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Stokes formula on the Wiener space and n-dimensional Nourdin–Peccati analysis

Hélène Airault [a,*], Paul Malliavin [b], Frederi Viens [c]

[a] *Université de Picardie Jules Verne, Insset, 48 rue Raspail, 02100 Saint-Quentin (Aisne), LAMFA, UMR 6140, 33, rue Saint-Leu, 80000 Amiens, France*

[b] *10, rue Saint-Louis-en-l'Ile, 75004 Paris, France*

[c] *Dept. Statistics and Dept. Mathematics, Purdue University, 150 N. University St., West Lafayette, IN 47907-2067, USA*

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Abstract

Extensions of the Nourdin–Peccati analysis to $\mathbf{R}^n$-valued random variables are obtained by taking conditional expectation on the Wiener space. Several proof techniques are explored, from infinitesimal geometry, to quasi-sure analysis (including a connection to Stein's lemma), to classical analysis on Wiener space. Partial differential equations for the density of an $\mathbf{R}^n$-valued centered random variable $Z = (Z^1, \dots, Z^n)$ are obtained. Of particular importance is the function defined by the conditional expectation given Z of the auxiliary random matrix $(-D\mathcal{L}^{-1}Z^i \mid DZ^j)$, $i, j = 1, 2, \dots, n$, where D and $\mathcal{L}$ are respectively the derivative operator and the generator of the Ornstein–Uhlenbeck semigroup on Wiener space.

Keywords: Wiener space; Quasi-sure analysis; Density formula; Nourdin–Peccati analysis; Stein's lemma; Conditional probability

1. Introduction

We set up various systems of partial differential equations for the density and other functionals of the distribution of a random variable, extending to the n-dimensional case a formula from [12]

* Corresponding author.
E-mail addresses: helene.airault@u-picardie.fr (H. Airault), paul.malliavin@upmc.fr (P. Malliavin), viens@stat.purdue.edu (F. Viens).

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based on the Nourdin–Peccati analysis introduced in [8]. The basic tool is the projection (or equivalently the image) of a vector field defined on the Wiener space through a centered non-degenerate map Z, as defined in [7, p. 70]; we connect this projection to [8] and [12] by extending the use of the random variable $A = (-D\mathcal{L}^{-1}Z \mid DZ)$ defined and employed in those references, where D is the derivative operator and $\mathcal{L}$ is the Ornstein–Uhlenbeck generator; these are defined precisely below, also see [7].

For instance, for an $\mathbf{R}^n$-valued random variable $Z = (Z^1, Z^2, \dots, Z^n)$ having a density ρ the one-dimensional density formula in [12] is generalized herein to the system of partial differential equations in $\mathbf{R}^n$

$$\frac{\partial}{\partial x_1}\left(\beta_j^1 \rho\right) + \frac{\partial}{\partial x_2}\left(\beta_j^2 \rho\right) + \dots + \frac{\partial}{\partial x_n}\left(\beta_j^n \rho\right) = -x_j \rho \quad \text{for } j = 1, 2, \dots, n, \tag{0.1}$$

where β_j^k is the function on $\mathbf{R}^n$ given by the conditional expectation

$$\beta_j^k(x) = E^{Z=x}\left[\left(-D\mathcal{L}^{-1}Z^j \mid DZ^k\right)\right]. \tag{0.2}$$

First, we introduce definitions and notations, including those needed for the above expressions.

1.1. Non-degenerate maps

We denote by Ω the Wiener space, by μ the standard Wiener measure and by $\mathcal{H}$ its Cameron–Martin space. For a function $h \in \mathcal{H}$, and $\omega \in \Omega$, we define the derivative operator D, see [6,7], by

$$D_h F(\omega) := \lim_{\epsilon \to 0} \frac{F(\omega + \epsilon h) - F(\omega)}{\epsilon} = \int_0^1 D_s F(\omega) h'(s)\, ds. \tag{1.1.1}$$

For a vector field V on the Wiener space and $F : \Omega \to \mathbf{R}$, we denote $D_V F := (V \mid DF)$ where $(. \mid .)$ is the scalar product in the Cameron–Martin space. The divergence operator δ is the dual of the operator D in L^2_μ. This means that for a vector field A and a random variable Ψ (i.e. Ψ is a real-valued measurable function defined on the Wiener space Ω), respectively in the domain of the operators δ and D,

$$E\left[\delta(A)\Psi\right] = E\left[(A \mid D\Psi)\right]. \tag{1.1.2}$$

We define the *Ornstein–Uhlenbeck operator* $\mathcal{L}$ by

$$\mathcal{L} := -\delta D. \tag{1.1.3}$$

We have

$$\mathcal{L}F = -\int_0^1 D_s F(\omega)\, d\omega(s). \tag{1.1.4}$$

The pseudo-inverse $\mathcal{L}^{-1}$ of the operator $\mathcal{L}$ plays an important role in our study. For a function $F:\Omega \to R$ such that $E[F]=0$, we put

$$\mathcal{L}^{-1} = -\int_0^{+\infty} e^{t\mathcal{L}}\,dt. \tag{1.1.5}$$

Using Wiener chaos, for any $F \in L^2_\mu(\Omega)$, we can find a sequence of symmetric functions $f_n \in L^2([0,1]^n)$, $n = 1,2,\ldots,$ such that $F = E[F] + \sum_{n=1}^{\infty} I_n(f_n)$ and $\mathrm{Var}[F] = \sum_{n=1}^{\infty} n!\|f_n\|^2_{L^2([0,1]^n)}$ where $I_n(f)$ is $n!$ times the iterated Itô integral of f w.r.t. the Wiener process ω:

$$I_n(f) = n!\int_0^1 \int_0^{s_{n-1}} \cdots \int_0^{s_2} f(s_1,\ldots,s_n)\,d\omega(s_1)\cdots d\omega(s_n).$$

Then it turns out that $\mathcal{L}F = -\sum_{n=1}^{\infty} nI_n(f_n)$ and $\mathcal{L}^{-1}F = -\sum_{n=1}^{\infty} n^{-1}I_n(f_n)$. Details are in Nualart's book [13, Chapter 1].

Let $Z = (Z^1, Z^2, \ldots, Z^n)$ be an $\mathbf{R}^n$-valued random variable defined on the Wiener space Ω. Assume that $Z \in \mathbf{D}^\infty(\Omega)$, i.e. Z is infinitely differentiable with respect to the operator D. Let M be the $n\times n$ Gram matrix having for coefficients $(DZ^i \mid DZ^j)$; assume that $[\det(M)]^{-1} \in L^p_\mu$ for every p: if all the above conditions are fulfilled, we say that *the map Z is non-degenerate*. It is known from [7] that the law of a non-degenerate map Z has a density ρ relatively to the volume measure of $\mathbf{R}^n$, and that ρ is infinitely differentiable. We denote $Z * \mu$ the image of μ through the map Z; thus $Z*\mu = \rho(x)\,dx$. For $f:\mathbf{R}^n \to \mathbf{R}$

$$\int f\big(Z(\omega)\big)\,d\mu(\omega) = \int f(x)\rho(x)\,dx \tag{1.1.6}$$

where $dx = dx_1\,dx_2\cdots dx_n$ is the volume measure on $\mathbf{R}^n$. Let Φ be an $\mathbf{R}$-valued function defined on the Wiener space and consider the measure $d\nu = \Phi(\omega)\,d\mu(\omega)$; we denote $Z*\Phi(\omega)\,d\mu$ the image measure of ν by Z. If this image measure has a density with respect to the volume measure dx, we denote this density $dZ*(\Phi\,d\mu)/dx$. The conditional expectation of Φ given $Z=(x_1,x_2,\ldots,x_n)$ is

$$E^{Z=(x_1,x_2,\ldots,x_n)}[\Phi] = \frac{dZ*(\Phi\,d\mu)/dx}{dZ*d\mu/dx}(x_1,x_2,\ldots,x_n). \tag{1.1.7}$$

