TWO-DIMENSIONAL STOCHASTIC NAVIER-STOKES EQUATIONS WITH FRACTIONAL BROWNIAN NOISE

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Abstract. We study the perturbation of the two-dimensional stochastic Navier-Stokes equation by a Hilbert-space-valued fractional Brownian noise. Each Hilbert component is a scalar fractional Brownian noise in time, with a common Hurst parameter H and a specific intensity. Because the noise is additive, simple Wiener-type integrals are sufficient for properly defining the problem. It is resolved by separating it into a deterministic nonlinear PDE, and a linear stochastic PDE. Existence and uniqueness of mild solutions are established under suitable conditions on the noise intensities when the Hurst parameter H is in $(1/8, 1)$, with different proof techniques depending on whether or not $H > 1/4$.

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1. Introduction

The stochastic Navier-Stokes system has been an important and active area of research, and has received considerable attention in recent years. The introduction of randomness in the Navier-Stokes equations arises from a need to understand (i) the velocity fluctuations observed in wind tunnels under identical experimental conditions, and (ii) the onset of turbulence. Random body forces also arise as control terms, or from random disturbances such as structural vibrations that act on the fluid. It was originally the idea of Kolmogorov (see Vishik and Fursikov [15]) to introduce white noise in the Navier-Stokes system in order to obtain an invariant measure for the system.

Of late, stochasticians have embraced this white-noise forcing for the 2-D Navier-Stokes system (see [3] and references therein), and in certain inertial scales, this is justifiable: in [5], Kuppiainen has shown that it is reasonable to model the uncertainty in velocity profiles by white noise. The independence of increments inherent in white noise is key to all these studies: this has been confirmed by the discovery and analysis of the solution's Markov semigroup (again see references listed in [3]). The only published article we have found in which there is a deviation from white noise for stochastic Navier-Stokes equations is [16]: a Levy process is used, but this is still confined to the realm of processes with independent increments.

In this paper, we study a case where the time-scaling in the random forces is not related to the state-space scaling, so that white noise is not appropriate. We consider the stochastic Navier-Stokes equation (NSE) on a bounded open domain G in $\mathbf{R}^2$, with a infinite-dimensional fractional Brownian noise $\mathbf{W}^H$.

Writing it in the abstract evolution setup, this is:

$$\frac{\partial \mathbf{u}}{\partial t} + \nu \mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u}) = \Phi \, \frac{d\mathbf{W}^H(t)}{dt} \tag{1.1}$$

with $\mathbf{u}(t,x) = 0$ for all $x \in \partial G$, with $\mathbf{u}(0,x) = \mathbf{u}_0(x)$ for all $x \in G$, with $\mathbf{A}$ being the negative Laplace operator, and $\mathbf{B}(\mathbf{u}) = (\mathbf{u} \cdot \nabla)\mathbf{u}$. In the next section we will define suitable Hilbert spaces in which to find mild-sense (evolution) solutions of this equation; we will consider incompressible flows with no-slip condition at the boundary. The process $\mathbf{W}^H$ is a space-time fractional Brownian motion (fBm) in a suitable Hilbert space, implying that for fixed space parameter x, it is a scalar fBm with Hurst parameter $H \in (0,1)$; see Tindel, Tudor and Viens [14].

Fractional Brownian motion is not a semimartingale, nor a Markov process, and its increments have medium- or long-range dependence. Therefore, the usual methods of solvability of stochastic Navier-Stokes equations such as energy equality, local monotonicity, Markov semigroups, and martingale problems, do not apply to the present system. In addition to noise memory length, fBm is self-similar with parameter H; when combining this with possible scaling in the space parameter, infinite-dimensional noises with specific multiscale properties can be achieved.

The theory of stochastic integration with respect to fBm is in sharp contrast to the Itô theory of integrals; it has been developed by several authors (see Nualart [11, Chapter 5] and the references therein). However, infinite-dimensional equations with additive noise, such as the one we consider here, can typically be expressed in a mild sense, making the required integration theory rather elementary, as we will see in Section 3. In particular, we certainly bypass the need for the noise term to be a semimartingale.

Stochastic partial differential equations of parabolic type perturbed by an fBm noise have been studied in recent years by several authors (see Tindel, Tudor and Viens [14], Maslowski and Schmalfuss [7], Nualart [9], [10], and the references therein). A major question solved in these works is to identify sharp sufficient conditions on the noise coefficient Φ that guarantee the existence and uniqueness of solutions.

The Navier-Stokes system is quite distinct from all these works since it is a nonlinear system with an unbounded, non-Lipschitz term $\mathbf{B}$. Because of this, calculations in $L^2(\Omega)$, which are typical of the above works (see [14]), are insufficient in our case. As we will see, another major quantitative difference between heat equations and our system is its linear second-order operator $\mathbf{A}$, which, because it is restricted to a divergence-free domain, has unbounded eigenfunctions.

In this article, we take advantage of the fact that the noise term in (1.1) is additive; using a fixed point argument, the existence and uniqueness of a mild solution is established by combining a solution of the non-linear equation with no noise, and a solution of the stochastic equation without the non-linearity using properties of the semigroup of the so-called Stokes operator. The question of finding conditions on Φ guaranteeing existence and uniqueness is dealt with in the linear stochastic portion of the analysis. These conditions are fully explicit, insofar as they depend in an elementary way on the eigenstructure of the Stokes operator. Our main result (Theorem 6.1 on page 14) states that under these conditions (which are different depending on whether $H \in (1/8, 1/4)$ or $H \in [1/4, 1)$), almost-surely w.r.t. the randomness of $\mathbf{W}^H$, there is a unique solution in $L^4([0,T] \times G)$ to the stochastic Navier-Stokes system.

Our method leaves the question of existence of a solution open when $H \leq 1/8$. Neither are we able to estimate any moment of the solution w.r.t. $\mathbf{W}^H$'s randomness. Regarding the first question, we suspect that there is no solution when $H < 1/8$, because the tools we use to solve the linear equation are sharp; this could presumably be proved for some specific cases of domains G where the eigenstructure of the Stokes operator is well-understood. For the section question, we conjecture that the solution is square-integrable w.r.t $\mathbf{W}^H$, but no more (see for instance the results in the white-noise case [8]). This issue appears to be non-trivial, and is beyond the scope of this concise article; we will investigate it, and its connections to path regularity of the solution, in a separate paper.

The organization of the paper is as follows. In Section 2, the evolution equation setup of the Navier-Stokes equations is presented. The development of integration with respect to fractional Brownian motion is briefly explained in Section 3. The $L^4(\Omega)$-integrability of a convolution Wiener integral is proved in Section 4 under suitable conditions on the noise coefficient; here two separate calculations must be performed, depending on whether $H > 1/4$ or $H \leq 1/4$. The solvability of the stochastic Navier-Stokes system is proved in the final Section 5.

2. Navier-Stokes Equations

In this section, we express the NSE using appropriate function spaces. Let G be a bounded open domain in $\mathbf{R}^2$ with a smooth boundary ∂G. For $t \in [0, T]$, consider the stochastic NSE for a viscous incompressible flow with no-slip condition at the boundary:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \Phi \frac{d\mathbf{W}^H(t)}{dt} \tag{2.1}$$

and

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

with initial and boundary data

$$\begin{aligned} \mathbf{u}(t,x) &= 0 \ \forall x \in \partial G, \forall t \geq 0 \\ \mathbf{u}(0,x) &= \mathbf{u}_0(x) \ \forall \ x \in G. \end{aligned}$$

In the above, p denotes the pressure field and is a scalar-valued function. The noise coefficient Φ is assumed to be deterministic and independent of t. This modeling assumption ensures that the driving noise has the memory length and self-similarity properties of fBm, in the parameter t.

To study the stochastic Navier-Stokes system (2.1), (2.2), we first write these stochastic partial differential equation in the abstract (variational, or evolution) form on suitable function spaces. For the functional analytic set up and the mathematical details, one can consult Ladyzhenskaya [6] and Temam [13]. Let $\mathcal{V}$ be the space of 2-dimensional vector functions $\mathbf{u}$ on G which are infinitely differentiable with compact support strictly contained in G, satisfying $\nabla \cdot \mathbf{u} = 0$. For any fixed $\alpha \in \mathbf{R}$, we can define the restriction of the standard Sobolev space $W^{\alpha,2}$ to those divergence-free 2-vectors by letting V_α denote the closure of $\mathcal{V}$ in $W^{\alpha,2}$.

