2(1+\rho)\gamma}\left(\hat{A}_2^{UIA}\right)\right] = 0, \tag{108}$$

$$
\begin{aligned}
&\left(\hat{B}_3^{UIA}\right)_t + (1-\gamma)\Bigg[(\lambda_2-\lambda_1)(\hat{\sigma}^2(1+\rho)\gamma) + \frac{2\eta\hat{B}_3}{\hat{\sigma}^2(1+\rho)\gamma} + \frac{(1+\eta\hat{A}_1)(\lambda_2-\lambda_1)}{\hat{\sigma}^2(1+\rho)\gamma} \\
&\quad + \hat{\sigma}^2(1-\rho)\Gamma_1\left(\frac{2\hat{B}_1\left(\hat{B}_3^{UIA}\right)}{\gamma} + \frac{2\hat{B}_3\left(\hat{B}_1^{UIA}\right)}{\gamma} - \frac{\left(\hat{B}_3^{UIA}\right)(\lambda_1+\lambda_2)}{\hat{\sigma}^2\gamma(1-\rho)}\right) \\
&\quad + \eta\Gamma_1\left(\frac{2\eta\hat{A}_1\left(\hat{B}_3^{UIA}\right) + 2\eta\hat{B}_3\left(\hat{A}_1^{UIA}\right)}{\hat{\sigma}^2(1+\rho)\gamma} + \frac{2\left(\hat{B}_3^{UIA}\right)}{\hat{\sigma}^2(1+\rho)\gamma} + \frac{\hat{A}_1(\lambda_2-\lambda_1)}{\hat{\sigma}^2(1+\rho)\gamma}\right) \\
&\quad + 2\left(\hat{B}_1^{UIA}\right)\left(\hat{B}_3^{UIA}\right)\hat{\sigma}^2\Gamma_1\frac{1-\rho}{1-\gamma} + 2\left(\hat{A}_1^{UIA}\right)\left(\hat{B}_3^{UIA}\right)\frac{(1-\rho)\eta^2\Gamma_1}{\rho_1^2(1-\gamma)\hat{\sigma}^2} \\
&\quad - \frac{2\Gamma(1+\eta\hat{A}_1)}{\gamma}\left(\frac{\lambda_2-\lambda_1}{2\hat{\sigma}^2(1+\rho)\gamma} + \frac{\eta\hat{B}_6}{\hat{\sigma}^2(1+\rho)\gamma}\right) - \frac{\hat{B}_3(\lambda_1+\lambda_2)}{\gamma} \\
&\quad - \frac{\kappa\left(\hat{B}_3^{UIA}\right)}{1-\gamma} - 2(1-\rho)\hat{\sigma}^2\Gamma\frac{\hat{B}_3}{\gamma}\left(\frac{\hat{B}_1}{\gamma} - \frac{\lambda_1+\lambda_2}{2\hat{\sigma}^2(1-\rho)\gamma}\right)\Bigg] = 0,
\end{aligned} \tag{109}
$$

with terminal conditions $\hat{A}_1^{UIA}(T) = \hat{A}_2^{UIA}(T) = \hat{A}_0^{UIA}(T) = \hat{B}_1^{UIA}(T) = \hat{B}_2^{UIA}(T) = \hat{B}_3^{UIA}(T) = 0$ and $\hat{A}_1$, $\hat{A}_2$, $\hat{A}_3$, $\hat{B}_1$, $\hat{B}_2$ and $\hat{B}_3$ satisfy (82)-(87).

Appendix E

The parameter estimation of the mispricing model (23)-(25) is quite straightforward since the likelihood function of the observed prices $X_1(t), X_2(t)$ can be computed explicitly. In fact, $a(t)$ and $X(t)$ are Ornstein-Uhlenbeck processes given by

$$
a(t) = \theta + (a_0-\theta)e^{-\kappa t} + \sigma_a e^{-\kappa t}\int_0^t e^{\kappa u}dZ_a(u), \tag{110}
$$

$$
X(t) = X_0 e^{-(\lambda_1+\lambda_2)t} + be^{-(\lambda_1+\lambda_2)t}\left(\int_0^t e^{(\lambda_1+\lambda_2)u}dZ_1(u) - \int_0^t e^{(\lambda_1+\lambda_2)u}dZ_2(u)\right). \tag{111}
$$