We denote this function by $E^Z[\Phi]$. By definition, for any integrable function $\psi:\mathbf{R}^n \to R$,

$$E\big[\psi\big(Z(\omega)\big)\Phi(\omega)\big] = \int \psi\big(Z(\omega)\big)\Phi(\omega)\,d\mu(\omega), \tag{1.1.8}$$

$$\int \psi(x)E^{Z=x}[\Phi]\rho(x)\,dx = \int \psi\big(Z(\omega)\big)\Phi(\omega)\,d\mu(\omega). \tag{1.1.9}$$

1.2. Inner action of vector fields on differential forms

Let v be a vector field on a differentiable manifold $\mathcal{M}$ of dimension n, we assume $n \geqslant 2$; denote $\bigwedge^p(\mathcal{M})$ the vector space of differential forms of degree p on $\mathcal{M}$; for $p > 1$, the *inner product*

$$\mathrm{i}(v) : \bigwedge^p(\mathcal{M}) \mapsto \bigwedge^{p-1}(\mathcal{M}) \tag{1.2.1}$$

is defined through the identity

$$\bigl\langle \mathrm{i}(v)(\Theta), e_1 \wedge \cdots \wedge e_{p-1} \bigr\rangle = \langle \Theta, v \wedge e_1 \wedge \cdots \wedge e_{p-1} \rangle \tag{1.2.2}$$

where $e_1, \ldots, e_{p-1}$ are generic vector fields on $\mathcal{M}$. In particular if $\mathcal{M} = \mathbf{R}^n$, and

$$\theta := dx^1 \wedge \cdots \wedge dx^n \tag{1.2.3}$$

is the canonical volume form of $\mathbf{R}^n$, then $\mathrm{i}(v)(\theta) \in \bigwedge^{n-1}(\mathbf{R}^n)$ and if we represent the vector field v as

$$v = \sum_{k=1}^n \beta^k \frac{\partial}{\partial x^k} \tag{1.2.4}$$

we get

$$\begin{aligned} \mathrm{i}(v)(\theta) &= \beta^1 \times dx^2 \wedge \cdots \wedge dx^n - \beta^2 \times dx^1 \wedge dx^3 \wedge \cdots \wedge dx^n + \cdots \\ &= \sum_{j=1}^n (-1)^{j+1} \beta^j \times \bigwedge_{k \neq j} dx^k. \end{aligned} \tag{1.2.5}$$

Let d be the classical differential: if $u : \mathbf{R}^n \to \mathbf{R}$ is a function, then $du = \sum_{k=1}^n \frac{\partial u}{\partial x^k} dx^k$ and for forms, if

$$\alpha = \sum_{i_1 < i_2 < \cdots < i_p} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$

then

$$d\alpha = \sum_{i_1 < i_2 < \cdots < i_p} (d\alpha_{i_1 \cdots i_p}) \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}. \tag{1.2.6}$$

We obtain

$$\begin{aligned} d\bigl(\mathrm{i}(v)(\theta)\bigr) &= d\beta^1 \wedge dx^2 \wedge \cdots \wedge dx^n - d\beta^2 \wedge dx^1 \wedge dx^3 \wedge \cdots \wedge dx^n + \cdots \\ &= \sum_k \frac{\partial \beta^k}{\partial x^k} \times \theta \end{aligned} \tag{1.2.7}$$

and

$$du \wedge \mathrm{i}(v)(\theta) = \left(\sum_{k=1}^{n} \frac{\partial u}{\partial x^k}\, dx^k\right) \wedge \mathrm{i}(v)(\theta)$$
$$= \sum_{k=1}^{n} \beta^k \frac{\partial u}{\partial x_k} \times \theta = D_v u \times \theta \tag{1.2.8}$$

where v is the vector field (1.2.4) and we denote

$$D_v u = \sum_{k=1}^{n} \beta^k \frac{\partial u}{\partial x_k}. \tag{1.2.9}$$

1.3. Image of a vector field V on the Wiener space through a non-degenerate map $Z : \Omega \to \mathbf{R}^n$

Given a measure $m(dx)$ and a vector field v on $\mathbf{R}^n$, $n \geqslant 1$, we define the function $\mathrm{div}_m(v)$ via the relation

$$\int \mathrm{div}_m(v)(x)\psi(x)\, dm = \int D_v\psi(x)\, dm, \quad \forall \psi : \mathbf{R}^n \to \mathbf{R},\ \psi \text{ integrable.} \tag{1.3.1}$$

Assume that $m(dx) = e^u\, dx$ where $dx = \theta$ is the volume measure and $u : \mathbf{R}^n \to \mathbf{R}$ is a function. If $v = \sum_k \beta^k \frac{\partial}{\partial x^k}$, then integration by parts yields

$$\int \sum_k \beta^k(x) \frac{\partial \psi}{\partial x^k}(x) e^{u(x)}\, dx = -\sum_k \int \beta^k(x)\psi(x) \frac{\partial u}{\partial x_k} e^{u(x)}\, dx - \sum_k \int \psi(x) \frac{\partial \beta^k}{\partial x_k}(x) e^{u(x)}\, dx$$

which is the same as

$$\mathrm{div}_m(v)(x) = -D_v u - \sum_k \frac{\partial \beta^k}{\partial x^k} \tag{1.3.2}$$

or equivalently, letting $m(dx) = \rho(x)\, dx$,

$$\mathrm{div}_{\rho\, dx}(v)(x) = -\frac{1}{\rho(x)} \sum_k \frac{\partial}{\partial x^k}\big(\beta^k(x)\rho(x)\big). \tag{1.3.3}$$

Let V be a vector field on Ω and let $Z = (Z^1, Z^2, \ldots, Z^n)$ be a non-degenerate map $Z : \Omega \to \mathbf{R}^n$ with density ρ with respect to the volume measure dx on $\mathbf{R}^n$, then for any integrable function $\psi : \mathbf{R}^n \to \mathbf{R}$,

$$\int \psi(x) E^{Z=x}[\delta V]\rho(x)\, dx = E\big[(\delta V)(\omega)\psi\big(Z(\omega)\big)\big] = \sum_{k=1}^{n} E\left[\frac{\partial \psi}{\partial x^k}(Z)\big(V \mid DZ^k\big)\right].$$

We define the vector field v on $\mathbf{R}^n$ as

$$v = \sum_{k=1}^{n} \beta^k(x) \frac{\partial}{\partial x^k} \quad \text{with } \beta^k(x) = E^{Z=x}\big[D_V Z^k\big] = E^{Z=x}\big[\big(V \mid DZ^k\big)\big]. \tag{1.3.4}$$

Then from Theorem 2.4, p. 70 in [7], we have

$$\operatorname{div}_{\rho\, dx}(v) = E^Z[\delta V]. \tag{1.3.5}$$

The relation (1.3.5) extends to differential forms, see Section 7.3, p. 142 in [7].

1.4. Stein and Nourdin–Peccati lemmas

Recall the following (see Lemma 1.2, part (iii) in [8], and Stein's original presentation [15]).