We will use the shorthand notation $H := V_0$ and $V := V_1$. The notation $L^2(G)$, $W_0^{1,2}(G)$, etc. denotes 2-vector functions on G with each coordinate in the scalar versions of $L^2(G)$, $W_0^{1,2}(G)$, etc. For instance, we simply have

$$W_0^{1,2}(G) = \{\mathbf{u} \in L^2(G, \mathbf{R}^2) : \nabla \mathbf{u} \in L^2(G, \mathbf{M}_2(\mathbf{R})), \ \mathbf{u}|_{\partial G} = 0\}.$$

Denoting by $\mathbf{n}$ the outward normal on ∂G, the following characterizations of the spaces H and V are well-known, and will be convenient:

$$H = \{\mathbf{u} \in L^2(G); \nabla \cdot \mathbf{u} = 0,\ \mathbf{u} \cdot \mathbf{n}|_{\partial G} = 0\},$$
$$V = \{\mathbf{u} \in W_0^{1,2}(G) : \nabla \cdot \mathbf{u} = 0\}.$$

Let V' be the dual of V. We will denote the norm in H by $|\cdot|$, and the inner product in H by $(\cdot,\cdot)$. We have the dense, continuous and compact embedding (see [13]):

$$V \subset_{\rightarrow} H = H^{'} \subset_{\rightarrow} V^{'}.$$

Let $\mathcal{D}(\mathbf{A}) = W^{2,2}(G) \cap V$. Define the linear operator $\mathbf{A} : \mathcal{D}(\mathbf{A}) \to H$ by $\mathbf{A}\mathbf{u} = -\Delta \mathbf{u}$. Since $V = \mathcal{D}(\mathbf{A}^{1/2})$, we can endow V with the norm $\|\mathbf{u}\| = \left|\mathbf{A}^{1/2}\mathbf{u}\right|$. The V-norm is equivalent to the $W^{1,2}$-norm by the Poincaré inequality. From now on, $\|\cdot\|$ will denote the V-norm. The pairing between V and its dual V' is denoted by $\langle \cdot,\cdot \rangle$. The operator $\mathbf{A}$ is known as the Stokes operator and is positive, self-adjoint with compact resolvent. The eigenvalues of $\mathbf{A}$ will be denoted by $0 < \lambda_1 < \lambda_2 \leq \cdots$, and the corresponding eigenfunctions by $e_1, e_2, \cdots$. The eigenfunctions form a complete orthonormal system for H. It is known (see [6]) that there are value $c, c' > 0$ such that

$$\lim_{j\to\infty} j/\lambda_j = c > 0 \text{ and } \|e_j\|_{L^4(G)} \leq c\lambda_j^{1/4} \text{ for all } j.$$

Define $b(\cdot,\cdot,\cdot) : V \times V \times V \mapsto \mathbf{R}$ by

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{2} \int_G u_i\,(x)\,\frac{\partial v_j}{\partial x_i}\,(x)\,w_j\,(x)\;dx.$$

This allows us to define $\mathbf{B} : V \times V \mapsto V'$ as the continuous bilinear operator such that

$$\langle \mathbf{B}(\mathbf{u},\mathbf{v}); \mathbf{w}\rangle = b(\mathbf{u},\mathbf{v},\mathbf{w}) \text{ for all } \mathbf{u},\mathbf{v},\mathbf{w} \in V.$$

Note that $b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v})$. We will denote $\mathbf{B}(\mathbf{u},\mathbf{u})$ by $\mathbf{B}(\mathbf{u})$. This $\mathbf{B}(\mathbf{u})$ satisfies the following estimate:

$$\|\mathbf{B}(\mathbf{u})\|_{V'} \leq 2\,|\mathbf{u}|\,||\mathbf{u}|| \tag{2.3}$$

We assume that $\mathbf{u}_0$ is H-valued. Let Π denote the Leray projection of $L^2(G)$ into H. By applying this projection to each term of the Navier-Stokes system, and invoking the Leray decomposition of $L^2(G)$ into divergence free and irrotational components, we can write the system (2.1) and (2.2) as

$$d\mathbf{u}\,(t) + [\nu \mathbf{A}\,\mathbf{u}\,(t) + \mathbf{B}(\mathbf{u}\,(t))]\,dt = \Phi\,d\mathbf{W}^H(t). \tag{2.4}$$

This is to be understood in the integral form $\mathbf{u}\,(t) = \mathbf{u}\,(0) - \nu \int_0^t \mathbf{A}\,\mathbf{u}\,(s)\,ds - \int_0^t \mathbf{B}(\mathbf{u}\,(s))ds + \int_0^t \Phi d\mathbf{W}^H(s)$. To write this in its evolution form, we will need $S\,(t)$ the semigroup generated by $\mathbf{A}$. Assuming for simplicity that $\nu = 1$, the stochastic NSE in H then writes as

$$\mathbf{u}\,(t) = S\,(t)\,\mathbf{u}\,(0) - \int_0^t S\,(t-s)\,\mathbf{B}(\mathbf{u}\,(s))ds + \int_0^t [S\,(t-s)\,\Phi]\,d\mathbf{W}^H(s). \tag{2.5}$$

This means we only need to explain how to define integrals of the form $\int_0^t \phi\,(s)\,d\mathbf{W}^H(s)$, where ϕ is a suitable non-random integrand. While the strategy to construct this type of integral is well-known, we detail in the next section for completeness, along with some general information about fBm.

3. **Fractional Brownian Motion**

Unfortunately, the letter H, used previously for Hilbert spaces, is also used for the so-called *Hurst* or self-similarity parameter for fBm, which is a number in $(0,1)$. The context dictates when one or the other is being used.

Definition 3.1. *A centered Gaussian process $\{\beta_t^H\}$ is called a fractional Brownian motion with Hurst parameter $H \in (0,1)$ if its covariance function is given by*

$$\mathbf{E}\left[\left(\beta_t^H \beta_s^H\right)\right] = R_H(t,s) := \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{3.1}$$

It is well-known that $\{\beta_t^H\}$ is not a semimartingale. Indeed, for $p>0$, consider the random variable $Y_{n,p} = \frac{1}{n}\sum_{j=1}^n |\beta_j - \beta_{j-1}|^p$. By self-similarity, this has the same distribution as $n^{pH-1}\sum|\beta_{j/n} - \beta_{(j-1)/n}|^p$. The stationary sequence $\{\beta_j - \beta_{j-1}\}$ is mixing, and hence ergodic. Thus, $\lim_{n\to\infty} Y_{n,p} = \mathbf{E}(|\beta_1|^p)$ a.s. and in L^1. Therefore, as $n\to\infty$, the approximate pth variation $V_{n,p} := \sum_{j=1}^n |\beta_{j/n} - \beta_{(j-1)/n}|^p$ converges to 0 if $pH>1$ and ∞ if $pH<1$. In particular for $p=2$, the quadratic variation, which by definition is $\lim_{n\to\infty} V_{n,2}$, is either null when $H>1/2$ or infinite when $H<1/2$. This non-semimartingale situation is closely related to the fact fBm is almost-surely Hölder-continuous with exponent α for any $\alpha<H$ but not for $\alpha=H$, a fact which holds typically for semimartingales with $\alpha=1/2$.

The fractional Brownian motion also exhibits medium or long range memory as described below. Let $H\neq 1/2$. Let $s,n>0$. Then, using the covariance formula (3.1), as $n\to\infty$, the correlation between two increments at a distance n is

$$\rho_H(n) := \mathbf{E}\left[\left(\beta_{s+n+1}^H - \beta_{s+n}^H\right)\left(\beta_{s+1}^H - \beta_s^H\right)\right] = H(2H-1)n^{2H-2}\left(1+o\left(1\right)\right).$$

- If $H>1/2$, then $\rho_H(n)>0$, and $\sum_n \rho_H(n) = +\infty$. This is known as aggregation behavior, or the long memory property.
- If $H<1/2$, then $\rho_H(n)<0$ and $\sum|\rho_H(n)|<\infty$. This behavior is sometimes known as antipersistence. The fact that the correlations decay to 0 slower than quadratically with n, but are still summable, can be described as a medium memory property.

Stochastic integration with respect to fBm is not straightforward as illustrated by the following example:

Example 3.1. *If one tries to approximate $\int_0^1 \beta_s^H d\beta_s^H$ by the Itô-type Riemann sum $\sum_{j=1}^n \beta_{t_{j-1}}^H(\beta_{t_j}^H - \beta_{t_{j-1}})$, using the covariance formula (3.1), it is straightforward to check that, for instance with $t_j = j/n$, the sum's expected value equals $(1/2)(1-n^{1-2H})$. This tends to $-\infty$ if $H<1/2$ and to a non-zero value of $H>1/2$, unlike the Itô integral for standard BM, which has zero expectation. However, the expected symmetric Riemann sums converges to $1/2$ for fBm.*

Gaussian integral with respect to fBm

Fortunately for our purposes, the distinctions between these various integrals will not be relevant, because, as seen in (2.5), we only need to explain how to integrate deterministic functions w.r.t. an fBm β^H (albeit perhaps an infnite-dimensional one).

Let S be the set of all step functions on $[0,T]$. For a step function $\phi = \sum_0^{n-1} a_j 1_{(t_j,t_{j+1}]}$ we pose $\int_0^T \phi(s) d\beta_s^H := \sum_{j=0}^{n-1} a_j (\beta_{t_{j+1}}^H - \beta_{t_j}^H)$. Let $\mathcal{H}$ be the closure of S w.r.t. the inner product $\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}} := R_H(t,s)$. Then $1_{[0,t]} \to \beta_t^H$ extends to an isometry between $\mathcal{H}$ and the $L^2(\Omega)$-closure of the linear span of $\{\beta_t^H : t \in [0,T]\}$. This extension is called the Wiener integral w.r.t. β, and can be denoted by $\mathcal{H}\phi \mapsto \int_0^T \phi(s)\, d\beta^H(s) \in L^2(\Omega)$. Note that the Wiener integral of any function $\phi \in \mathcal{H}$ w.r.t. β^H is a centered Gaussian random variable, and that for $\phi, \psi \in \mathcal{H}$ we have that $\int_0^T \phi d\beta^H$ and $\int_0^T \psi d\beta^H$ are jointly Gaussian with covariance equal to $\langle \phi, \psi \rangle_{\mathcal{H}}$, thereby extending the Wiener integral for standard Brownian motion.