Substituting back to (23)-(24), we have

$$
\begin{aligned}
\ln\left(\frac{P_1(t)}{P_1(0)}\right) =&(2\tilde{r}-\sigma^2-b^2)t/2+\int_0^t a(s)ds-\lambda_1\int_0^t X(s)ds+\sigma Z(t)+bZ_1(t)\\
=&(2\tilde{r}-\sigma^2-b^2)t/2+\int_0^t\left(\theta+(a_0-\theta)e^{-\kappa s}+\sigma_a e^{-\kappa s}\int_0^s e^{\kappa u}dZ_a(u)\right)ds\\
&-\lambda_1\int_0^t\left(X_0e^{-(\lambda_1+\lambda_2)s}+be^{-(\lambda_1+\lambda_2)s}\left(\int_0^s e^{(\lambda_1+\lambda_2)u}dZ_1(u)\right.\right.\\
&\left.\left.-\int_0^s e^{(\lambda_1+\lambda_2)u}dZ_2(u)\right)\right)ds+\sigma Z(t)+bZ_1(t)\\
=&(2\tilde{r}+2\theta-\sigma^2-b^2)t/2+(a_0-\theta)(1-e^{-\kappa t})/\kappa-X_0\lambda_1(1-e^{-(\lambda_1+\lambda_2)t})/(\lambda_1+\lambda_2)\\
&+\int_0^t\int_u^t\sigma_a e^{\kappa(u-s)}ds\ dZ_a(u)+\int_0^t b\left(1-\int_u^t\lambda_1 e^{(\lambda_1+\lambda_2)(u-s)}ds\right)dZ_1(u)\\
&+\int_0^t\int_u^t\lambda_1 be^{(\lambda_1+\lambda_2)(u-s)}ds\ dZ_2(u)+\sigma Z(t),
\end{aligned}
$$

$$
\begin{aligned}
\ln\left(\frac{P_2(t)}{P_2(0)}\right) =&(2\tilde{r}-\sigma^2-b^2)t/2+\int_0^t a(s)ds+\int_0^t\lambda_2 X(s)ds+\sigma Z(t)+bZ_2(t)\\
=&(2\tilde{r}+2\theta-\sigma^2-b^2)t/2+(a_0-\theta)(1-e^{-\kappa t})/\kappa+X_0\lambda_2(1-e^{-(\lambda_1+\lambda_2)t})/(\lambda_1+\lambda_2)\\
&+\int_0^t\int_u^t\sigma_a e^{\kappa(u-s)}ds\ dZ_a(u)+\int_0^t b\left(1-\int_u^t\lambda_2 be^{(\lambda_1+\lambda_2)(u-s)}ds\right)dZ_2(u)\\
&+\int_0^t\int_u^t\lambda_2 be^{(\lambda_1+\lambda_2)(u-s)}ds\ dZ_1(u)+\sigma Z(t).
\end{aligned}
$$

Let $Q_1(t)=\ln(P_1(t)/P_1(0))$ and $Q_2(t)=\ln(P_2(t)/P_2(0))$. The joint distribution of
$(Q_1(s),Q_2(s),Q_1(t),Q_2(t))$ with $s<t$ is bivariate Normal with mean and covariance given by

$$
\begin{aligned}
E(Q_1(t)) &= (2\tilde{r}+2\theta-\sigma^2-b^2)t/2+(a_0-\theta)(1-e^{-\kappa t})/\kappa\\
&\quad -X_0\lambda_1(1-e^{-(\lambda_1+\lambda_2)t})/(\lambda_1+\lambda_2), && (112)\\
E(Q_2(t)) &= (2\tilde{r}+2\theta-\sigma^2-b^2)t/2+(a_0-\theta)(1-e^{-\kappa t})/\kappa\\
&\quad +X_0\lambda_2(1-e^{-(\lambda_1+\lambda_2)t})/(\lambda_1+\lambda_2), && (113)\\
Cov(Q_1(s),Q_1(t)) &= I_1(s,t)+I_2(s,t,\lambda_1,\lambda_1)+I_5(s,t,\lambda_1,\lambda_1)+\sigma^2 s, && (114)\\
Cov(Q_2(s),Q_2(t)) &= I_1(s,t)+I_2(s,t,\lambda_2,\lambda_2)+I_5(s,t,\lambda_2,\lambda_2)+\sigma^2 s, && (115)\\
Cov(Q_1(s),Q_2(t)) &= I_1(s,t)+I_3(s,t,\lambda_1,\lambda_2)+I_4(s,t,\lambda_1,\lambda_2)+\sigma^2 s, && (116)\\
Cov(Q_2(s),Q_1(t)) &= I_1(s,t)+I_3(s,t,\lambda_2,\lambda_1)+I_4(s,t,\lambda_2,\lambda_1)+\sigma^2 s, && (117)
\end{aligned}
$$

where

$$I_1(s,t) = \int_0^s \left(\int_u^s \sigma_a \exp(\kappa(u-v))\, dv \right) \left(\int_u^t \sigma_a \exp(\kappa(u-v))\, dv \right) du$$
$$= -\frac{\sigma_a^2 e^{-\kappa(s+t)} \left((2-2\kappa s) e^{\kappa(s+t)} - 2e^{\kappa s} + e^{2\kappa s} - 2e^{\kappa t} + 1 \right)}{2k^3}, \tag{118}$$