Classical Stein equation. For a measurable function $s : \mathbf{R} \mapsto \mathbf{R}$ such that $\|s\|_\infty \leqslant 1$, denote

$$E\big[s(N)\big] = \int_{\mathbf{R}} (2\pi)^{-1/2} s(y) e^{-y^2/2}\, dy$$

where N is a centered gaussian variable with variance 1, then

$$f(x) = e^{x^2/2} \int_{-\infty}^{x} \big[s(t) - E\big[s(N)\big]\big] e^{-t^2/2}\, dt \tag{1.4.1}$$

is the unique bounded solution of the differential equation

$$f'(x) - x f(x) = s(x) - E\big[s(N)\big] \tag{1.4.2}$$

for a.e. x; it satisfies $\|f\|_\infty \leqslant 4\sqrt{2\pi}$ and $\|f'\|_\infty \leqslant 4$.

The differential equation for f is called Stein's equation. One main interest of Stein's equation comes from the boundedness properties of f and f'. See Lemma 2.5, p. 594 in [15]. Nourdin and Peccati [8] proved that the existence of a bounded solution for Stein's equation (Stein's lemma), an analytic result, is equivalent to the following non-analytic (probabilistic) interpretation.

Nourdin–Peccati lemma. For a measurable function $s : \mathbf{R} \mapsto \mathbf{R}$, such that $\|s\|_\infty \leqslant 1$, there exists a continuous and Lebesgue almost everywhere differentiable function f with a derivative bounded by 4 which satisfies

$$E\big[f'(Z)\big(1 - h(Z)\big)\big] = E\big[s(Z)\big] - E\big[s(N)\big] \tag{1.4.3}$$

for every non-degenerate map $Z : \Omega \to \mathbf{R}$ and its corresponding function h defined by

$$h(x) = E^{Z=x}\big[\big(-D\mathcal{L}^{-1} Z \,\big|\, DZ\big)\big] \tag{1.4.4}$$

and N denotes a standard normal r.v.

The Nourdin–Peccati lemma is useful when one wishes to compare the distribution of a random variable Z to the normal distribution, by considering its action on all the functions s in the unit ball: indeed, the boundedness of f' in the above lemma shows that an upper bound on the difference between the expectation of $s(Z)$ and the corresponding normal expectation, for any such s, is 4 times the quantity $E[|h(Z)-1|]=\int_{\mathbf{R}}|h(x)-1|\rho(x)\,dx$. Since the function h does not depend on s, but only on the law of Z, this device identifies the proximity of Z to a normal r.v. by how close the function h is to the constant 1. Many details on this technique can be found in [8–11].

The equivalence of Stein's equation with Nourdin–Peccati identity (1.4.3) can be seen via the key formula (1.1.2). Starting from Stein's equation, we replace the variable x by Z and we take the expectation, it gives

$$E\big[f'(Z)-Zf(Z)\big]=E\big[s(Z)\big]-E\big[s(N)\big]. \tag{1.4.5}$$

Since $Z=-\delta D\mathcal{L}^{-1}Z$, with (1.1.2), we deduce

$$E\big[Zf(Z)\big]=E\big[\big(-D\mathcal{L}^{-1}Z \mid DZ\big)f'(Z)\big]=E\big[h(Z)f'(Z)\big]$$

and we obtain (1.4.3). Conversely, from (1.4.3), with (1.1.2), we deduce (1.4.5). Indeed, for any non-degenerate map Z with density ρ,

$$\int\limits_{\mathbf{R}}\big(xf(x)-f'(x)\big)\rho(x)\,dx=-\int\limits_{\mathbf{R}}\big(s(x)-E\big[s(N)\big]\big)\rho(x)\,dx.$$

This implies that f satisfies Stein's equation.

Other applications are estimates for the distribution function of Z, see (3.14) in [17]. The Nourdin–Peccati lemma applies to areas as diverse as mathematical physics and theoretical statistics, this can be found in [3,14,17]: the first deals with estimating the long-memory parameter of a fractional Brownian motion, the second finds upper and lower bounds for the density of the solution of a stochastic heat equation with non-linear drift, the third proves that Brownian polymers in some spatially correlated white-noise environments have diffusive fluctuation.

In Section 2, the analysis of Nourdin and Peccati, via the random variable $A=(-D\mathcal{L}^{-1}Z \mid DZ)$, is extended to $\mathbf{R}^n$ in a general geometric setting thanks to an infinitesimal proof and the functional identity (1.3.5). Section 3 explains the relation between Stein's lemma and a lemma of Nourdin and Peccati, by employing the quasi-sure analysis on Wiener space. An extension of the Nourdin–Peccati analysis for $\mathbf{R}^n$-valued random variables is presented in Section 4, thanks again to the quasi-sure analysis on Wiener space. In Section 5, we introduce an approach to the n-dimensional Nourdin–Peccati analysis via partial differential equations for the density of $\mathbf{R}^n$-valued random variables. The results of Section 5 are used in Section 6 to propose a way of comparing conditional probabilities of a pair of random variables to Gaussian conditional probabilities.

It is possible to obtain all the results in this paper as corollaries of the main Theorem 2.1. We have chosen to present various other proofs of the results in Sections 3–5.

2. Extension of the Nourdin–Peccati analysis to $\mathbf{R}^n$

Theorem 2.1. *Let $Z = (Z^1, Z^2, \dots, Z^n)$, $Z : \Omega \to \mathbf{R}^n$ be a non-degenerate map; given a smooth function $f : \mathbf{R}^n \to \mathbf{R}$. Set $F = f \circ Z$; assume that $F \in \mathbf{D}^\infty(\Omega)$ and that $E[F] = 0$. Define on Ω the gradient vector field*

$$V^f = -D\mathcal{L}^{-1}(F),$$

where $\mathcal{L}$ is the Ornstein–Uhlenbeck generator; define the image of V^f through Z by

$$v_x^f = \sum_{k=1}^{n} \beta^k(x) \frac{\partial}{\partial x^k}, \quad \text{where } \beta^k(x) := E^{Z=x}\big[D_{V^f} Z^k\big]$$

then if $\theta := dx_1\, dx_2 \cdots dx_n$ is the volume measure on $\mathbf{R}^n$,

$$f = \operatorname{div}_{\rho\times\theta}\big(v^f\big). \tag{2.1}$$

Proof. From Theorem (2.4) on p. 70 in [7], $E^Z(\delta(V)) = \operatorname{div}_{\rho\times\theta}(v)$: see the identity (1.3.5). To see that (2.1) is true, it is thus enough to verify that

$$f(x) = E^{Z=x}[\delta V]. \tag{2.2}$$

This results immediately from the definition of $V = -D\mathcal{L}^{-1}(f \circ Z)$, and the identity $\delta D = -\mathcal{L}$ which implies $\delta V = f \circ Z$. $\square$

Lemma 2.2. *If $n > 1$, let $\theta := dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ be the volume form in $\mathbf{R}^n$, then the identity $f = \operatorname{div}_{\rho\times\theta}(v^f)$ is equivalent to*

$$-(f\rho) \times \theta = d\big(\rho \times \mathrm{i}\big(v^f\big)(\theta)\big). \tag{2.3}$$

Proof. The notations are those of Section 1.2. Set $\rho = \exp(u)$, then (2.3) becomes

$$-f\theta = du \wedge \mathrm{i}\big(v^f\big)(\theta) + d\big(\mathrm{i}\big(v^f\big)(\theta)\big). \tag{2.4}$$

We calculate each term in the right-hand side of (2.3) with the help of Section 1.2: $d(\mathrm{i}(v^f)(\theta)) = \sum_k \frac{\partial \beta^k}{\partial x^k} \times \theta$ and $du \wedge \mathrm{i}(v^f)(\theta) = D_{v^f} u \times \theta$. Then (2.4) becomes

$$-f = D_{v^f} u + \sum_k \frac{\partial \beta^k}{\partial x^k} \tag{2.5}$$

thus the identity (2.3) is equivalent to $f = \operatorname{div}_{\rho\times\theta}(v^f)$. $\square$

By integration and by classical Stokes theorem, we immediately obtain the following.