There is a connection between the standard Wiener process and fractional Brownian motions. One begins by noting that R_H is, by definition, a non-negative definite kernel, which means that there exists a kernel function K_H such that $R_H(t,s) = \int_0^{t\wedge s} K_H(t,u) K_H(s,u) du$. In fact, its expression is explicit:

$$K_H(t,s) = c_H \left(\frac{t}{s}\right) (t-s)^{H-1/2} + s^{1/2-H} F\left(\frac{t}{s}\right)$$

where $F(z) = c_H(1/2-H) \int_0^{z-1} r^{H-3/2} \left(1-(1+r)^{H-1/2}\right) dr$. Using these facts, one proves that there exists a standard Brownian motion W such that

$$\beta_t^H = \int_0^t K_H(t,s) dW_s.$$

For $s < T$, if we define the adjoint operator K^* on a possible subset of $L^2([0,T])$ by

$$(K_T^* \phi)(s) = K(T,s)\phi(s) + \int_s^T (\phi(r) - \phi(s)) \frac{\partial K}{\partial r}(r,s) dr,$$

a result of Alos, Mazet, Nualart [1] then guarantees that K_T^* is an isometry between $\mathcal{H}$ and $L^2[0,T]$, and that the Wiener integral w.r.t β^H can be represented in the following convenient way: for all $\phi \in \mathcal{H}$, $K_T^* \phi \in L^2[0,T]$ and

$$\int_0^T \phi(s) d\beta^H(s) = \int_0^T (K_T^* \phi)(s) dW_s$$

where the last integral is a Wiener integral w.r.t. standard Brownian motion. It is easy to check that $K_T^*[\phi\, 1_{[0,t]}] = K_t^*[\phi] 1_{[0,t]}$. Therefore,

$$\int_0^t \phi(s) d\beta^H(s) = \int_0^t (K_t^* \phi)(s) dW_s. \tag{3.2}$$

Cylindrical fBm

As announced at the end of the Introduction, we now only need to define integrals of the form $\int_0^t \phi(s) d\mathbf{W}^H(s)$ where $\mathbf{W}^H$ is a cylindrical H-valued fBm. This $\mathbf{W}^H$ is an infinite-dimensional stochastic process with a fBm behavior in time, taking values in the Hilbert space H, with equal weights on all directions of H. Specifically let $\{e_n\}$ be the complete orthonormal basis in H, formed by the eigenfunctions of the Stokes operator $\mathbf{A}$ on G. Define

$$\mathbf{W}^H(t) = \sum_{j=1}^{\infty} e_n\, \beta_n^H(t),$$

where $\left\{\beta_n^H\right\}_n$ is a family of IID scalar fBm's. Strictly speaking, $\mathbf{W}^H(t)$ is not a member of $L^2(\Omega, H)$, since its norm is infinite, but it will be easy to guarantee that an integral w.r.t. $\mathbf{W}^H$ will be.

Indeed, let $\{\phi(s) : s \in [0,T]\}$ be a deterministic measurable function such that for every s, $\phi(s) \in H$. So we can write $\phi(s) e_n = \sum_m (\phi(s) e_n, e_m) e_m$, and this is a deterministic measurable function on $[0,T]$. We may now define

$$\begin{aligned}\int_0^t \phi(s)\, d\mathbf{W}^H(s) &:= \sum_{n=1}^{\infty} \int_0^t \phi(s)\, e_n\, d\beta_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t (K^*(\phi(\cdot)\, e_n))(s)\ dW_n(s) \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} e_j \int_0^t (K^*((\phi(\cdot)\, e_n, e_j)))(s)\ dW_n(s)\end{aligned}$$

where the second line follows from the convenient representation (3.2), in which W_n is the standard Brownian motion used to represent β_n^H.

Since all the terms in the last expression above are independent centered Gaussian r.v.'s, we can immediately give a necessary and sufficient condition for the above integral to exist: it is a Gaussian random variable in $L^2(\Omega)$ if and only if

$$\begin{aligned}\mathbf{E}\left[\left\|\int_0^t \phi(s)\, d\mathbf{W}^H(s)\right\|^2\right] &= \mathbf{E}\left[\sum_{j=1}^{\infty}\left|\sum_{n=1}^{\infty}\int_0^t (K^*((\phi(\cdot)\, e_n, e_j)))(s)\ dW_n(s)\right|^2\right] \\ &= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \mathbf{E}\left[\left|\int_0^t (K^*((\phi(\cdot)\, e_n, e_j)))(s)\ dW_n(s)\right|^2\right] \\ &= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\int_0^t |(K^*((\phi(\cdot)\, e_n, e_j)))(s)|^2\, ds < \infty. \qquad (3.3)\end{aligned}$$

4. **Integrability of the convolution integral when $H > 1/4$**

In equation (2.5), we announced that we only need to be able to define the convolution integral $\mathbf{z}(t) = \int_0^t [S(t-s)\Phi]\, d\mathbf{W}^H(s)$ where $S(t)$ is the semigroup of the Stokes operator $\mathbf{A}$, and Φ is a linear function from H to H. In other words, we apply the result of the end of the previous section with the function $\phi(s) = S(t-s)\Phi$. Using the eigenstructure of $\mathbf{A}$ we thus have

$$\begin{aligned}\mathbf{z}(t) := \int_0^t S(t-s)\Phi d\mathbf{W}_s^H &= \sum_{n=1}^{\infty}\int_0^t S(t-s)\Phi e_n d\beta_n^H(s) \\ &= \int_0^t \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} (S(t-s)\Phi e_n, e_j) d\beta_n^H(s)\cdot e_j \\ &= \int_0^t \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} e^{-(t-s)\lambda_j}(\Phi e_n, e_j) d\beta_n^H(s)\cdot e_j \qquad (4.1)\end{aligned}$$

It is elementary to check that if $\mathbf{A}\mathbf{z}$ is defined as an H-valued function on $[0,T]$, then $\mathbf{z}$ is the solution of

$$u(t) - u(0) - \int_0^t \mathbf{A}u(s)\,ds = \Phi \mathbf{W}^H(t) \tag{4.2}$$

with $\mathbf{z}(0) = 0$. This justifies our claim that $\mathbf{z}$ in (4.1) is the mild- (evolution-) sense solution to the stochastic evolution equation (4.2), modulo finiteness of the expression in (3.3), even if $\mathbf{A}\mathbf{z}$ is not defined.

In order to see under what conditions on Φ we may have

$$\mathbf{z} \in L^4(\Omega \times [0,T] \times G),$$

we may attempt to use the results from [14] in which necessary and sufficient conditions for $\mathbf{z}(t,x)$ to be in $L^2(\Omega)$ for fixed t,x, are given. Since the space in which we seek to find $\mathbf{z}$ is smaller than the space of pointwise existence used in [14], we cannot apply the results therein directly, but several of the calculations used in the their proofs can still be used. In fact, we state the following.

Let $H \in (0,1)$. Recall $\mathcal{H}$ the canonical Hilbert space of fBm β^H, such that for any $g,h \in \mathcal{H}$, $\mathbf{E}\left[\left(\int_0^\infty h(s)d\beta^H(s)\right)\left(\int_0^\infty g(s)d\beta^H(s)\right)\right] = \langle g,h\rangle_{\mathcal{H}}$.

Lemma 4.1. *For any $\lambda, t \geq 0$, there is a constant $c_{t,H}$ such that*

$$\mathbf{E}\left[\left(\int_0^t e^{-\lambda(t-s)}d\beta^H(s)\right)^2\right] = |\mathbf{1}_{[0,t]}e^{-\lambda(t-\cdot)}|^2_{\mathcal{H}} \leq c_{t,H}\lambda^{-2H}.$$

In fact, there exists a constant $C(H)$ depending only on H such that $c_{t,H}$ is given as follows:

$$\begin{cases} c_{t,H} \leq C(H) \text{ for all } H > 1/2, \\ c_{t,H} \leq C(H)(1+t^{2H-1}) \text{ for all } H < 1/2. \end{cases}.$$

Let us now calculate the norm of $\mathbf{z}$ in $L^4(\Omega \times [0,T] \times G)$. With the shorthand notation

$$p_s^t(n,j) := \mathbf{1}_{[0,t]}(s)e^{-(t-s)\lambda_j}(\Phi e_n, e_j)e_j,$$

using standard Gaussian calculations, we get

$$\begin{aligned}
&\mathbf{E}[\mathbf{z}^4(t,x)] \\
&= \sum_n \mathbf{E}\left[\left(\int_0^t [\textstyle\sum_j p_s^t(n,j)(x)]d\beta_n^H(s)\right)^4\right] \\
&+ 3\sum_{n\neq m}\mathbf{E}\left[\left(\int_0^t [\textstyle\sum_j p_s^t(n,j)(x)]d\beta_n^H(s)\right)^2\left(\int_0^t [\textstyle\sum_j p_s^t(m,j)(x)]d\beta_n^H(s)\right)^2\right] \\
&= 3\sum_n\left(\mathbf{E}\left[\left(\int_0^t [\textstyle\sum_j p_s^t(n,j)(x)]d\beta_n^H(s)\right)^2\right]\right)^2 \\
&+ 3\sum_{n\neq m}\mathbf{E}\left[\left(\int_0^t [\textstyle\sum_j p_s^t(n,j)(x)]d\beta_n^H(s)\right)^2\right]\mathbf{E}\left[\left(\int_0^t [\textstyle\sum_j p_s^t(m,j)(x)]d\beta_n^H(s)\right)^2\right]
\end{aligned}$$

$$= 3 \sum_{n=m} \mathbf{E} \left[\left(\int_0^t [\textstyle\sum_j p_s^t(n,j)(x)] d\beta_n^H(s) \right)^2 \right] \mathbf{E} \left[\left(\int_0^t [\textstyle\sum_j p_s^t(m,j)(x)] d\beta_n^H(s) \right)^2 \right].$$

With the inner product notation in $\mathcal{H}$, this now reads as

$$\mathbf{E}[\mathbf{z}^4(t,x)] = 3 \sum_{n,m} \left| \textstyle\sum_j p_\cdot^t(n,j)(x) \right|_{\mathcal{H}}^2 \left| \textstyle\sum_j p_\cdot^t(m,j)(x) \right|_{\mathcal{H}}^2 = 3 \left(\sum_n \left| \textstyle\sum_j p_\cdot^t(n,j)(x) \right|_{\mathcal{H}}^2 \right)^2.$$