$$\begin{aligned} I_2(s,t,c,d) &= \int_0^s b^2 \left(1 - c \int_u^s \exp\left(-\left(\lambda_1+\lambda_2\right)(u-v)\right)\, dv \right) \\ &\times \left(1 - d \int_u^t \exp\left(-\left(\lambda_1+\lambda_2\right)(u-v)\right)\, dv \right) du \\ &= \frac{b^2}{2\left(\lambda_1+\lambda_2\right)^3} \Big(cd \Big(-2e^{(\lambda_1+\lambda_2)s} + e^{(\lambda_1+\lambda_2)(t-s)} + e^{(\lambda_1+\lambda_2)(s+t)} \\ &-2e^{(\lambda_1+\lambda_2)t} + 2 \Big) + 2\left(\lambda_1+\lambda_2\right)\left(\left(\lambda_1+\lambda_2\right) s \left(c+d+\lambda_1+\lambda_2\right) + cds \right. \\ &\left. +c\left(-e^{(\lambda_1+\lambda_2)s}\right) + c + de^{(\lambda_1+\lambda_2)(t-s)} - de^{(\lambda_1+\lambda_2)t} \right) \Big), \end{aligned} \tag{119}$$

$$\begin{aligned} I_3(s,t,c,d) &= \int_0^s b^2 \left(1 - c \int_u^s \exp\left(-\left(\lambda_1+\lambda_2\right)(u-v)\right)\, dv \right) \\ &\times \left(d \int_u^t \exp\left(-\left(\lambda_1+\lambda_2\right)(u-v)\right)\, dv \right) du \\ &= \frac{b^2 d}{2\left(\lambda_1+\lambda_2\right)^3} \Big(2ce^{(\lambda_1+\lambda_2)s} - ce^{(\lambda_1+\lambda_2)(s+t)} + 2\left(c+\lambda_1+\lambda_2\right) e^{(\lambda_1+\lambda_2)t} \\ &+ e^{(\lambda_1+\lambda_2)(-s)} \Big(-2ce^{(\lambda_1+\lambda_2)s} - 2\left(\lambda_1+\lambda_2\right) s \left(c+\lambda_1+\lambda_2\right) e^{(\lambda_1+\lambda_2)s} \\ &+ce^{(\lambda_1+\lambda_2)t} - 2\left(c+\lambda_1+\lambda_2\right) e^{(\lambda_1+\lambda_2)t} \Big) \Big), \end{aligned} \tag{120}$$

$$\begin{aligned} I_4(s,t,c,d) &= \int_0^s b^2 \left(c \int_u^s \exp\left(-\left(\lambda_1+\lambda_2\right)(u-v)\right)\, dv \right) \\ &\times \left(1 - d \int_u^t \exp\left(-\left(\lambda_1+\lambda_2\right)(u-v)\right)\, dv \right) du \\ &= \frac{b^2 c}{2\left(\lambda_1+\lambda_2\right)^3} \Big(2\left(d+\lambda_1+\lambda_2\right) e^{(\lambda_1+\lambda_2)s} - de^{(\lambda_1+\lambda_2)(s+t)} + 2de^{(\lambda_1+\lambda_2)t} \\ &+ e^{(\lambda_1+\lambda_2)(-s)} \Big(-2\left(d+\lambda_1+\lambda_2\right) e^{(\lambda_1+\lambda_2)s} \\ &-2\left(\lambda_1+\lambda_2\right) s \left(d+\lambda_1+\lambda_2\right) e^{(\lambda_1+\lambda_2)s} - de^{(\lambda_1+\lambda_2)t} \Big) \Big), \end{aligned} \tag{121}$$

$$I_5(s,t,c,d) = \int_0^s b^2 \left(c \int_u^s \exp\left(-\left(\lambda_1+\lambda_2\right)\left(u-v\right)\right) dv \right) \tag{122}$$
$$\times \left(d \int_u^t \exp\left(-\left(\lambda_1+\lambda_2\right)\left(u-v\right)\right) dv \right) du$$
$$= \frac{b^2 cd}{2\left(\lambda_1+\lambda_2\right)^3} \left(2\lambda_1 s + 2\lambda_2 s + e^{(\lambda_1+\lambda_2)(-(s-t))} \left(\left(e^{(\lambda_1+\lambda_2)s} - 1 \right)^2 - 2e^{(\lambda_1+\lambda_2)(2s-t)} \right) + 2 \right).$$

Given the historical stock prices $\{P_1(t_i), P_2(t_i)\}_{i=1}^n$, we can obtain $\{Q_1(t_i), Q_2(t_i)\}_{i=1}^n$ and they have multivariate Normal distribution with mean and covariance matrix by (112)-(122) . The parameters can be then estimated by maximizing the likelihood function of $\{Q_1(t_i), Q_2(t_i)\}$ subject to the constraints $\lambda_1 + \lambda_2 > 0$ and $b, \sigma, \sigma_a, \kappa > 0$.

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