Corollary 2.3. *For every domain $\mathcal{O} \subset \mathbf{R}^n$ defined by $\mathcal{O} = \{\phi(\xi) \geqslant 0\}$ where $\phi : \mathbf{R}^n \to R$ is a smooth map, denoting $\partial\mathcal{O}$ the boundary of $\mathcal{O}$, we have*

$$\int_{\mathcal{O}} (f\rho) \times dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = -\int_{\partial\mathcal{O}} \rho \times \mathrm{i}\big(v^f\big)\big(dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n\big). \tag{2.6}$$

The case $n = 1$. When $n = 1$ and $f(x) = x$, we have $V^f = -D\mathcal{L}^{-1}Z$, and therefore β in Theorem 2.1 takes the simpler expression

$$\beta(x) := E^{Z=x}\big[\big(DZ \mid -D\mathcal{L}^{-1}Z\big)\big] \tag{2.7}$$

which is the fundamental function introduced by Nourdin and Peccati in their analysis [8] for the purpose of comparisons of random variable laws to Normal and Gamma laws via Stein's lemmas (see Section 1.4 for a description of this comparison). When $n = 1$, $f(x) = x$ and dx is the Lebesgue measure on $\mathbf{R}$, Theorem 2.1 gives

$$x = \mathrm{div}_{\rho\, dx}\left(\beta(x)\frac{\partial}{\partial x}\right) \tag{2.8}$$

that is for any smooth integrable function $\psi : \mathbf{R} \to \mathbf{R}$,

$$\int x\psi(x)\rho(x)\,dx = \int \beta(x)\psi'(x)\rho(x)\,dx. \tag{2.9}$$

Let x_0 be fixed, then this last identity stays valid for the function $\psi = \psi_\epsilon$ such that $\psi_\epsilon(x) = 0$ if $x \leqslant x_0 - \epsilon$, $\psi_\epsilon(x) = \frac{1}{\epsilon}(x - x_0 + \epsilon)$ if $x_0 - \epsilon \leqslant x \leqslant x_0$ and $\psi_\epsilon(x) = 1$ if $x \geqslant x_0$. Passing to the limit when ϵ goes to zero, we deduce

$$\int_{x_0}^{+\infty} x\rho(x)\,dx = \beta(x_0)\rho(x_0). \tag{2.10}$$

In [12] the function β in (2.7) (called g in [12]) was subsequently used to derive a density formula. Writing $\varphi(x) = \int_x^\infty t\rho(t)\,dt$, this yields $\varphi'(x) = -x\varphi(x)/\beta(x)$. It implies the following density formula (Eq. (3.14) in [12]).

Lemma 2.4. *With Z a non-degenerate real-valued map with a density ρ with respect to Lebesgue measure, we have*

$$\rho(x) = \frac{\varphi(x)}{\beta(x)} = \frac{\varphi(0)}{\beta(x)}\exp\left(-\int_0^x \frac{y}{\beta(y)}\,dy\right) \tag{2.11}$$

on its support, where $\beta = E^Z[(DZ \mid -D\mathcal{L}^{-1}Z)]$. This function β was called g in [12].

The case $n > 1$. For $n > 1$, the identity $f = \operatorname{div}_{\rho\times\theta}(v^f)$ in (2.1) from Theorem 2.1 can be written as

$$v^f\big(\log\big(\rho(x)\big)\big) = -f(x) - \sum_{k=1}^{n} \frac{\partial}{\partial x^k}\big(\beta^k(x)\big). \tag{2.12}$$

If we vary f, taking successively $f(x_1, x_2, \dots, x_n) = x_j$ for $j = 1, \dots, n$, we obtain the system (0.1)–(0.2) which is the n-dimensional analogue of the density formula of [12] (Lemma 2.4 herein). We can also give a direct proof of (0.1)–(0.2). Summarize we have the following result.

Theorem 2.5. *Let $(Z^1, Z^2, \dots, Z^n) = Z : \Omega \to \mathbf{R}^n$ be a random variable. Assume that under the Wiener measure μ, Z has a density ρ with respect to Lebesgue measure. We define*

$$h^{i,j}(x) = E^{Z=x}\big[H^{i,j}\big] \quad \text{where } H^{i,j} = -\big(D\mathcal{L}^{-1}Z^i \mid DZ^j\big).$$

Then ρ satisfies the system of partial differential equations:

$$\frac{\partial}{\partial x_1}\big(h^{j,1}\rho\big) + \frac{\partial}{\partial x_2}\big(h^{j,2}\rho\big) + \dots + \frac{\partial}{\partial x_n}\big(h^{j,n}\rho\big) = -x_j\rho \quad \text{for } j = 1, 2, \dots, n.$$

Proof. For any suitable test function $g : \mathbf{R}^n \to \mathbf{R}$, using the relation $\mathcal{L} = -\delta D$, the duality relation for δ and D, and the definition of conditional expectation, we have

$$\begin{aligned}
I &:= E\big[Z^j g(Z^1, Z^2, \dots, Z^n)\big] = -E\big[\delta\big(D\mathcal{L}^{-1}Z^j\big)g\big(Z^1, Z^2, \dots, Z^n\big)\big] \\
&= -E\left[\sum_{k=1}^{n}\big(D\mathcal{L}^{-1}Z^j \mid DZ^k\big)\left(\frac{\partial}{\partial x_k}g\right)\big(Z^1, Z^2, \dots, Z^n\big)\right] \\
&= \sum_{k=1}^{n}\int\left(\frac{\partial}{\partial x_k}g\right)(x_1, x_2, \dots, x_n)h^{j,k}(x_1, x_2, \dots, x_n)\rho(x_1, x_2, \dots, x_n)\,dx.
\end{aligned}$$

We integrate by parts, to get

$$I = -\sum_{k=1}^{n}\int g(x_1, x_2, \dots, x_n)\frac{\partial}{\partial x_k}\big(h^{j,k}(x_1, x_2, \dots, x_n)\rho(x_1, x_2, \dots, x_n)\big)\,dx.$$

On the other hand, by definition of I, we have

$$I = \int x_j g(x_1, x_2, \dots, x_n)\rho(x_1, x_2, \dots, x_n)\,dx.$$

The result follows by identifying the last two expressions for I. $\square$

3. Density formula and quasi-sure analysis on Wiener space

The proof of Theorem 2.1 is done without quasi-sure analysis. However it is interesting to relate Stein's lemma to the coarea formula and quasi-sure analysis. See [7, pp. 86–148] for a survey of this theory. In the following, we show how the density identity in the case $n = 1$ can be deduced from the Stokes formula on a tube in the Wiener space. Let a random variable be given by a non-degenerate map $Z : \Omega \to \mathbf{R}$. We consider the random variable

$$H := (DZ \mid A) \quad \text{and its conditional expectation} \quad h(x) := E^{Z=x}(H) \tag{3.1}$$

where A is the vector field on the Wiener space defined as

$$A := -D\mathcal{L}^{-1}Z. \tag{3.2}$$

Here the notation h coincides with β used in Lemma 2.4. Since $\mathcal{L} = -\delta D$, it holds that

$$\delta(A) = Z. \tag{3.3}$$

Recall the expression of conditional expectation through the coarea formula established in [2] and exposed in [7, Theorem 6.3.1, p. 140], where Z is any non-degenerate map from the Wiener space to $\mathbf{R}^n$:

$$E\big[\psi(Z)\det\big(Z'\big) \times u\big] = \int_{\mathbf{R}^n} \psi(x) \times \Bigg[\int_{Z^{-1}(x)} u(x)\, a(dx)\Bigg]\, dx \tag{3.4}$$

where a is the area measure on the submanifold $Z^{-1}(x)$ of the Wiener space. (See [7] for a detailed definition and [4] for the classical coarea formula.)