Now let us reintroduce the terms in the notation $p_\cdot^t(n,j)$. As a shorthand, we will omit the factor $\mathbf{1}_{[0,t]}$ in $p^t(n,j)$. We thus get

$$\|\mathbf{z}\|_{L^4(\Omega\times[0,T]\times G)}^4 = 3 \int_0^T \int_G dtdx \left(\sum_n \textstyle\sum_{j,k} \langle e^{-(t-\cdot)\lambda_j}; e^{-(t-\cdot)\lambda_k} \rangle_{\mathcal{H}} (\Phi e_n, e_j)(\Phi e_n, e_k) e_j(x) e_k(x) \right)^2.$$

At this point, one notes that to do this computation exactly, it would be necessary to evaluate an inner product in $\mathcal{H}$ of two exponentials relative to two different modes λ_j and λ_k. There is a wide class of examples, that of noise spatial covariances which are co-diagonalizable with the Stokes operator, where this is unnecessary, since the sum over j,k reduces to a single term where $j=k=n$. Therefore, there is not much loss of power in invoking the Cauchy-Schwarz inequality to write:

$$\|\mathbf{z}\|_{L^4(\Omega\times[0,T]\times G)}^4 \leq 3 \int_0^T \int_G dtdx (\sum_n \textstyle\sum_{j,k} |e^{-(t-\cdot)\lambda_j}|_{\mathcal{H}} |e^{-(t-\cdot)\lambda_k}|_{\mathcal{H}} (\Phi e_n, e_j)(\Phi e_n, e_k) e_j(x) e_k(x))^2.$$

Now, in order to reunite the space base functions e_j etc... with their space integral, in principle it is necessary to expand the above square, resulting in terms of the form $\int_G e_i(x)e_j(x)e_k(x)e_\ell(x)dx$. Unfortunately, nothing is known about the values of these integrals for the Stokes operator's eigenfunctions e_i. The best we can do is to say that this integral is bounded above by the product of $\|e_i\|_{L^4(G)}$ with the other three. Then it is known that

$$\|e_j\|_{L^4(G)} \leq c\lambda_j^{1/4}.$$

Together with Lemma 4.1, this yields

$$\begin{aligned}
&\|\mathbf{z}\|_{L^4(\Omega\times[0,T]\times G)}^4 \\
&\leq 3c^4 \int_0^T (c_{t,H})^2 dt \sum_{n,m} \textstyle\sum_{i,j,k,\ell} (\lambda_i\lambda_j\lambda_k\lambda_\ell)^{-H+1/4} (\Phi e_n, e_j)(\Phi e_n, e_k)(\Phi e_m, e_i)(\Phi e_n, e_\ell) \\
&= 3c^4 \int_0^T (c_{t,H})^2 dt (\sum_n (\textstyle\sum_j \lambda_j^{-H+1/4} (\Phi e_n, e_j))^2)^2
\end{aligned}$$

where the constant $(c_{t,H})^2$ is either bounded ($H > 1/2$) or is bounded away from $t=0$ and behaves like t^{4H-2} near $t=0$. Since the time integral then converges for all $H > 1/4$, we have proved the following.

Theorem 4.2. *Assume $H > 1/4$. The evolution solution $\mathbf{z}$ of the stochastic parabolic equation (4.2) on $[0,T]\times G$ with the Stokes operator $\mathbf{A}$ and driven by the additive noise ΦdW^H, which is given by the formula (4.1), satisfies $\|\mathbf{z}\|_{L^4(\Omega\times[0,T]\times G)} < \infty$ as soon as*

$$\sum_n (\textstyle\sum_j \lambda_j^{-H+1/4} (\Phi e_n, e_j))^2 < \infty, \tag{4.3}$$

where $(\lambda_j, e_j)_j$ are the eigen-elements of $\mathbf{A}$.

Remark 4.1. *Since λ_j is asymptotically linear, Condition (4.3) is equivalent to*

$$\sum_n (\textstyle\sum_j j^{-H+1/4}(\Phi e_n, e_j))^2 < \infty,$$

We state the following corollary, which is specific to the co-diagonalizable case.

Corollary 4.3. *Under the assumptions of Theorem 4.2, if in addition Φ is co-diagonalizable with $\mathbf{A}$ in the sense that $(\Phi e_n, e_j) = 0$ if $j \neq n$, then, denoting $q_n = (\Phi e_n, e_n)$ the nth eigenvalue of Φ, Condition (4.3) in Theorem 4.2 becomes*

$$\sum_n \lambda_n^{-2H+1/2}(q_n)^2 < \infty,$$

or equivalently

$$\sum_n n^{-2H+1/2}(q_n)^2 < \infty,$$

This corollary is to be compared to the necessary and sufficient condition for pointwise existence of a solution to the stochastic heat equation driven by ΦdW^H. We leave it to the reader to check that the method of Section 3.1 in [14] can be used to prove that this solution exists if and only if $\sum_n n^{-2H} q_n < \infty$. One can check that this is a weaker condition than (4.3) when $H > 1/4$. This is accounted for by the fact that the eigenfunctions of the Laplacian are bounded, unlike those of the Stokes operator $\mathbf{A}$ which grow like the 4th root of the eigenvalues. However, we do not claim that our result for $\mathbf{A}$ is optimal; indeed, this optimality will not be known until one discovers the asymptotic behavior of the inner products of the eigenfunctions of $\mathbf{A}$.

5. Integrability when $1/8 < H \leq 1/4$

We consider the integrability of the convolution integral $\mathbf{z}$ in the case $H \in (\frac{1}{8}, \frac{1}{4})$. We first recall the following basic inequality.

Minkowski's Inequality for Integrals: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let f be an $\mathcal{M} \otimes \mathcal{N}$-measurable function on $X \times Y$. If $f \geq 0$ and $1 \leq p < \infty$, then

$$\left[\int \left(\int f(x,y)d\nu(y)\right)^p d\mu(x)\right]^{1/p} \leq \int \left[\int f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y).$$

We may now state and prove our existence theorem via the following calculatory lemma, proved in the Appendix.

Lemma 5.1. *For $0 < H < \frac{1}{2}$, $\int_0^t e^{-2x}\left(\int_0^x y^{H-\frac{3}{2}}(e^y - 1)dy\right)^2 dx$ is bounded for all $t \geq 0$.*

Theorem 5.2. *Let $\mathbf{z}$ be defined in (4.1). Assume $H \in (1/8; 1/4]$. The evolution solution $\mathbf{z}$, given by (4.1), of the stochastic parabolic equation (4.2), satisfies $\|\mathbf{z}\|_{L^4(\Omega\times[0,T]\times G)} < \infty$ as soon as*

$$\sum_n \left(\sum_i |(\Phi e_n, e_i)| \lambda_i^{1/4}\right)^2 < \infty.$$

Proof.

Step 0: setup.

$$\begin{aligned}\mathbf{E}\int_0^T\int_G \mathbf{z}^4(t,x)dxdt =&\mathbf{E}\int_0^T\int_G\left(\int_0^t e^{-(t-s)\mathbf{A}}\Phi dW^H(s)\right)^4 dxdt\\ =&\mathbf{E}\int_0^T\int_G\left(\sum_n\int_0^t e^{-(t-s)\mathbf{A}}\Phi e_n d\beta_n^H(s)\right)^4 dxdt\\ =&\mathbf{E}\int_0^T\int_G\left(\sum_n\int_0^t a_s^t(n)+b_s^t(n)dW_n(s)\right)^4 dxdt,\end{aligned}$$

where

$$a_s^t(n)=K(t,s)e^{-(t-s)\mathbf{A}}\Phi e_n$$

and

$$b_s^t(n)=\int_s^t(e^{-(t-r)\mathbf{A}}-e^{-(t-s)\mathbf{A}})\Phi e_n\frac{\partial K(r,s)}{\partial r}dr.$$

Then, standard properties of Gaussian r.v.'s imply

$$\begin{aligned}&\mathbf{E}\int_0^T\int_G\mathbf{z}^4(t,x)dxdt\\ =&\mathbf{E}\int_0^T\int_G\left[\sum_n\left(\int_0^t a_s^t(n)dW_n(s)\right)^4+\sum_n\left(\int_0^t b_s^t(n)dW_n(s)\right)^4\right.\\ &\left.+3\sum_{n\neq m}\left(\int_0^t a_s^t(n)dW_n(s)\right)^2\left(\int_0^t b_s^t(m)dW_m(s)\right)^2\right]dxdt\\ =:&G_1+G_2+G_3.\end{aligned}\tag{5.1}$$

Step 1: the term G_1.

The first term G_1 of (5.1) is

$$\begin{aligned}G_1=&\int_0^T\int_G\sum_n\mathbf{E}(\int_0^t K(t,s)e^{-(t-s)\mathbf{A}}\Phi e_n dW_n(s))^4dxdt\\ =&\int_0^T\int_G\sum_n 3\left(\mathbf{E}\left[\left(\int_0^t K(t,s)e^{-(t-s)\mathbf{A}}\Phi e_n dW_n(s)\right)^2\right]\right)^2dxdt.\end{aligned}$$

Since $K(t,s)\le c(H)(t-s)^{H-\frac12}s^{H-\frac12}$ (given in Decreusefond and Ustunel [4]), we get

$$\begin{aligned}&\mathbf{E}\left[\left(\int_0^t K(t,s)e^{-(t-s)\mathbf{A}}\Phi e_n dW_n(s)\right)^2\right]\\ \le&\int_0^t\left(\sum_j(\Phi e_n,e_j)e_j(x)(t-s)^{H-1/2}s^{H-1/2}e^{-(t-s)\lambda_j}\right)^2ds\end{aligned}$$

$$\leq \int_0^t \sum_{i,j} |(\Phi e_n, e_i)| \,.\, |(\Phi e_n, e_j)| \,.\, |e_i(x)| \,.\, |e_j(x)| \;\; (t-s)^{2H-1} s^{2H-1} e^{-(t-s)(\lambda_i+\lambda_j)} ds$$

$$\leq \sum_{i,j} |(\Phi e_n, e_i)| \,.\, |(\Phi e_n, e_j)| \,.\, |e_i(x)| \,.\, |e_j(x)| \cdot \left(\int_0^t (t-s)^{2H-1} s^{2H-1} e^{-2(t-s)\lambda_i} ds \right)^{1/2}$$

$$\cdot \left(\int_0^t (t-s)^{2H-1} s^{2H-1} e^{-2(t-s)\lambda_j} ds \right)^{1/2}.$$

We calculate the last integrals above, via the next lemma, with proof in the Appendix.

Lemma 5.3. $\int_0^t (t-s)^{2H-1} s^{2H-1} e^{-2(t-s)\lambda_j} ds \leq \frac{2^{-4H}}{H}(1+e^{-\lambda_j t}) t^{4H-1}$.

Let C be a generic constant which may depend on H and change from line to line. By the above calculation and lemma,

$$\mathbf{E}\left[\left(\int_0^t K(t,s) e^{-(t-s)\mathbf{A}} \Phi e_n dW_n(s) \right)^2 \right]$$
$$\leq C \sum_{i,j} |(\Phi e_n, e_i)||(\Phi e_n, e_j)||e_i(x)||e_j(x)| t^{4H-1}.$$

Therefore,

$$G_1 \leq C \int_0^T \int_G \sum_n \left(\sum_{i,j} |(\Phi e_n, e_i)||(\Phi e_n, e_j)||e_i(x)||e_j(x)| t^{4H-1} \right)^2 dx dt$$

$$\leq C \int_G \sum_n \left(\sum_{i,j} |(\Phi e_n, e_i)||(\Phi e_n, e_j)||e_i(x)||e_j(x)| \right)^2 \frac{T^{8H-1}}{8H-1} dx$$

$$= C \sum_n \sum_{i_1,i_2,j_1,j_2} \left| \prod_{k=1}^2 (\Phi e_n, e_{i_k}) \right| |(\Phi e_n, e_{j_k})||e_{i_k}(x)||e_{j_k}(x)| dx$$

$$\leq C \sum_n \left(\sum_i |(\Phi e_n, e_i)| \lambda_i^{1/4} \right)^4 < \infty$$

by the assumption on Φ. So G_1 is finite under the Theorem's assumptions, and in fact the assumptions are stronger than is needed here. However, the full strength of the assumption will be needed to control G_2.

Step 2: the term G_2.

Let

$$b_s^t(n,j) := \int_s^t (e^{-(t-r)\lambda_j} - e^{-(t-s)\lambda_j})(\Phi e_n, e_j) e_j \frac{\partial K(r,s)}{\partial r} dr,$$

then the second term is

$$
\begin{aligned}
G_2 =& 3\int_0^T \int_G \sum_n \left(\mathbf{E}\left[\left(\sum_j \int_0^t b_s^t(n,j) dW_n(s) \right)^2 \right] \right)^2 dxdt \\
=& 3\int_0^T \int_G \sum_n I^2 dxdt
\end{aligned}
$$

where $I := \mathbf{E}\left[\left(\sum_j \int_0^t b_s^t(n,j)dW_n(s)\right)^2\right]$. Since $|\frac{\partial K(r,s)}{\partial r}| \le C(H)(r-s)^{H-\frac{3}{2}}$ (again see [4]), we get that

$$
\begin{aligned}
I =& \mathbf{E}\sum_{i,j} \int_0^t b_s^t(n,i) b_s^t(n,j) ds \\
\le& C\sum_{i,j} \int_0^t \left(\int_s^t (e^{-(t-r_1)\lambda_i} - e^{-(t-s)\lambda_i}) |(\Phi e_n, e_i)||e_i(x)|(r-s)^{H-\frac{3}{2}} dr_1 \right. \\
& \left. \cdot \int_s^t (e^{-(t-r_2)\lambda_j} - e^{-(t-s)\lambda_j}) |(\Phi e_n, e_j)||e_j(x)|(r-s)^{H-\frac{3}{2}} dr_2 \right) ds \\
=& C\sum_{i,j} |(\Phi e_n, e_i)||e_i(x)||(\Phi e_n, e_j)||e_j(x)| \int_0^t ds \int_s^t dr_1 \int_s^t dr_2 (r_1-s)^{H-3/2}(r_2-s)^{H-3/2} \\
& \cdot (e^{-(t-r_1)\lambda_i} - e^{-(t-s)\lambda_i})(e^{-(t-r_2)\lambda_j} - e^{-(t-s)\lambda_j})
\end{aligned}
$$

Using the change of variables $u = t-s$, $v_1 = t-r_1$, $v_2 = t-r_2$, the above expression becomes:

$$
\begin{aligned}
& C\sum_{i,j} |(\Phi e_n, e_i)||e_i(x)||(\Phi e_n, e_j)||e_j(x)| \int_0^t du \int_0^u dv_1 \int_0^u dv_2 (u-v_1)^{H-3/2}(u-v_2)^{H-3/2} \\
& \cdot (e^{-v_1\lambda_i} - e^{-u\lambda_i})(e^{-v_2\lambda_j} - e^{-u\lambda_j}) \\
=& C\sum_{i,j} |(\Phi e_n, e_i)||e_i(x)||(\Phi e_n, e_j)||e_j(x)| \int_0^t du \,(\int_0^u (u-v_1)^{H-3/2}(e^{-v_1\lambda_i} - e^{-u\lambda_i}) dv_1) \\
& \cdot (\int_0^u (u-v_2)^{H-3/2}(e^{-v_2\lambda_j} - e^{-u\lambda_j}) dv_2)
\end{aligned} \tag{5.2}
$$

Setting $r = u - v_1$, we can write

$$
\int_0^u (u-v_1)^{H-3/2}(e^{-v_1\lambda_i} - e^{-u\lambda_i}) dv_1 = e^{-\lambda_i u} \int_0^u r^{H-3/2}(e^{\lambda_i r} - 1) dr.
$$

Using this twice, we get that the expression on the right side of (5.2) is equal to

$$
\begin{aligned}
& C\sum_{i,j} |(\Phi e_n, e_i)||e_i(x)||(\Phi e_n, e_j)||e_j(x)| \int_0^t e^{-(\lambda_i+\lambda_j)u} \\
& \cdot (\int_0^u r_1^{H-3/2}(e^{\lambda_i r_1} - 1) dr_1)(\int_0^u r_2^{H-3/2}(e^{\lambda_j r_2} - 1) dr_2) du \\
\le& K_H \sum_{i,j} |(\Phi e_n, e_i)||e_i(x)||(\Phi e_n, e_j)||e_j(x)| \lambda_i^{-H} \lambda_j^{-H}
\end{aligned}
$$

where we have used Lemma 2 in Tindel, Tudor and Viens [14].

Thus the square of the expression in (5.2) can be bounded above by

$$K_H^2 \sum_{i_1,i_2,j_1,j_2} \prod_{k=1}^{2} |(\Phi e_n, e_{i_k})||e_{i_k}(x)||(\Phi e_n, e_{j_k})||e_{j_k}(x)|\lambda_{i_k}^{-H}\lambda_{j_k}^{-H}.$$

Therefore,

$$G_2 = 3\int_0^T \int_G \sum_n I^2 dxdt \le 3\sum_n \left(\sum_i |(\Phi e_n, e_i)|\lambda_i^{1/4-H}\right)^4$$

which is finite by the hypothesis on Φ.

Step 3: the term G_3.

Finally, the third term G_3 in (5.1) is

$$\begin{aligned} G_3 =& 3\mathbf{E}\int_0^T \int_G \sum_{m,n,m\neq n} \left(\int_0^t a_s^t(n)dW_n(s)\right)^2 \left(\int_0^t b_s^t(m)dW_m(s)\right)^2 dxdt \\ \le& C(H)\int_T \int_G \sum_{m,n} \mathbf{E}\left[\left(\int_0^t a_s^t(n)dW_n(s)\right)^2\right] \cdot \mathbf{E}\left[\left(\int_0^t b_s^t(m)dW_m(s)\right)^2\right] dxdt \end{aligned}$$

The basic inequality $2ab \le a^2 + b^2$ means this is finite by the calculations in the previous two steps, concluding the proof of the theorem. □

6. **Existence and uniqueness of solutions**

The existence and uniqueness of mild solutions to Navier-Stokes evolution systems have been studied by a number of authors (cf. Da Prato and Zabczyk [2], Sohr [12], Temam [13]). The method of solvability in the stochastic case, consists in breaking up the system (2.5) into a linear stochastic system and a nonlinear partial differential equation. Since our system is perturbed by an additive fractional noise term, this approach works in a straightforward way. The theorem below is this article's main result. Its proof is given in full detail, and divided into several steps, for the reader's convenience. Note that this theorem falls short of proving that the solution exists in $L^4([0,T]\times G\times\Omega)$, showing only that almost surely, it belongs to the space $L^4([0,T]\times G)$. We will investigate the stronger, former statement in a separate publication, conjecturing here that the solution is only square-integrable with respect to Ω.