In the case $n = 1$ we have

$$\det\big(Z'\big) = \|DZ\| = \sqrt{(DZ \mid DZ)}. \tag{3.5}$$

Denote N the vector of norm 1 defined as

$$N := \frac{1}{\|DZ\|} DZ. \tag{3.6}$$

Note that N_ω is the unit normal at the hypersurface $Z^{-1}(x)$ for $Z(\omega) = x$. Taking $u = (A \mid N) = (A \mid DZ)/\|DZ\|$ and writing $\rho(x)$ for the density of the law of Z relatively to the volume measure, it holds

$$\begin{aligned} E\big[\psi(Z)\|DZ\| \times u\big] &= E\big[\psi(Z)(A \mid DZ)\big] \\ &= \int \psi(x) E^{Z=x}\big[(A \mid DZ)\big]\rho(x)\, dx = \int \psi(x) E^{Z=x}[H]\rho(x)\, dx. \end{aligned}$$

Therefore from (3.4),

$$E^{Z=x}[H] = \frac{1}{\rho(x)} \int\limits_{Z^{-1}(x)} (A_u \mid N_u)\, a(du). \tag{3.7}$$

Theorem 3.1. *Consider the tube*

$$D(x, x') := \{\omega \in \Omega;\ Z(\omega) \in [x, x']\}, \quad x' > x, \tag{3.8}$$

then

$$-\int\limits_{Z^{-1}(x')} (A_u \mid N_u)\, a(du) + \int\limits_{Z^{-1}(x)} (A_u \mid N_u)\, a(du) = \int\limits_{D(x,x')} Z\, \mu(d\omega). \tag{3.9}$$

Proof. From the Stokes formula in [7, p. 143], identifying vector fields with 1-differential forms we get

$$-\int\limits_{Z^{-1}(x')} (A_u \mid N_u)\, a(du) + \int\limits_{Z^{-1}(x)} (A_u \mid N_u)\, a(du) = \int\limits_{D(x,x')} \delta(A)\, \mu(d\omega) \tag{3.10}$$

which, together with (3.3), proves the theorem. We also give the following direct proof of (3.10) via an approximation: if $x < x'$, define the continuous function $\phi_\varepsilon : R \to R$ with $\phi_\varepsilon(\eta) = 1$ if $x \leqslant \eta \leqslant x'$, $\phi_\varepsilon(\eta) = 0$ if $\eta \leqslant x - \varepsilon$ or $\eta \geqslant x' + \varepsilon$ and linear otherwise. Since $\phi'_\varepsilon(\eta) = 0$ if $x \leqslant \eta \leqslant x'$, we have

$$\begin{aligned} &E\big[(\delta A) 1_{x \leqslant Z(\omega) \leqslant x'}\big] \\ &\quad = \lim_{\varepsilon \to 0} \int_{x-\varepsilon}^{x} \phi'_\varepsilon(u) \Bigg(\int\limits_{Z^{-1}(u)} (A \mid N)\, da(\omega) \Bigg) du + \lim_{\varepsilon \to 0} \int_{x'}^{x'+\varepsilon} \phi'_\varepsilon(u) \Bigg(\int\limits_{Z^{-1}(u)} (A \mid N)\, da(\omega) \Bigg) du. \end{aligned}$$

Taking into account that $\phi'_\varepsilon(\eta) = 1/\varepsilon$ if $x - \varepsilon < \eta < x$ and $\phi'_\varepsilon(\eta) = -1/\varepsilon$ if $x' < \eta < x' + \varepsilon$, we obtain (3.10). Then with (3.10) and (3.3), we get (3.9). □

Combining (3.9) with (3.7) yields the following.

Corollary 3.2. *Let Z be a real valued non-degenerate map with density ρ. Then for any $x_1, x_2 \in \mathbf{R}$, with the function h defined in* (3.1),

$$-\rho(x_2) h(x_2) + \rho(x_1) h(x_1) = \int_{x_1}^{x_2} x \rho(x)\, dx. \tag{3.11}$$

Letting $x_2 \to +\infty$ leads immediately to the density formula of Lemma 2.4.

4. Nourdin–Peccati analysis for $\mathbf{R}^n$-valued random variables and quasi-sure analysis on Wiener space

The above quasi-sure analysis and resulting theorem can be generalized to a non-degenerate $\mathbf{R}^n$-valued random variable $Z = (Z^1, Z^2, \dots, Z^n)$. Consider the $\mathbf{R}^n$-valued function $F = (F^1, F^2, \dots, F^n)$ defined by

$$F^i(x_0) = \int_{D(x_0,+\infty)} x^i \,\rho(dx) = \int_{Z^{-1}(D(x_0,+\infty))} Z^j(\omega)\,d\mu(\omega), \quad i = 1, 2, \dots, n, \tag{4.1}$$

where $D(x_0, +\infty)$ is the positive orthant with corner x_0, i.e.

$$D(x_0, +\infty) := \{x \in \mathbf{R}^n;\ x^i > x_0^i,\ \forall i\}. \tag{4.2}$$

The function F is similar to a cumulative distribution function: it would be thus if one removed x^i from the integrand. The presence of the factor x^i is to facilitate comparisons to Gaussian r.v.'s, just as is the case when $n = 1$: see (2.10) and Lemma 2.4.

Assume $E[Z] = 0$. Similarly to A in (3.2), let A^i be the vector fields defined by

$$A^i := -D(\mathcal{L}^{-1})(Z^i). \tag{4.3}$$

The relevant analogue of the scalar r.v. H in (3.1) from the Nourdin–Peccati analysis is the random matrix

$$H^{i,j} := D_{A^i} Z^j = (A^i \mid DZ^j), \tag{4.4}$$

along with its conditional expectation

$$h^{i,j}(x) = E^{Z=x}[H^{i,j}]. \tag{4.5}$$

Since

$$\delta A^i = Z^i$$

we can rewrite

$$F^i(x_0) = \int_{Z^{-1}(D(x_0,+\infty))} \delta A^i(\omega)\,d\mu(\omega). \tag{4.6}$$

We can use an approximation technique to prove the following theorem when $n = 2$.