Theorem 6.1. *Let $\{e_n : n\in\mathbf{N}\}$ be an orthonormal basis in the Hilbert space H of eigenfunctions of the Stokes operator $\mathbf{A}$. Under the following two conditions, there exists a unique mild solution of the stochastic Navier-Stokes system, i.e. $\mathbf{P}$-almost surely, there is a unique solution in $L^4([0,T]\times G)$ to equation (2.5) driven by the infinite-dimensional fractional Brownian noise $\Phi\mathbf{W}^H$:*

1. *$H > 1/4$, and $\sum_n\left(\sum_j j^{1/4-H}\,|(\Phi e_n, e_j)|\right)^2 < \infty$, or*
2. *$1/8 < H \le 1/4$, and $\sum_n\left(\sum_j j^{1/4}\,|(\Phi e_n, e_j)|\right)^2 < \infty$.*

Proof. Step 1. Consider the system

$$d\mathbf{u} + [\nu \mathbf{A}\, \mathbf{u} + \mathbf{B}(\mathbf{u})]\, dt = \Phi d\mathbf{W}_t^H,$$

as in (2.1). In order to find the solution $\mathbf{u}$, we will use the previous theorems, which tell us how to find the unique evolution (mild) solution $\mathbf{z}(t)$ of

$$d\mathbf{z}\,(t) + \mathbf{Azdt} = \Phi d\mathbf{W}_t^H,$$

with $\mathbf{z}(0) = 0$. If $\mathbf{u}$ existed, say in a strong sense, we would denote $\mathbf{v} := \mathbf{u} - \mathbf{z}$, and notice that

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} &= \frac{\partial \mathbf{u}}{\partial t} - \frac{\partial \mathbf{z}}{\partial t} \\
&= (-\mathbf{Au} - \mathbf{B}(\mathbf{u}) + \Phi \frac{d\mathbf{W}^H}{dt}) - (-\mathbf{Az} + \Phi \frac{d\mathbf{W}^H}{dt}) \\
&= -\mathbf{A}(\mathbf{u} - \mathbf{z}) - \mathbf{B}(\mathbf{u}) = -\mathbf{Av} - \mathbf{B}(\mathbf{v} + \mathbf{z})
\end{aligned}$$

Therefore, with $\mathbf{z}$ given, solving for $\mathbf{u}$ in (2.5) would be equivalent to solving for $\mathbf{v}$ in

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{Av} + \mathbf{B}(\mathbf{v} + \mathbf{z}) = 0 \tag{6.1}$$

with initial data $\mathbf{v}(0) = \mathbf{u}_0 \in H$.

More precisely, Theorems 4.2 and 5.2 guarantee the existence (and uniqueness) in $L^4(\Omega \times [0,T] \times G)$ of $\mathbf{z}$ as a mild solution of (4.2) given by formula (4.1); therefore the evolution equation (2.5) has a unique solution mild in that same space (starting from $\mathbf{u}_0$) if the evolution (mild) version of equation (6.1) admits a solution in $L^4(\Omega \times [0,T] \times G)$ as well. This evolution solution $\mathbf{v}$, when it exists in that space, satisfies

$$\mathbf{v}(t) = S(t)\mathbf{u}_0 - \int_0^t S(t-s)\mathbf{B}(\mathbf{v}(s) + \mathbf{z}(s))ds \tag{6.2}$$

where $S(t) = e^{-t\mathbf{A}}$ is the semigroup generated by the operator $\mathbf{A}$. Let us introduce notation meant to signify that equation (6.2) is a fixed point problem:

$$\Lambda\,(\mathbf{w}) := S(t)\mathbf{u}_0 - \int_0^t S(t-s)\mathbf{B}(\mathbf{w}(s) + \mathbf{z}(s))ds.$$

Studying the properties of this operator Λ is the main subject of this proof.

Step 2: Let $\mathbf{w} \in L^4([0,T] \times G) \cap V$. We will show that $\mathbf{B}(\mathbf{w} + \mathbf{z}) \in L^2(0,T;V')$. Indeed, for any $\phi \in L^2(0,T;V)$, and suppressing time in the argument of functions, and denoting $\mathbf{w} + \mathbf{z}$ as y,

$$\begin{aligned}
|\langle \mathbf{B}(y), \phi \rangle| &= |b(y, \phi, y)| \\
&= \left| \sum_{i=1}^{2} \int_G y_i \frac{d\phi_j}{dx_i} y_j dx \right| \\
&\le |y|_{L^4(G)} |\nabla \phi|_{L^2(G)} |y|_{L^4(G)}.
\end{aligned} \tag{6.3}$$

By the Poincaré inequality which applies by the boundedness of the domain G and the zero boundary condition, we get the equivalence of $|\nabla \phi|_{L^2(G)}$ and $\|\phi\|_V$. Hence,

$$\int_0^T |\langle \mathbf{B}(y), \phi \rangle|^2 ds \le \int_0^T |y|_{L^4(G)}^2 |\nabla \phi|_{L^2(G)} ds. \tag{6.4}$$

By the Schwarz inequality, the assertion of this step is obtained.

Step 3: We show here that $\Lambda(\mathbf{w}) \in L^\infty(0,T;H) \cap L^2(0,T;V) =: Y$, and in fact,

$$||\Lambda(\mathbf{w})||_Y \le ||\mathbf{B}(\mathbf{w}+\mathbf{z})||_{L^2(0,T;V')}.$$

Indeed, by the Sobolev embedding theorem $H^{1/2} = W^{1/2,2}(G) \hookrightarrow L^4(G)$, there is a non-random constant C depending only on the bounded domain G such that $\|\mathbf{u}\|_{L^4(G)} \le C\|\mathbf{u}\|_{W^{1/2,2}(G)}$. Using this in (6.3),

$$\begin{aligned}|\langle \mathbf{B}(y), \phi\rangle| &\le C|y|^2_{W^{\frac{1}{2},2}} \|\phi\|_V \\ & C|y|_H \|y\|_V \|\phi\|_V\end{aligned}$$

by the interpolation theorem. Thus $\|\mathbf{B}(y)\|_{V'} \le C|y|_H \|y\|_V$.

Now define $h(t) = -\int_0^t S(t-s)\mathbf{B}(y(s))ds$, and $y \in L^4([0,T]\times G)$. Then $h(0)=0$, and by the energy equality,

$$\begin{aligned}|h(t)|^2_{L^2} =& -2\int_0^t |\nabla h|^2_{L^2} ds - 2\int_0^t \langle \mathbf{B}(y(s)), h(s)\rangle_{V'\times V} ds \\ \le& -2\int_0^t |h|^2_V ds + 2\int_0^t |\mathbf{B}(y(s))|_{V'} \cdot |h(s)|_V ds \\ \le& -2\int_0^t |h|^2_V ds + \int_0^t |\mathbf{B}(y(s))|^2_{V'} ds + \int_0^t |h(s)|^2_V ds.\end{aligned}$$

So

$$|h(t)|^2_H + \int_0^t |h(s)|^2_V ds \le \int_0^t |\mathbf{B}(y)|^2_{V'} ds,$$

and thus

$$\sup_{0\le t\le T} |h(t)|^2_H + \int_0^t |h(s)|^2_V ds \le 2\int_0^t |\mathbf{B}(y)|^2_{V'} ds,$$

which is bounded. Therefore, $h(t) \in L^\infty(0,T;H) \cap L^2(0,T;V)$. Therefore, $\Lambda(\mathbf{w}) \in L^\infty(0,T;H) \cap L^2(0,T;V)$.

Step 4: Let L^4 denote $L^4([0,T]\times G) = L^4(0,T;L^4(G))$. We now show that for any $\mathbf{w}_1, \mathbf{w}_2 \in L^4([0,T]\times G) \cap V$, we have

$$|\Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)|_{L^4} \le CC' |\mathbf{w}_1 - \mathbf{w}_2|_{L^4} (|\mathbf{w}_1 + \mathbf{z}|_{L^4} + |\mathbf{w}_2 + \mathbf{z}|_{L^4}).$$

Here C is the universal (G-dependent) constant from the Sobolev embedding theorem used in Step 3, and C' is another constant which depends only on G.

For any $\mathbf{u}_1$, $\mathbf{u}_2 \in L^4$, and $\phi \in V$, we have

$$\begin{aligned}&|\langle \mathbf{B}(\mathbf{u}_1) - \mathbf{B}(\mathbf{u}_2), \psi\rangle| \\ &= |b(\mathbf{u}_1,\mathbf{u}_1,\psi) - b(\mathbf{u}_2,\mathbf{u}_2,\psi)| \\ &\le |b(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1, \psi)| + |b(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2, \psi)| \\ &= |b(\mathbf{u}_1 - \mathbf{u}_2, \psi, \mathbf{u}_1)| + |b(\mathbf{u}_2, \psi, \mathbf{u}_1 - \mathbf{u}_2)| \\ &\le |\mathbf{u}_1 - \mathbf{u}_2|_{L^4} |\nabla\psi|_H |\mathbf{u}_1|_{L^4} + |\mathbf{u}_2|_{L^4} |\nabla\psi|_H |\mathbf{u}_1 - \mathbf{u}_2|_{L^4} \\ &= |\mathbf{u}_1 - \mathbf{u}_2|_{L^4} |\psi|_V (|\mathbf{u}_1|_{L^4} + |\mathbf{u}_2|_{L^4}),\end{aligned}$$

which implies, by Jensen's inequality for some C' depending only on G,

$$|\mathbf{B}(\mathbf{u}_1)-\mathbf{B}(\mathbf{u}_2)|_{L^2(0,T;V')}\leq C'|\mathbf{u}_1-\mathbf{u}_2|_{L^4}(|\mathbf{u}_1|_{L^4}+|\mathbf{u}_2|_{L^4})$$

and thus

$$|\mathbf{B}(\Lambda(\mathbf{w}_1)+\mathbf{z})-\mathbf{B}(\Lambda(\mathbf{w}_2)+\mathbf{z})|_{L^2(0,T;V')}\leq C'|\Lambda(\mathbf{w}_1)-\Lambda(\mathbf{w}_2)|_{L^4}(|\Lambda(\mathbf{w}_1)+\mathbf{z}|_{L^4}+|\Lambda(\mathbf{w}_2)+\mathbf{z}|_{L^4}).$$