Theorem 4.1. *With Z a centered non-degenerate random variable in $\mathbf{R}^2$, with F^i defined in* (4.1)*, and $h^{i,j}$ defined in* (4.5)*, we have for each $i = 1, 2$, and each $x = (x^1, x^2) \in \mathbf{R}^2$,*

$$F^i(x) = \int_{x^2}^{\infty} h^{i,1}(x^1, u^2)\rho(x^1, u^2)\,du^2 + \int_{x^1}^{\infty} h^{i,2}(u^1, x^2)\rho(u^1, x^2)\,du^1. \tag{4.7}$$

Proof. By (4.6) we have

$$F^j(x^1, x^2) = \int\limits_{Z^{-1}(D(x_0,+\infty))} \delta A^i(\omega)\, d\mu(\omega) = E\big[\delta A^j 1_{Z^1>x^1} 1_{Z^2>x^2}\big], \quad j = 1, 2.$$

Then we proceed as in the direct proof of (3.10). We define an approximation of the set function $1_{D(x,+\infty)}$. For small $\varepsilon > 0$, we define the continuous and almost everywhere differentiable function $\phi_\varepsilon : \mathbf{R}^2 \to \mathbf{R}$ by $\phi_\varepsilon(\eta_1, \eta_2) = 1$ if $\eta = (\eta_1, \eta_2) \in D(x, +\infty)$ and $\phi_\varepsilon(\eta_1, \eta_2) = 0$ if $\eta = (\eta_1, \eta_2) \notin D(x - (\varepsilon, \varepsilon), +\infty)$. We join these two pieces by planes. For that we put $\phi_\varepsilon(\eta_1, \eta_2) = \varepsilon^{-1}(\eta^1 - (x^1 - \varepsilon))$ if $x^1 - \varepsilon < \eta^1 < x^1$ and $\eta^1 - x^1 < \eta^2 - x^2$, then $\phi_\varepsilon(\eta_1, \eta_2) = \varepsilon^{-1}(\eta^2 - (x^2 - \varepsilon))$ if $x^2 - \varepsilon < \eta^2 < x^2$ and $\eta^1 - x^1 \geqslant \eta^2 - x^2$. We have

$$1_{D(x,+\infty)} = \lim_{\varepsilon\to 0} \phi_\varepsilon \quad \text{almost everywhere w.r.t. the measure } dx$$

and the derivatives of ϕ_ε exist almost everywhere. We deduce

$$\begin{aligned} F^j(x^1, x^2) &= \lim_{\varepsilon\to 0} E\big[\delta A^j \phi_\varepsilon\big(Z^1(\omega), Z^2(\omega)\big)\big] \\ &= \lim_{\varepsilon\to 0} E\bigg[\big(A^j \mid DZ^1\big) \frac{\partial\phi_\varepsilon}{\partial\eta_1}\big(Z^1(\omega), Z^2(\omega)\big)\bigg] \\ &\quad + \lim_{\varepsilon\to 0} E\bigg[\big(A^j \mid DZ^2\big) \frac{\partial\phi_\varepsilon}{\partial\eta_2}\big(Z^1(\omega), Z^2(\omega)\big)\bigg]. \end{aligned} \tag{4.8}$$

The partial derivatives of ϕ_ε are zero for $\eta \in D(x, +\infty)$ and $\eta \notin D(x - (\varepsilon, \varepsilon), +\infty)$. On the strip (of width ε) $x^1 - \varepsilon < \eta^1 < x^1$ and $\eta^1 - x^1 < \eta^2 - x^2$, we have $\partial\phi_\varepsilon/\partial\eta_1 = \varepsilon^{-1}$. We thus obtain the theorem from (4.8). □

For fixed $x_0 \in \mathbf{R}^2$, we proved the above theorem approximating the boundary of the domain $D(x_0, +\infty)$. However we may define the boundary of the submanifold $Z^{-1}(D(x_0, +\infty))$ in Wiener space. The boundary $\partial D(x_0, +\infty)$ is constituted by the two half-lines l_1, l_2, starting from x_0 satisfying $dx_0^2|_{l_1} = 0$, and $dx_0^1|_{l_2} = 0$, i.e.

$$l_1 = \big\{\big(\eta^1, x_0^2\big),\ \eta^1 \geqslant x_0^1\big\} \quad \text{and} \quad l_2 = \big\{\big(x_0^1, \eta^2\big),\ \eta^2 \geqslant x_0^2\big\}. \tag{4.9}$$

For $k = 1, 2$ and $j = 1, 2$, $j \neq k$, we let $L_k = Z^{-1}(l_k)$, i.e.

$$L_k = \big\{Z^k(\omega) \geqslant x_0^k,\ Z^j(\omega) = x_0^j\big\} \tag{4.10}$$

thus L_1, L_2 are two submanifolds of codimension 1 of the Wiener space and the boundary of $Z^{-1}(D(x_0, +\infty))$ is

$$\partial\big[Z^{-1}\big(D(x_0, +\infty)\big)\big] = L_1 \cup L_2. \tag{4.11}$$

Then with quasi-sure analysis [5,16], we can consider the previous theorem as a projection on $\mathbf{R}^2$ of the Stokes formula on the Wiener space. Let A be a vector field on the Wiener space. For a differentiable function $\psi : \Omega \to R$, it holds $\int \psi(\omega)(\delta A)\, d\mu(\omega) = \int (A \mid D\psi)\, d\mu(\omega)$. If ψ is not differentiable but is the set function $1_{Z^{-1}(D(x_0,+\infty))}$, we have Stokes theorem

$$\int_{Z^{-1}(D(x_0,+\infty))} (\delta A)\, d\mu(\omega) = \int_{\partial[Z^{-1}(D(x_0,+\infty))]} A$$

where we specify the meaning of the boundary integral of the vector field A on the right-hand side by

$$\int_{\partial[Z^{-1}(D(x_0,+\infty))]} A = \int_{L_1} A + \int_{L_2} A.$$

This interpretation therefore generalizes to the n-dimensional case, as follows.

Theorem 4.2. *Let $Z : \Omega \to \mathbf{R}^n$ be a non-degenerate map and A be a vector field on Ω. Let $x_0 = (x^1, x^2, \ldots, x^n) \in \mathbf{R}^n$. The boundary $\partial[Z^{-1}(D(x_0, +\infty))]$ is*

$$\partial\big[Z^{-1}\big(D(x_0, +\infty)\big)\big] = \bigcup_{1 \leqslant k \leqslant n} P_k$$

where P_k is the subset of the Wiener space defined by

$$P_k = \big\{Z^k(\omega) = x^k,\ Z^j(\omega) \geqslant x^j,\ \forall j \neq k\big\}.$$

It is a submanifold of codimension one in the Wiener space. We have

$$\int_{Z^{-1}(D(x_0,+\infty))} (\delta A)\, d\mu(\omega) = \int_{\partial[Z^{-1}(D(x_0,+\infty))]} A = \sum_{j=1}^{n} \int_{P_k} A$$

with

$$\int_{P_k} A = -\frac{\partial}{\partial \eta^k}\bigg|_{\eta = x_0} \int_{Z^{-1}(D(\eta,+\infty))} \big(A \mid DZ^k\big)\, d\mu(\omega)$$

where we put $\eta = (\eta^1, \eta^2, \ldots, \eta^n)$.

Proof. Following [18], one only needs to consider distributions on the Wiener space: we have

$$\int_{Z^{-1}(D(x_0,+\infty))} (\delta A)\, d\mu(\omega) = \sum_{j=1}^{n} E\bigg[\big(A \mid DZ^j\big) \times \prod_{k,\, k \neq j} 1_{Z^k(\omega) \geqslant x^k} \times \frac{\partial}{\partial x^j} 1_{Z^j(\omega) \geqslant x^j}\bigg]. \qquad \square$$

5. Approach to n-dimensional density formulae via partial differential equations

In dimension $n \geqslant 2$, as we have seen that (0.1)–(0.2) are consequence of Section 2, it is possible to find implicit expressions for the density of an $\mathbf{R}^n$-valued random variable Z using partial differential equations as a consequence of Theorem 2.1.

5.1. A system of PDEs employing DZ and $\mathcal{L}Z$

If one is able to calculate functions based on DZ and $\mathcal{L}Z$, a system was already known in [1, pp. 355–360]. Let $Z : \Omega \to \mathbf{R}^n$ be an $\mathbf{R}^n$-valued random variable, $Z = (Z^1, Z^2, \dots, Z^n)$. We assume that Z has a density $\rho(x_1, x_2, \dots, x_n)$ with respect to the Lebesgue measure dx. We denote

$$\beta_{ij}(x) = E^{Z=x}\big[\big(DZ^i \bigm| DZ^j\big)\big] \quad \text{and} \quad \gamma_j = -E^{Z=x}\big[\mathcal{L}Z^j\big]$$

then according to [1], the density ρ satisfies the system of partial differential equations (S):

$$\frac{\partial}{\partial x_1}(\beta_{1j}\rho) + \frac{\partial}{\partial x_2}(\beta_{2j}\rho) + \cdots + \frac{\partial}{\partial x_n}(\beta_{nj}\rho) = -\gamma_j\rho \quad \text{for } j = 1, 2, \dots, n. \tag{S}$$

Like in Section 2, we can deduce the system (S) from a more general result, as we now see.

Proposition 5.1. *Let $Z = (Z^1, Z^2, \dots, Z^n) : \Omega \to \mathbf{R}^n$ be an $\mathbf{R}^n$-valued random variable with density ρ. Let $\psi : \mathbf{R}^n \to \mathbf{R}$, and denote*

$$\beta_\psi^k(x) = E^{Z=x}\big[\big(D(\psi \circ Z) \bigm| DZ^k\big)\big] \quad \textit{and} \quad \gamma_\psi(x) = -E^{Z=x}\big[\mathcal{L}(\psi \circ Z)\big]$$

then

$$\sum_{k=1}^n \frac{\partial}{\partial x^k}\big(\beta_\psi^k \rho\big) = -\gamma_\psi \rho. \tag{5.1.1}$$

Proof. For any test function $g : \mathbf{R}^n \to \mathbf{R}$ with suitable boundedness and smoothness assumptions, and μ the Wiener measure, we have by definition of the density and the conditional expectation, first using $-\mathcal{L} = \delta D$, then integration by parts, and finally the duality relation between δ and D and the chain rule for D,

$$\begin{aligned}\int g(x)\gamma_\psi(x)\rho(x)\,dx &= -\int g(Z)\mathcal{L}(\psi \circ Z)\,d\mu = \int g(Z)\delta D(\psi \circ Z)\,d\mu \\ &= \sum_k \int \frac{\partial g}{\partial x_k}(Z)\big(DZ^k \bigm| D(\psi \circ Z)\big)\,d\mu = \int \sum_k \frac{\partial g}{\partial x_k}(x)\beta_\psi^k(x)\rho(x)\,dx.\end{aligned}$$

Integrating again by parts yields (5.1.1). Another proof of this proposition is to apply Theorem 2.1 with $f(x) = \gamma_\psi(x)$. $\square$

Example 5.2. Let $Z = (\omega(t_1), \omega(t_2))$, then $\beta_{ij} = t_i \wedge t_j$ and $\mathcal{L}\omega(t_j) = \omega(t_j)$, thus $\gamma_j(x_1, x_2) = E^{Z=(x_1,x_2)}[\mathcal{L}\omega(t_j)] = E^Z[\omega(t_j)] = x_j$. The system (S) becomes

$$t_1 \frac{\partial}{\partial x_1}\rho + t_1 \wedge t_2 \frac{\partial}{\partial x_2}\rho = -x_1\rho, \qquad t_1 \wedge t_2 \frac{\partial}{\partial x_1}\rho + t_2 \frac{\partial}{\partial x_2}\rho = -x_2\rho.$$

The solution is given by $\rho(x_1, x_2) = \exp(-\frac{x_1^2}{2t_1}) \exp(-\frac{(x_2-x_1)^2}{2(t_2-t_1)})$.

As can be seen in the above example, the proposition is easily interpreted when Z is jointly Gaussian: one notes that then the system (S) becomes

$$\sum_{i=1}^{n} \beta^{i,j} \frac{\partial \rho}{\partial x_i} = -x_j \rho(x) \quad \text{for } j = 1, 2, \ldots, n,$$

whose solution is evidently the density of Z. An economy of functional parameters can be achieved, and a greater ability to compare the law of an arbitrary random variable Z to a Gaussian law, if one reverts to the use of the matrix h defined in (0.2), see Theorem 2.5. The Gaussian case is equivalent to the case where h is a constant matrix, equal to the covariance matrix of Z. We easily see in this case that the system (S) is identical to the system (0.1)–(0.2). In general, this is not the case. Indeed, while the matrix α in (S) is always symmetric, the matrix h, which coincides with the matrix α only in the Gaussian case, is typically non-symmetric when Z is not Gaussian. On the other hand, from the point of view of PDEs, assume that the $h^{j,k}$ are constants and that the matrix $(h^{j,k})$ is invertible; then the system (0.1)–(0.2) has a solution if and only if the matrix $(h^{j,k})$ is symmetric. This is proved writing the integrability conditions for the system as follows. Let h^{-1} be the inverse of the matrix h. We have $\partial \log \rho / \partial x_j = \sum_k (h^{-1})^{j,k} x_k$ and $\partial^2 \log \rho / \partial x_p \partial x_j = (h^{-1})^{j,p}$. The condition that $\partial^2 \log \rho / \partial x_p \, \partial x_j$ is symmetric in j, p implies that the matrix $(h^{-1})^{j,p}$ is symmetric.

5.2. A general system. Comparison of two random variables

The following proposition is also a consequence of Theorem 2.1. It covers both system (S) and system (0.1)–(0.2), see Theorem 2.5. Let $Y = (Y^1, Y^2, \ldots, Y^p)$ and $Z = (Z^1, Z^2, \ldots, Z^n)$ be two random variables with values respectively in $\mathbf{R}^p$ and in $\mathbf{R}^n$. Let $f : \mathbf{R}^p \to \mathbf{R}$. In the next proposition, to obtain the system (S) of Section 5.1, we take $n = p$ and $Y^j = Z^j$ and to obtain (0.1)–(0.2), we take $Y^j = \mathcal{L}^{-1} Z^j$.