If we let $\mathbf{y}_j=\Lambda(\mathbf{w}_j)+\mathbf{z}$ for $j=1,2$, then, using again the Sobolev embedding of $L^4(G)$ in $W^{1/2,2}$,

$$|h(\mathbf{y}_1)-h(\mathbf{y}_2)|_{L^4(G)}\leq C|h(\mathbf{y}_1)-h(\mathbf{y}_2)|_{W^{\frac{1}{2},2}}\leq C|h(\mathbf{y}_1)-h(\mathbf{y}_2)|_H^{\frac{1}{2}}\cdot|h(\mathbf{y}_1)-h(\mathbf{y}_2)|_V^{\frac{1}{2}}$$

Note that $h(\mathbf{y}_1)-h(\mathbf{y}_2)=\Lambda(\mathbf{w}_1)-\Lambda(\mathbf{w}_2)$ so that by the above estimate

$$\begin{aligned}|\Lambda(\mathbf{w}_1)-\Lambda(\mathbf{w}_2)|_{L^4}&=|h(\mathbf{y}_1)-h(\mathbf{y}_2)|_{L^4}\\&=(\int_0^T|h(\mathbf{y}_1)-h(\mathbf{y}_2)|_{L^4}^4dt)^{\frac{1}{4}}\\&\leq C(|h(\mathbf{y}_1)-h(\mathbf{y}_2)|_H^2\cdot\int_0^T|h(\mathbf{y}_1)-h(\mathbf{y}_2)|_V^2dt)^{\frac{1}{4}}\\&\leq C(\sup_{0\leq t\leq T}|h(\mathbf{y}_1)-h(\mathbf{y}_2)|_H^2\cdot\int_0^T|h(\mathbf{y}_1)-h(\mathbf{y}_2)|_V^2dt)^{\frac{1}{4}}\\&\leq C[(\int_0^T|\mathbf{B}(\mathbf{y}_1)-\mathbf{B}(\mathbf{y}_2)|_{V'}^2dt)^2]^{\frac{1}{4}}\\&\leq CC'|\mathbf{y}_1-\mathbf{y}_2|_{L^4}(|\mathbf{y}_1|_{L^4}+|\mathbf{y}_2|_{L^4})\\&=CC'|\mathbf{w}_1-\mathbf{w}_2|_{L^4}(|\mathbf{w}_1+\mathbf{z}|_{L^4}+|\mathbf{w}_2+\mathbf{z}|_{L^4}).\end{aligned}$$

Step 5:

The previous step proves that the operator

$$\Lambda:\begin{cases}L^4\mapsto L^4\\ \mathbf{w}\ \mapsto\Lambda(\mathbf{w}):=S(\cdot)\mathbf{u}_0-\int_0^\cdot S(\cdot-s)\mathbf{B}(\mathbf{w}(s)+\mathbf{z}(s))ds\end{cases}$$

is well-defined as mapping L^4 to itself. As mentioned in Step 1, from Theorems 4.2 and 5.2, $\mathbf{z}$ is in $L^4(\Omega\times[0,T]\times G)$, which implies that $\mathbf{z}\in L^4$ almost surely. Fix any ω in this almost sure set. Note that

$$\Lambda(0)=S(\cdot)\mathbf{u}_0-\int_0^\cdot S(\cdot-s)\mathbf{B}(\mathbf{z}(s))ds$$

is then a fixed function on $[0,T]\times G$, and a member of L^4. Let $\eta=\frac{1}{4CC'}$. Replacing T in the definition of L^4 by a smaller value, one can choose a time $T_1>0$ small enough, which may depend on ω, so that

$$|\mathbf{\Lambda}(0)|_{L^4}\leq\frac{\eta}{2}\quad\text{and}\quad|\mathbf{z}|_{L^4}\leq\frac{\eta}{4}.$$

Define $L:=\{\mathbf{w}\in L^4:|\mathbf{w}+\mathbf{z}|_{L^4}\leq\eta\}$. Then $0\in L$ and $\Lambda(0)\in L$. In fact we have $|\Lambda(0)|\leq\frac{\eta}{2}$. Therefore, denoting by Λ^j the jth iteration of the map Λ, we get by the result of the previous step,

$$\begin{aligned}|\Lambda^2(0)|_{L^4}&\leq|\Lambda(\Lambda(0))-\Lambda(0)|_{L^4}+\frac{\eta}{2}\\&\leq CC'|\Lambda(0)|_{L^4}(|\Lambda(0)+z|_{L^4}+|z|_{L^4})+\frac{\eta}{2}\end{aligned}$$

$$\leq CC'|\Lambda(0)|_{L^4}\eta + \frac{\eta}{2}$$
$$\leq \frac{\eta}{8} + \frac{\eta}{2}.$$

Thus $|\Lambda^2(0) + z| \leq \eta(1/2 + 1/4 + 1/8)$ which means $\Lambda^2(0) \in L$.

More generally we can prove by induction that $|\Lambda^n(0)| < \frac{3}{4}\eta$ for all $n \geq 1$. Indeed, this is true for $n = 1, 2$, and repeating the above calculation we get

$$\begin{aligned}|\Lambda^n(0)|_{L^4} &\leq \left|\Lambda(\Lambda^{n-1}(0)) - \Lambda(0)\right|_{L^4} + |\Lambda(0)|_{L^4} \leq C\frac{3}{4}\eta(1+\frac{1}{4})\eta + \frac{\eta}{2} \\ &= C\frac{3}{4}\eta(1+\frac{1}{4})\frac{1}{4C} + \frac{\eta}{2} < \frac{\eta}{4} + \frac{\eta}{2} = \frac{3}{4}\eta.\end{aligned}$$

This proves that $|\Lambda^n(0) + z| < \eta$ for all $n \geq 1$. Thus, $\{\Lambda^n(0)\}$ is a sequence that remains within the closed set L. With our choice of η, Λ is a contraction in L: indeed, the result of Step 4 implies, for points $\mathbf{w}_1$ and $\mathbf{w}_2$ restricted to L, that

$$\begin{aligned}|\Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)|_{L^4} &\leq CC'|\mathbf{w}_1 - \mathbf{w}_2|_{L^4}(|\mathbf{w}_1 + \mathbf{z}|_{L^4} + |\mathbf{w}_2 + \mathbf{z}|_{L^4}) \\ &\leq |\mathbf{w}_1 - \mathbf{w}_2|_{L^4} CC' \cdot 2 \cdot \eta = \frac{1}{2}|\mathbf{w}_1 - \mathbf{w}_2|_{L^4}.\end{aligned}$$

Thus $\{\Lambda^n(0) : n \geq 1\}$ converges to a function $\mathbf{v} \in L$, which is the unique fixed point of the map Λ in L; this is the unique solution in L of equation (6.2) restricted to $[0, T_1]$, i.e. the unique evolution solution in L of the stochastic Navier Stokes equation (2.1) on $[0, T_1]$.

Step 6:

If there existed another distinct solution $\tilde{\mathbf{v}}$ to equation (6.2) on $[0, T_1]$, it would have to not be in L. Then by replacing T_1 by a sufficiently smaller time $T_0 < T_1$, the other solution $\tilde{\mathbf{v}}$ can be made to be in L also. Therefore, $\mathbf{v}$ and $\tilde{\mathbf{v}}$ coincide on $[0, T_0]$. In other words, for almost every fixed ω, we have existence and uniqueness of the solution to equation (6.2) up to some time T_0 which may depend on ω.

Now considering $\mathbf{v}(T_0)$ instead of $\mathbf{u}_0$ as a new initial condition of the evolution equation (6.2), one can find $T_2 > T_0$ such that the time interval can be extended to $[0, T_2]$ on which the unique solution exists. Continuing this way, suppose R is the maximum time up to which the unique solution exists, and suppose $R < T$, then there exists $R_1 > R$ such that the unique solution exists on $[0, R_1]$. Therefore, $R = T$. Therefore, for almost every ω, the unique solution exists on the entire time interval $[0, T]$, and belongs to L^4. □

7. Appendix

Proof of Lemma 4.1:

Case 1: $H > 1/2$. Using the notation in [14] (see equation (23) therein), we have for any $\lambda \geq 0$,

$$\begin{aligned}A(\lambda, t) := |\mathbf{1}_{[0,t]}e^{-\lambda(t-\cdot)}|^2_{\mathcal{H}} &= \int_0^{\lambda t} v^{2H-2}e^{-v}[1 - e^{-2(t\lambda - v)}]dv \\ &\leq \int_0^{\infty} v^{2H-2}e^{-v}dv =: C_0(H).\end{aligned}$$

Case 2: $H < 1/2$. Using the notation in [14] (see the calculation immediately preceding equation (26) therein), we have for all $\lambda \geq 0$,

$$A(\lambda, t) := |\mathbf{1}_{[0,t]} e^{-\lambda(t-\cdot)}|^2_{\mathcal{H}} \leq B_1(\lambda, t) + B_2(\lambda, t)$$

where

$$B_1(\lambda, t) := \lambda^{-2H} c(H) \int_0^{2\lambda t} e^{-v} v^{2H-1} (t - v/(2\lambda))^{2H-1} dv$$

with the well-known constant $c(H)$ defined for instance in [4], Theorem 3.2, and

$$B_2(\lambda, t) := C(H) \int_0^t e^{-2\lambda s} (\int_0^s (e^{\lambda r} - 1) r^{H-3/2} dr)^2 ds$$

where $C(H) := c(H)(H - 1/2)$.