Proposition 5.3. *Let $\psi : \mathbf{R}^p \to \mathbf{R}$. With $Y = (Y^1, Y^2, \ldots, Y^p)$ and $Z = (Z^1, Z^2, \ldots, Z^n)$ as above, we denote, for $x \in \mathbf{R}^n$,*

$$\gamma_\psi(x) = E^{Z=x}\big[\mathcal{L}(\psi \circ Y)\big] \quad \textit{and} \quad \beta_\psi^k(x) = -E^{Z=x}\big[\big(D(\psi \circ Y) \bigm| DZ^k\big)\big].$$

We assume that the variable Z has a density ρ with respect to the n-dimensional Lebesgue measure. Then ρ satisfies the following system of PDEs for $j = 1, 2, \ldots, n$:

$$\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\big(\beta_\psi^k \rho\big) = -\gamma_\psi \rho.$$

Proof. The proof is similar to those of Theorem 2.5 or of Proposition 5.1. For a bounded differentiable test function $g:\mathbf{R}^n\to\mathbf{R}$, we calculate

$$E\big[\mathcal{L}(\psi\circ Y)g\big(Z^1,Z^2,\dots,Z^n\big)\big]$$

in two different ways:

$$\begin{aligned}
E\big[\mathcal{L}(\psi\circ Y)g\big(Z^1,Z^2,\dots,Z^n\big)\big] &= \int g(x_1,x_2,\dots,x_n)Z*\big(\mathcal{L}(\psi\circ Y)\big)\,d\mu\\
&= \int g(x_1,x_2,\dots,x_n)\frac{Z*(\mathcal{L}(\psi\circ Y))d\mu}{Z*d\mu}Z*d\mu\\
&= \int g(x_1,x_2,\dots,x_n)E^Z\big[\mathcal{L}(\psi\circ Y)\big](x_1,x_2,\dots,x_n)Z*d\mu\\
&= \int g(x)\gamma_\psi(x)\rho(x)\,dx.
\end{aligned}$$

On the other hand

$$\begin{aligned}
E\big[\mathcal{L}(\psi\circ Y)g\big(Z^1,Z^2,\dots,Z^n\big)\big] &= -E\big[\delta\big(D(\psi\circ Y)\big)g\big(Z^1,Z^2,\dots,Z^n\big)\big]\\
&= -\sum_{k=1}^n E\bigg[\big(D(\psi\circ Y)\mid DZ^k\big)\frac{\partial g}{\partial x_k}\big(Z^1,Z^2,\dots,Z^n\big)\bigg]\\
&= \sum_{k=1}^n\int\beta_\psi^k(x)\frac{\partial g}{\partial x_k}(x)\rho(x)\,dx\\
&= -\sum_{k=1}^n\int\frac{\partial}{\partial x_k}\big(\beta_\psi^k\rho\big)(x)g(x)\,dx
\end{aligned}$$

finishing the proof of the proposition. □

The above general proposition gives information when $n=1$, $p=1$, for calculating conditional expectations for D-differentiable r.v.'s. Indeed, assume Z and V are D-differentiable, and let $Y=\mathcal{L}^{-1}V$. Then

$$\gamma(x)=E[V\mid Z=x]\quad\text{and}\quad\beta(x)=-E^{Z=x}\big[\langle D\mathcal{L}^{-1}V,DZ\rangle\big].$$

Let ρ be the density of Z. The proposition yields $(\beta\rho)'=-\gamma\rho$. In particular,

$$\beta(x)\rho(x)=\int_x^\infty\gamma(y)\rho(y)\,dy=E[V\mathbf{1}_{Z>x}].$$

This relation helps to see how β and γ are connected to the issue of how Z and V are correlated. For instance one way to signify that V and Z are from the same distribution but are non-trivially correlated, is to say that there is some constant $K \in (0,1)$ such that for $x > 0$, $\gamma(x) = E^{Z=x}[Y] \leqslant Kx$. Then for $x > 0$,

$$\beta(x)\rho(x) \leqslant K \int_x^{+\infty} z\rho(z)\,dz = K\varphi(x)$$

where we used the notation φ as in Lemma 2.4. Recall from therein that the density formula of Lemma 2.4 is equivalent to $\varphi = \rho h$ where $h(x) = -E^{Z=x}[\langle D\mathcal{L}^{-1}Z, DZ\rangle]$ as usual. Therefore on the support of Z, with $x > 0$ therein,

$$\beta(x) \leqslant Kh(x).$$

In other words, the non-trivial correlation of Y and Z can be read off of the above inequality as well.

6. Estimating conditional probabilities

Theorem 4.1 is a special case, in dimension 2, of a corollary of the PDE-based Theorem 2.5 (i.e. of the system of PDEs (0.1)–(0.2)), which we now give. Theorem 4.1 and this corollary have the advantage of not referring to the derivatives of ρ.

Corollary 6.1. *Under the assumptions of Theorem* 2.5, *recall the distribution-moment-type function F defined in* (4.1), *i.e.*

$$F^i(x) = \int_{x_1}^{\infty}\int_{x_2}^{\infty}\cdots\int_{x_n}^{\infty} y_i\,\rho(y)\,dy_n\cdots dy_2\,dy_1 = E[Z_i \mathbf{1}_{Z_1>x_1}\mathbf{1}_{Z_2>x_2}\cdots\mathbf{1}_{Z_n>x_n}]$$

for $i = 1, 2, \ldots, n$, and the corresponding matrix h from (4.5). *Then for each i, and each $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$,*

$$F^i(x) = \sum_{j=1}^{n} I^{i,j}$$

where

$$I^{i,j} := \int_{x_1}^{\infty}\cdots\widehat{\int_{x_j}^{\infty}}\cdots\int_{x_n}^{\infty} h^{i,j}(z_1,\ldots,x_j,\ldots,z_n)\rho(z_1,\ldots,x_j,\ldots,z_n)\,dz_1\cdots\widehat{dz_j}\cdots dz_n,$$

where the symbol $\widehat{\cdot}$ means that the corresponding expression is to be omitted.

Proof. For each fixed $i = 1, 2, \ldots, n$, if we integrate the corresponding equation in the system of PDEs (0.1)–(0.2), over the orthant $D(x, +\infty)$, the expression on the left-hand side of (0.1) becomes precisely the sum $-\sum_{j=1}^{n} I^{i,j}$ above, while the expression on the right-hand side of (0.1) becomes precisely $-F^i(x)$, proving the corollary. □

Let us now go back to the special case $n = 2$. Consider the conditional distribution function of Z^2 given Z^1, and conversely: with ρ_{Z^i} the density of Z^i, we define

$$\psi^1(x) := \int_{x_2}^{\infty} \rho(x_1, z_2)\, dz_2 = \rho_{Z^1}(x_1) P^{Z^1 = x_1}\big[Z^2 > x_2\big],$$

$$\psi^2(x) := \int_{x_1}^{\infty} \rho(z_1, x_2)\, dz_1 = \rho_{Z^2}(x_2) P^{Z^2 = x_2}\big[Z^1 > x_1\big].$$

Note that F^i is an antiderivative with respect to x_i of $-z_i \psi^i$. Given prior information about the marginal densities ρ_{Z_1} and ρ_{Z_2}, estimates on H translate into relations on the two functions ψ_i, as the next proposition shows.

Proposition 6.2. *With the notation of Theorem* 2.5 *with $n = 2$, assume that, for some $c \in \mathbf{R}$, for all $x \in \mathbf{R}^2$, $h^{i,i}(x) \leqslant 1$ for $i = 1, 2$, and $h^{i,j}(x) \leqslant c$ when $i \neq j$. Then*

$$\int_{x_1}^{\infty} \psi^1(z_1, x_2) z_1\, dz_1 \leqslant \psi^1(x) + c\psi^2(x),$$

$$\int_{x_2}^{\infty} \psi_2(x_1, z_2) z_2\, dz_2 \leqslant c\psi^1(x) + \psi^2(x).$$

Similarly, if the inequalities in the assumptions are both reversed, then so are the inequalities in the conclusions.

Proof. This follows immediately from Corollary 6.1 (or Theorem 4.1), the fact that $F^1(x_1, x_2) = \int_{x_1}^{\infty} \psi^1(z_1, x_2) z_1\, dz_1$ (similarly for F^2), the non-negativity of ρ, and the definitions of ψ^i. □

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