By a linear change of variables, and then using Lemma 2 in [14] with the constant K_A defined therein, we get

$$\begin{aligned} B_2(\lambda, t) &= C(H) t^{2H} \int_0^1 e^{-2\lambda t s} (\int_0^s (e^{\lambda t r} - 1) r^{H-3/2}) \\ &\leq C(H) K_{H-1/2}\ t^{2H} (\lambda t)^{-2H} \\ &=: C_2(H) \lambda^{-2H}. \end{aligned}$$

Now for the term B_1, splitting the integral up at the midpoint λt, and changing the variable for the second half of the interval, we can write,

$$\begin{aligned} B_1(t, \lambda) &\leq c(H) \lambda^{-2H} [(t/2)^{2H-1} \int_0^\infty e^{-v} v^{2H-1} dv + (\lambda t)^{2H-1} \int_0^{\lambda t} e^{-(2\lambda t - v)} (v/(2\lambda))^{2H-1} dv] \\ &\leq c(H) \lambda^{-2H} (t/2)^{2H-1} [\int_0^\infty e^{-v} v^{2H-1} dv + e^{-\lambda t} (\lambda t)^{2H} / (2H)]. \end{aligned}$$

The function $x \mapsto e^{-x} x^{2H}$ attains its maximum value of $e^{-2H}(2H)^{2H}$ on $\mathbf{R}_+$ at the point $x = 2H$. Therefore we can write

$$B_1(t, \lambda) \leq \lambda^{-2H} c(H) (t/2)^{2H-1} d(H)$$

where $d(H) := \int_0^\infty e^{-v} v^{2H-1} dv + e^{-2H}(2H)^{2H}$.

We now have for all $\lambda, t \geq 0$, when $H < 1/2$, with $c(H)$, $d(H)$, and $C_2(H)$ defined above,

$$A(\lambda, t) \leq \lambda^{-2H} (2^{1-2H} c(H) d(H) t^{2H-1} + C_2(H)).$$

Gathering our results, the lemma now follows, with $C(H) = \max(C_0(H), 2^{1-2H} c(H) d(H), C_2(H))$. ■

Proof of Lemma 5.1.

Define two measures on $[0,\infty)$, $d\nu(y) = y^{H-\frac{3}{2}}(e^y-1)dy$ and $d\mu(x) = e^{-2x}dx$, then both ν and μ are σ-finite. To see this, it suffices to show that for fixed $n = 0, 1, 2, \cdots$, $\nu([n, n+1))$ and $\mu([n, n+1))$ are finite. Clearly, $\mu([n, n+1)) < \infty$. For ν,

$$\begin{aligned}
\nu([0,1)) &= \int_0^1 y^{H-\frac{3}{2}}(e^y-1)dy \\
&= \lim_{\epsilon\to 0}\int_\epsilon^1 y^{H-\frac{3}{2}}(e^y-1)dy \\
&= \frac{e-1}{H-\frac{1}{2}} - \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{y^{H-\frac{1}{2}}e^y}{H-\frac{1}{2}}dy \qquad \text{by integration by parts} \\
&= \frac{e-1}{H-\frac{1}{2}} - \frac{1}{H-\frac{1}{2}}\Big(\frac{e}{H+\frac{1}{2}} - \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{y^{H+\frac{1}{2}}e^y}{H+\frac{1}{2}}dy\Big) \\
&= -\frac{e-1}{\frac{1}{2}-H} + \frac{e}{(\frac{1}{2}-H)(\frac{1}{2}+H)} - \frac{1}{\frac{1}{2}-H}\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{y^{H+\frac{1}{2}}e^y}{H+\frac{1}{2}}dy\Big) \\
&\le -\frac{e-1}{\frac{1}{2}-H} + \frac{e}{(\frac{1}{2}-H)(\frac{1}{2}+H)} \\
&= \frac{e}{\frac{1}{2}+H} + \frac{1}{\frac{1}{2}-H} < \infty
\end{aligned}$$

For $n \ge 1$,

$$\begin{aligned}
\nu([n, n+1)) &= \int_n^{n+1} y^{H-\frac{3}{2}}(e^y-1)dy \\
&\le \int_n^{n+1} n^{H-\frac{3}{2}}(e^y-1)dy \\
&= n^{H-\frac{3}{2}}(e^{n+1}-e^n-1) < \infty
\end{aligned}$$

Therefore, by the Minkowski's inequality for integrals,

$$\begin{aligned}
&\Big[\int_0^t e^{-2x}\Big(\int_0^t 1_{(0,x)}(y)y^{H-\frac{3}{2}}(e^y-1)dy\Big)^2dx\Big]^{\frac{1}{2}} \\
&\le \int_0^t\Big[\int_0^t 1_{(0,x)}(y)e^{-2x}dx\Big] \\
&\le \int_0^t e^{-y}y^{H-\frac{3}{2}}(e^y-1)dy < \infty
\end{aligned}$$

So the result is obtained. The bound is independent of t. $\square$

Proof of Lemma 5.3.

$$\begin{aligned}
&\int_0^t (t-s)^{2H-1}s^{2H-1}e^{-2(t-s)\lambda_j}ds \\
=&\int_{2t\lambda_j}^0 \Big(\frac{w}{2\lambda_j}\Big)^{2H-1}\Big(t-\frac{w}{2\lambda_j}\Big)^{2H-1}e^{-w}\Big(-\frac{1}{2\lambda_j}\Big)dw, \qquad \text{with } (t-s) = \frac{w}{2\lambda_j} \\
=&(2\lambda_j)^{-2H}\int_0^{2t\lambda_j} e^{-w}w^{2H-1}\Big(t-\frac{w}{2\lambda_j}\Big)^{2H-1}dw
\end{aligned}$$

$$
\begin{aligned}
=&(2\lambda_j)^{-2H}(\int_0^{\lambda_j t} + \int_{\lambda_j t}^{2\lambda_j t})e^{-w}w^{2H-1}(t-\frac{w}{2\lambda_j})^{2H-1}dw \\
\leq&(2\lambda_j)^{-2H}[(\frac{t}{2})^{2H-1}\int_0^{\lambda_j t} e^{-w}w^{2H-1}dw + (\lambda_j t)^{2H-1}\int_{\lambda_j t}^{2\lambda_j t} e^{-w}(t-\frac{w}{2\lambda_j})^{2H-1}dw], \text{since} 2H-1<0 \\
\leq&(2\lambda_j)^{-2H}[(\frac{t}{2})^{2H-1}\int_0^{\lambda_j t} w^{2H-1}dw + (\lambda_j t)^{2H-1}\int_{t/2}^{0} e^{-2(t-s)\lambda_j}s^{2H-1}(-2\lambda_j)ds] \\
=&(2\lambda_j)^{-2H}[(\frac{t}{2})^{2H-1}\cdot\frac{(\lambda_j t)^{2H}}{2H} + (\lambda_j t)^{2H-1}(2\lambda_j)\int_0^{\frac{t}{2}} e^{-2(t-s)\lambda_j}s^{2H-1}ds] \\
\leq&(2\lambda_j)^{-2H}[(\frac{t}{2})^{2H-1}\cdot\frac{(\lambda_j t)^{2H}}{2H} + (\lambda_j t)^{2H-1}(2\lambda_j)(e^{-t\lambda_j}\frac{(\frac{t}{2})^{2H}}{2H})] \\
=&\frac{2^{-4H}}{H}(1+e^{-\lambda_j t})t^{4H-1}.
\end{aligned}
$$

□

References

[1] Alos, E.; Mazet, O.; Nualart, D. (1999) Stochastic calculus with respect to Gaussian processes. *Ann. Probab.*, **29**, no 2, 766-801.

[2] Da Prato, G., and Zabczyk, J. (1992) *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, Cambridge.

[3] Da Prato, G., and Debussche, A. (2008). *On the martingale probem associated to the 2D and 3D Stochastic Navier-Stokes equations.* Preprint, at http://arxiv.org/abs/0805.1906 .

[4] Decreusefond, L. and Ustunel, A.-S. (1997) Stochastic Analysis of the fractional Brownian motion, *Poten. Anal.*, **10**, 177-214.

[5] Kuppiainen, A. (2000) Statistical theories of turbulence; Lecture Notes *Random Media 2000*, Madralin.

[6] Ladyzhenskaya, O.A. (1969) *The Mathematical Theory of Viscous Incompressible Flow*, Gordon Breach Science Publishers, New York.

[7] Maslowski, B. and Schmalfuss, B. (2004) Random dynamical systems and stationary solutions of differential equations driven by fractional Brownian motions; *Stoch. Anal. Appl.* **22**, 1577-1607.

[8] Menaldi, J.-L., and Sritharan, S.S. (2002). Stochastic 2-D Navier–Stokes Equation. *Appl Math Optim* **46**, 31-53.

[9] Nualart, D. (2003) Stochastic integration with respect to fractional Brownian motion and applications; *Stochastic models* (Mexico City, 2002), 3–39, Contemp. Math., 336, AMS, Providence, RI.

[10] Nualart, D. (2006) Stochastic calculus with respect to fractional Brownian motion; *Ann. Fac. Sci. Toulouse Math.*(6), **15**, 63–78.

[11] Nualart, D. (2006) *Malliavin Calculus and Related Topics. Second Edition.* Springer.

[12] Sohr, H. (2001) *The Navier-Stokes Equations*, Birkhäuser Verlag, Basel.

[13] Temam, R. (1984) *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam.

[14] Tindel, S., Tudor, C., and Viens, F. (2003) Stochastic evolution equations with fractional Brownian motion; *Prob. Th. and Rel. Fields* 127, 186–204.

[15] Vishik, M. J., and Fursikov, A. V. (1988) *Mathematical Problems in Statistical Hydromechanics*, Kluwer Academic Publ., Boston.

[16] Xua, T., and Zhang, T. (2006) Large deviation principles for 2-D stochastic Navier–Stokes equations driven by Lévy processes. *Journal of Functional Analysis*, **257** (5) 1519-1545.

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